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ON MULTIDIMENSIONAL DETERMINATE MOMENT SEQUENCES

Abstract. Let $\mathfrak{X}$ be a vector subspace of $\mathbb{R}^N$, where $1 \leq N \leq \infty$ and let $\lambda_j > 0$ be a strictly positive sequence. It is proved that if two random vectors $\eta = (\eta_j)$ and $\xi = (\xi_j)$, on a finite dimensional $\mathfrak{X}$, satisfy $Ee^{\sum_{j=1}^{N} \lambda_j |\eta_j|} < \infty$ and $Ee^{\sum_{j=1}^{N} \lambda_j |\xi_j|} < \infty$, and distributions of η and ξ are continuous, then they are the same if and only if

$$E\eta_1^{n_1} \ldots \eta_N^{n_N} e^{-\sum_{j=1}^{N} \lambda_j |\eta_j|} = E\xi_1^{n_1} \ldots \xi_N^{n_N} e^{-\sum_{j=1}^{N} \lambda_j |\xi_j|}$$

holds eventually for all large multiindices $(n_1, n_2, \ldots, n_N)$. Finally we characterize those finite signed measures μ on $\mathfrak{X}$ so that

$$\begin{aligned} j \to\ & m_{n_1,\ldots,n_{i-1},j,n_{i+1},\ldots,n_k} \\ &= \int x_1^{n_1} x_2^{n_2} \cdots x_{i-1}^{n_{i-1}} x_i^j x_{i+1}^{n_{i+1}} \cdots x_k^{n_k} e^{-\sum_{j=1} \lambda_j |x_j|} \, d\mu(\underline{x}) \end{aligned}$$

is eventually constant or periodic. Analogous results are obtained for $N = \infty$.

1. Introduction

It has been recently proved by G.D. Lin and Y.H. Too (see [LT]) that if

$$\int_a^\infty g(x) x^n e^{-\lambda x} dx = \text{const}, \tag{1}$$

for all $n \geq n_0$, and if g is integrable, then $g = 0$ almost everywhere on (a, ∞). Following [Fu] we say that eventually constant moment sequences are determinate in the class of densities $g(x)e^{-\lambda x}$, where $g \in L^1(a, \infty)$. Our

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main goal is to extend results of [LT] to a multidimensional case and on some infinite dimensional Banach spaces.

Let (X, ρ) be a Polish metric space (i.e. separable and complete). By $M(X)$ we denote the Banach lattice of all real, signed, finite, σ additive and Borel measures μ on X. The set of all continuous measures is denoted by $M_c(X)$. If X is a subset of a finite dimensional Euclidean space then $M_{ac}(X)$ stands for all absolutely continuous signed (with respect to the Lebesgue measure) measures concentrated on X. The following generalizes the notion of complete sequences (see [LT]).

DEFINITION 1. Let $\mathfrak{L}$ be a subset of $M(X)$ and $\underline{m} = (m_n)_{n\geq 0}$ be a sequence of real numbers ($\underline{m} = (m_{n_1,\ldots,n_N})_{n_j\geq 0}$ be a multisequence $\underline{m} : \mathbb{N}_0^N \to \mathbb{R}$). A sequence $\mathcal{F}$ of Borel functions $f_n : X \to \mathbb{R}$ (an indexed family $\mathcal{F} = \{f_{n_1,\ldots,n_N} : n_j \in \mathbb{N}_0\}$ of Borel functions) is said to be *$\underline{m}$-complete* on $\mathfrak{L}$ if there exists a **unique** $\nu \in \mathfrak{L}$ such that every measure $\mu \in \mathfrak{L}$ satisfying the system of equations

$$\int_X f_n \, d\mu = m_n, \tag{2}$$

$$\left(\int_X f_{n_1,\ldots,n_N} d\mu = m_{n_1,\ldots,n_N} \text{ respectively}\right) \tag{3}$$

has the representation $\mu = \mu_0 + \nu$, where $\mu_0 \in M(X)$ is concentrated on the set

$$Z_{\mathcal{F}} = \{x \in X : f(x) = 0 \quad \text{for all} \quad f \in \mathcal{F}\}.$$

We say that $\mathcal{F}$ is *strictly $\underline{m}$–complete* if $Z_{\mathcal{F}}$ is the empty set (in particular the system of identities (2) has a unique solution in $\mathfrak{L}$).

The following two problems are addressed in this paper:

- • Given a family $\mathcal{M}$ of real sequences $\underline{m}$ and a class $\mathfrak{L} \subseteq M(X)$ find a sequence (indexed family) of Borel functions on X which is $\underline{m}$–complete on $\mathfrak{L}$ for every $\underline{m} \in \mathcal{M}$.
- •• Given a class $\mathfrak{L} \subseteq M(X)$ and a $\underline{m}$–complete family of Borel functions on X, where $\underline{m}$ is an element of a fixed family $\mathcal{M}$ of real sequences, find the formula

$$\mathcal{M} \ni \underline{m} \to \nu_{\underline{m}} \in \mathfrak{L}.$$

The result by Lin and Too, mentioned in the beginning of the paper, may be formulated as follows

THEOREM 1. (see [LT]) *Let $X = (a, b)$, where $-\infty < a < b \leq +\infty$, and $L^1(a, b)$ be the class of all Lebesgue integrable functions on (a, b). For every $n_0 \in \mathbb{N}_0$ and $\lambda > 0$ the family $\mathfrak{F}_{\lambda,n_0} = \{x^n e^{-\lambda x} : n \geq n_0\}$ is $\underline{c}$–complete on*

$L^1(a,b)$, where $\underline{c}$ is a constant sequence $c_n = c$. Moreover, if it holds, then $c = 0$ and (2) has a unique solution $\nu_{\underline{0}} = 0$.

REMARK 1. It has been actually proved in [LT] that $\mathfrak{F}_{\lambda,n_0}$ is $\underline{c}$–complete on the class $M_c(a,b)$.

REMARK 2. Our restriction to study $\underline{m}$-completeness, only for some specified classes $\mathcal{M}$ and $\mathfrak{L}$, is essential in the light of Boas's theorem (see [C] Theorem 6.3, page 74), which asserts that given any sequence of real numbers m_n there exists a signed measure μ, such that $\int_0^\infty x^n\, d\mu(x) = m_n$.

2. One dimensional case

In this section we generalize and simplify the proof of Theorem 1 from [LT]. The idea of studying behaviour of derivatives of the transform of measures is also inherited from [LT]. We discuss eventually periodic sequences $\underline{m}$, instead of constants. We say that a sequence $\underline{m} = (m_n)_{n\geq 1}$ of real numbers is eventually d periodic, if there exists a positive L, such that $m_{n+d} = m_n$ holds for all $n \geq L$. We begin with the following commonly known fact (see Proposition 43.1 in [P]). For the sake of completeness of the paper and the conveninence of the reader a detailed proof is included.

LEMMA 1. *Let $-\infty \leq a < b \leq +\infty$ and $F : (a,b) \to \mathbb{R}$ satisfies*

$$0 < |F(x)| \leq Ae^{-\varepsilon|x|} \tag{4}$$

for some $A, \varepsilon > 0$. Then the only signed measure $\mu \in M(a,b)$ satisfying

$$\int_{(a,b)} x^n F(x)\, d\mu(x) = 0 \tag{5}$$

for all $n \geq 0$ is the zero measure.

If instead of (4) we assume the weaker condition

$$0 \leq |F(x)| \leq Ae^{-\varepsilon|x|} \tag{6}$$

then a measure μ satisfies (5) if and only if it is concentrated on the set $\{x \in (a,b) : F(x) = 0\}$.

P r o o f. We define

$$\Psi(z) = \int_{(a,b)} e^{izx} F(x)\, d\mu(x).$$

By (4) the function Ψ is well defined and analytic on the complex halfplane $E = \{z \in \mathbb{C} : \operatorname{Im}(z) > -\varepsilon\}$, containing $\mathbb{R}$. Clearly its $\mathrm{n^{th}}$ derivative is given by

$$\Psi^{(n)}(z) = i^n \int_{(a,b)} e^{izx} x^n F(x)\, d\mu(x).$$

Using the Lebesgue dominated convergence theorem (which is applicable because of (4)) we obtain

$$\Psi^n(z) = i^n \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{k!} \int_{(a,b)} x^{n+k} F(x)\, d\mu(x) = 0$$

for all $z = i\beta$, where $\beta \in (-\varepsilon, \varepsilon)$. This implies $\Psi(z) = \text{const} = \Psi(0) = 0$ on the halfplane E. In particular

$$\Psi(t) = \int_{(a,b)} e^{itx} F(x)\, d\mu(x) = 0$$

for all $t \in \mathbb{R}$. This implies that $F(x)d\mu(x)$ is the zero measure. If we assume (4) then obviously $\mu = 0$ as $F(x) \neq 0$. If (6) holds we only get $\text{supp}(\mu) \subseteq \{x : F(x) = 0\}$. ■

Immediately we obtain

COROLLARY 1. *Let a, b and F be as in Lemma 1. If $\mu \in M(a,b)$ satisfies*

$$\int_{(a,b)} x^n F(x)\, d\mu(x) = 0 \tag{7}$$

for $n \geq n_0$, where n_0 nonnegative, then $\mu = t\delta_0$ for some scalar t. If only (6) is assumed, then $supp(\mu) \subseteq \{x : F(x) = 0\} \cup \{0\}$.

Proof. It is enough to substitute in Lemma 1 the measure $x^{n_0} d\mu(x)$ instead of μ and adjust constants A and ε. ■

The following is the main result of this chapter.

THEOREM 2. *Let $-\infty \leq a < b \leq +\infty$ and $F : (a,b) \to \mathbb{R}$ satisfies the property (4) on X. Suppose that $\underline{m}$ is eventually d periodic and a signed measure $\mu \in M(a,b)$ satisfies*

$$\int_{(a,b)} x^n F(x)\, d\mu(x) = m_n \quad \text{for all} \quad n \geq n_0. \tag{8}$$

Then the following hold:

(i) *If d is even then μ is concentrated on $\{-1, 0, 1\} \cap (a,b)$.*
If $a < -1$ and $b > 1$ then $m_{n+2} = m_n$ for all $n \geq 1$ (hence $d = 2$) and

$$\mu = \nu_{\underline{m}} = \frac{m_2 - m_3}{2F(-1)}\delta_{-1} + \frac{m_0 - \frac{m_2 - m_3}{2} - \frac{m_2 + m_3}{2}}{F(0)}\delta_0 + \frac{m_2 + m_3}{2F(1)}\delta_1.$$

If $-1 \leq a$ then $m_1 = m_2 = \ldots$ and $d = 1$ is actually odd.

(ii) *If d is odd then $d = 1$ and μ is concentrated on $\{0, 1\} \cap (a,b)$, and $m_1 = m_2 = \ldots$.*

If $\{0,1\} \subset (a,b)$ *then*

$$\mu = \nu_{\underline{m}} = \frac{m_0 - m_1}{F(0)}\delta_0 + \frac{m_1}{F(1)}\delta_1.$$

If $\{0,1\} \cap (a,b) = \{0\}$ *then* $0 = m_1 = m_2 = \dots$ *and*

$$\mu = \nu_{\underline{m}} = \frac{m_0}{F(0)}\delta_0.$$

If $\{0,1\} \cap (a,b) = \{1\}$ *then* $m_0 = m_1 = \dots$

$$\mu = \nu_{\underline{m}} = \frac{m_1}{F(1)}\delta_1.$$

If $\{0,1\} \cap (a,b) = \emptyset$ *then* $0 = m_0 = m_1 = \dots$ *and* $\mu = 0$.

Proof. Applying Corollary 1 we obtain $x^{n_0}(1 - x^d)F(x)d\mu(x) = 0$. Since $F(x) \neq 0$ on (a,b) thus μ is concentrated on $\{-1,0,1\} \cap (a,b)$ if d is even, or it is concentrated on $\{0,1\} \cap (a,b)$ if d is odd. Moreover, if d is even then

$$\begin{aligned} m_{n+2} &= \int\limits_{(a,b)} x^{n+2} F(x)\, d\mu(x) \\ &= (-1)^n(-1)^2 F(-1)\mu(\{-1\}) + 1^{n+2}F(1)\mu(\{1\}) \\ &= (-1)^n F(-1)\mu(\{-1\}) + 1^n F(1)\mu(\{1\}) = m_n \end{aligned}$$

hold for all $n \geq 1$. This means that $d = 2$ or $d = 1$. For an arbitrary nonzero n we obtain

$$\left\{ \begin{aligned} m_{2n+1} &= -F(-1)\mu(\{-1\}) + F(1)\mu(\{1\}) \\ m_{2n} &= F(-1)\mu(\{-1\}) + F(1)\mu(\{1\}). \end{aligned} \right. \tag{9}$$

It follows from (9) that if $-1 \notin (a,b)$ or $1 \notin (a,b)$, then the sequence $\underline{m}$ is eventually constant. Hence it is eventually 1–periodic. Assuming that both -1 and 1 belong to (a,b) we can easily evaluate

$$\mu(\{-1\}) = \frac{m_2 - m_1}{2F(-1)}$$

and

$$\mu(\{1\}) = \frac{m_2 + m_1}{2F(1)}.$$

Now it easily follows from $m_0 = \mu(\{-1\}) + \mu(\{0\}) + \mu(\{1\})$ that

$$\mu = \nu_{\underline{m}} = \frac{m_2 - m_3}{2F(-1)}\delta_{-1} + \frac{m_0 - \frac{m_2 - m_3}{2} - \frac{m_2 + m_3}{2}}{F(0)}\delta_0 + \frac{m_2 + m_3}{2F(1)}\delta_1.$$

In the case (ii), when d is odd, for every $n \geq 1$ we have

$$m_n = \int\limits_{(a,b)} x^n F(x)\, d\mu(x) = F(1)\mu(\{1\}) = \text{const},$$

whenever $1 \in (a,b)$, or $m_n \equiv 0$ if $1 \notin (a,b)$. Similarly we obtain that

$$m_0 = \int_{(a,b)} F(x)\,d\mu(x) = F(0)\mu(\{0\}) + F(1)\mu(\{1\})$$

if both 0 and 1 belong to (a,b). After elementary transformations we obtain:

$$\mu = \nu_{\underline{m}} = \frac{m_0 - m_1}{F(0)}\delta_0 + \frac{m_1}{F(1)}\delta_1.$$

The reader finds the remaining cases easy to verify. ∎

REMARK 3. Some cases discussed in the above theorem are included only for the sake of completeness. If $-\infty < a$ and $b < \infty$, then much stronger results are well known (see [B], Chapter 12 or [S], pages 400–403).

The next result is an easy application of our theorem and, which is a further generalization of Theorem 1 from [LT], as absolutely continuous distributions are continuous measures. We have:

COROLLARY 2. *Let $-\infty \le a < b \le +\infty$ and $F : (a,b) \to \mathbb{R}$ satisfies the property* (4). *Suppose that $\underline{m}$ is eventually d periodic and a signed measure $\mu \in M(a,b)$ satisfies*

$$\int_{(a,b)} x^n F(x)\,d\mu(x) = m_n \quad \text{for all} \quad n \ge n_0. \tag{10}$$

If μ is continuous, then $\mu = 0$.

In the sequel we will need the following extension of Theorem 2. Its proof is omitted as it is a modification of the previous proof.

COROLLARY 3. *Let $-\infty \le a < b \le +\infty$ and $F : (a,b) \to \mathbb{R}$ be a function satisfying* (4). *If for a signed measure $\mu \in M((a,b))$ the sequence of moments*

$$\int_{(a,b)} x^n F(x)\,d\mu(x) = m_n$$

is eventually periodic, then the measure μ is concentrated on the set $(\{x : F(x) = 0\} \cup \{-1,0,1\}) \cap (a,b)$. Moreover, if $m_n \equiv 0$ for all $n \ge n_0$, then μ is concentrated on $(\{x : F(x) = 0\} \cup \{0\}) \cap (a,b)$.

The functions $F_{\lambda,\alpha}(x) = e^{-\lambda|x|^\alpha}$, where $\lambda > 0$ and $\alpha \ge 1$, satisfy (4). We note that if $a > -\infty$, then for every $\lambda > 0$ the function $F_\lambda(x) = e^{-\lambda x}$ again satisfies (4) on $(a,+\infty)$. In particular, the sequence of functions $\{x^n e^{-\lambda x}\}$ (considered in [LT]) forms a complete family. Sequences of functions $\{x^n F_{\lambda,\alpha}(x)\}$ or $\{x^n F_\lambda(x)\}$ may be substituted in Theorem 2 as well. We obtain an extension of Theorems 4 and 5 from [LT].

THEOREM 3. *Let η and ξ be two random variables defined on the same probability space $(\Omega, \mathcal{A}, \text{Prob}\,)$, such that for some $\lambda > 0$ and $\alpha \ge 1$ the* *sequence*

$$E(\eta^n e^{-\lambda|\eta|^\alpha} - \xi^n e^{-\lambda|\xi|^\alpha}) = m_n, \quad n \geq 0 \tag{11}$$

becomes eventually periodic. Then

$$\text{Prob}(\eta \in A) = \text{Prob}\ (\xi \in A) \tag{12}$$

holds for every Borel $A \subseteq \mathbb{R} \setminus \{-1, 0, 1\}$. *If* $m_n = 0$ *for all* n *large enough, or if* η *and* ξ *have continuous distributions, then* η *and* ξ *have the same distributions.*

Proof. We define $\mu = \mu_\eta - \mu_\xi$ to be a signed measure, where μ_η and μ_ξ are distributions of η and ξ respectively. The condition (11) is equivalent to (8). Applying Theorem 2 we obtain (12). If both η and ξ have continuous distributions, then μ is continuous. Therefore (1) can be extended to all Borel $A \subseteq \mathbb{R}$.

Now suppose that $m_n \equiv 0$ for all $n \geq n_0$. It follows from Corollary 3 that $\text{Prob}(\eta \in A) = \text{Prob}(\xi \in A)$, for all Borel A not containing 0. But $\text{Prob}(\eta = 0) = 1 - \text{Prob}(\eta \in \mathbb{R} \setminus \{1\}) = 1 - \text{Prob}(\xi \in \mathbb{R} \setminus \{1\}) = \text{Prob}(\xi = 0)$. Hence η and ξ have the same distributions. ∎

To complete this section we briefly mention completeness on $L^p(a, b)$, where $p > 1$. Assume that for some $f \in L^p(a, b)$ the sequence

$$m_n = \int_{(a,b)} x^n F(x) f(x)\, dx \tag{13}$$

is eventually periodic, where F satisfies (4). By the Hölder inequality $f(x)e^{-\frac{\varepsilon}{2}|x|} \in L^1(a, b)$ and $e^{\frac{\varepsilon}{2}|x|}F(x)$ satisfies (4), with the coefficient $\varepsilon/2$ instead of ε. It follows from Corollary 1 that $f(x)e^{-\frac{\varepsilon}{2}|x|} = 0$ a.e., hence $f(x) = 0$ a.e..

We also notice that an arbitrary sequence $\{w_n(x)\}_{n \geq 1}$ of polynomials such that

$$\text{lin}\{w_n : n \geq 1\} = \text{lin}\{x^n : n \geq n_0\}$$

may be used to construct $\underline{0}$–complete families. For instance $\{L_n(x)e^{-|x|} : n = 0,\ 1, \ldots\}$, where L_n denote the Laguerre polynomials, and the Hermite functions $\{x^n e^{-x^2} : n = 0,\ 1, \ldots\}$ are $\underline{0}$–complete on $L^p(a, b)$. This is however well known (see [B] Chapter 12, [P] pages 214-217, or [S] pages 400–403).

3. A multidimensional moment problem

We begin this section with a brief introduction to conditional distributions. Most of the material we quote comes from [P] (see §45 and 46) and if necessary the reader is referred to this book for more details. Let X and X_1 be Polish spaces with Borel σ–algebras $\mathcal{B}$ and $\mathcal{B}_1$ respectively. Suppose that there is given a nonnegative measure P on $(X, \mathcal{B})$ and a Borel mapping $\Pi : X \to X_1$. The image of P is denoted by $Q_{P,\Pi} = P \circ \Pi^{-1}$. A *regular conditional distribution of* Π is a mapping $x_1 \to P_{x_1}$ such that:

- for each $x_1 \in X_1$, P_{x_1} is a measure on $(X, \mathcal{B})$
- there exists a set $I \in \mathcal{B}_1$ such that $Q_{P,\Pi}(I) = 0$ and for each $x_1 \in X_1 \setminus I$ we have $P_{x_1}(X \setminus X_{x_1}) = 0$, where $X_{x_1} = \{x \in X : \Pi(x) = x_1\}$
- for every set $A \in \mathcal{B}$ the mapping $X_1 \ni x_1 \to P_{x_1}(A)$ is $\mathcal{B}_1$ measurable and

$$P(A) = \int_{X_1} P_{x_1}(A)\, dQ_{P,\Pi}(x_1). \tag{14}$$

It is well known that on Polish spaces regular conditional distributions do exist (see Proposition 46.3, page 239 in [P]). The formula (14) can be easily extended to

$$\int_X h(x)\, dP(x) = \int_{X_1} \int_X h(x)\, dP_{x_1}(x)\, dQ_{P,\Pi}(x_1), \tag{15}$$

where $h \in L^1(P)$.

If μ is a signed measure on $(X, \mathcal{B})$ then

$$\mu = \frac{d\mu}{d|\mu|} |\mu|,$$

where $\frac{d\mu}{d|\mu|}$ is the Radon Nikodym derivative. Without loss of generality we will assume that $\frac{d\mu}{d|\mu|}(x) = K(x)$ for $|\mu|$ almost all $x \in X$, where K is a Borel function on X such that $|K(x)| = 1$ for all $x \in X$. Now we set $\tilde{\mu} = \frac{|\mu|}{|\mu|(X)}$ and $|\mu|_{x_1} = |\mu|(X)\tilde{\mu}_{x_1}$ and finally

$$\mu_{x_1}(A) = \int_A K(x)\, d|\mu|_{x_1}(x), \tag{16}$$

where $A \in \mathcal{B}$. Note that μ_{x_1} is a signed Borel measure concentrated on X_{x_1}. We have

$$\begin{aligned} \int_{X_1} \mu_{x_1}(A) dQ_{|\mu|,\Pi}(x_1) &= \int_{X_1} \int_A K(x)\, d|\mu|_{x_1}(x)\, dQ_{|\mu|,\Pi}(x_1) \\ &= \int_X K(x) \mathbf{1}_A(x)\, d|\mu|(x) = \mu(A). \end{aligned} \tag{17}$$

Again (17) can be extended to

$$\int_X h(x)\, d\mu(x) = \int_{X_1} \int_X h(x)\, d\mu_{x_1}(x)\, dQ_{|\mu|,\Pi}(x_1), \tag{18}$$

where $h \in L^1(|\mu|)$. We write

(19) $$\mu = \int_{X_1} \mu_{x_1}\, dQ_{|\mu|,\Pi}(x_1).$$

Now let us return to the finite dimensional moment problem and consider $\mathbb{R}^N$, the N–dimensional (real) vector space with a fixed norm $\|\cdot\|_N$. Elements of $\mathbb{R}^N$ will be denoted by $\underline{x} = (x_1, x_2, \ldots, x_N)$. Let X be a subset of $\mathbb{R}^N$. In the finite dimesional case condition (6) is naturally replaced by

(20) $$|F(\underline{x})| \leq Ae^{-\varepsilon\|\underline{x}\|_N}$$

where $F : X \to \mathbb{R}$, and $A,\ \varepsilon > 0$ are some fixed constants. Now we are in a position to formulate a finite dimensional version of Theorem 2. Namely we have

THEOREM 4. *Let X be a Borel subset of $(\mathbb{R}^N, \|\cdot\|_N)$ and μ be a finite Borel signed measure on X. Let F be a Borel function on X satisfying* (20). *Given natural numbers $n_1, \ldots, n_N$ we define*

(21) $$m_{n_1,n_2,\ldots,n_N} = \int_X x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N} F(x_1, \ldots, x_N)\, d\mu(x_1, x_2, \ldots, x_N).$$

(i) *If there exists L_N such that for every $1 \leq j \leq N$*

$$m_{n_1,\ldots,n_{j-1},n,n_{j+1},\ldots,n_N} = 0$$

for all $n \geq L_N$ and all $n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_N \geq 0$, then μ is concentrated on $\{\underline{x} : F(\underline{x}) = 0\} \cup \{(0, \ldots, 0)\}$. In particular, if $F(\underline{x}) \neq 0$ for all $\underline{x} \in X$, then $\mu = t\delta_{\underline{0}}$ for some scalar t.

(ii) *If $m_{n_1,n_2,\ldots,n_N} = 0$ for all $n_1, n_2, \ldots, n_N \geq 0$, and $F(\underline{x}) \neq 0$ for all $\underline{x} \in X$, then μ is the zero measure.*

(iii) *If for every $1 \leq j \leq N$ and all fixed $n_1, n_2, \ldots, n_{j-1}, n_{j+1}, \ldots, n_N \geq 0$ the sequence*

(22) $$n \to m_{n_1,n_2,\ldots,n_{j-1},n,n_{j+1},\ldots,n_N}$$

is eventually periodic, and $F(\underline{x}) \neq 0$ for all $\underline{x} \in X$, then μ is a discrete measure concentrated on the finite set $\{-1, 0, 1\}^N \cap X$.

Proof. We proceed with the induction for N. If $N = 1$ then (i), (ii), (iii) hold by Theorem 2. Now let us assume that they hold for all $N-1$ finite dimensional vector spaces $\mathfrak{X}$ (we notice that in our considerations the geometry of the norm on $\mathfrak{X}$ does not play any role as long as $\mathfrak{X}$ remains finite dimensional). Let us denote the projection on the k^{th} coordinate by π_k. If $m_{n_1,n_2,\ldots,n_N} \equiv 0$ for all $n_1, n_2, \ldots, n_N \geq L_N$ then

$$\int_{\mathbb{R}} x_1^{n_1} F_{n_2,\ldots,n_N}(x_1)\, dQ_{|\mu|,\pi_1}(x_1) = 0,$$

where

$$F_{n_2,\dots,n_N}(x_1) = \int_{\mathbb{R}^{N-1}}\!\!\dots\!\int x_2^{n_2}\cdots x_N^{n_N} F(x_1,x_2,\dots,x_N)\,d\mu_{x_1}(x_2,\dots,x_N),$$

and $Q_{|\mu|,\pi_1}$, μ_{x_1} come from the desintegration of μ associated with the projection $\mathbb{R}^N \ni (x_1,x_2,\dots,x_N) \to \pi_1(x_1,x_2,\dots,x_N) = x_1 \in \mathbb{R}$. The norm $\|\cdot\|_N$ is equivalent to $\|(x_1,x_2,\dots,x_N)\|_{N,1} = \sum_{j=1}^{N} |x_j|$. In particular there exists a constant $\gamma > 0$ such that

$$\begin{aligned}|F((x_1,x_2,\dots,x_N))| &\le Ae^{-\varepsilon\|(x_1,x_2,\dots,x_N)\|_N}\\ &\le Ae^{-\varepsilon\gamma\|(x_1,x_2,\dots,x_N)\|_{N,1}}\\ &= Ae^{-\varepsilon\gamma\|(x_2,\dots,x_N)\|_{N-1,1}}\cdot e^{-\varepsilon\gamma|x_1|}.\end{aligned}$$

Since $|\mu_{x_1}| \le \|\mu\|$ it follows that

$$A' = \sup_{x_1\in\mathbb{R}} A\int_{\mathbb{R}^{N-1}}\!\!\dots\!\int |x_2^{n_2}\cdots x_N^{n_N}|e^{-\varepsilon\gamma\|(x_2,\dots,x_N)\|_{N-1,1}}\,d|\mu_{x_1}|(x_2,\dots,x_N) < +\infty.$$

Finally we obtain condition (20), i.e.

$$|F_{n_2,\dots,n_N}(x_1)| \le A'e^{-\varepsilon\gamma|x_1|}$$

holds for all $x_1 \in \mathbb{R}$. By Corollary 2 the measure $Q_{|\mu|,\pi_1}$ is concentrated on $\{x_1 : F_{n_2,\dots,n_N}(x_1) = 0\} \cup \{0\}$. Since neither $Q_{|\mu|,\pi_1}$ nor μ_{x_1} depend on $n_2,\dots,n_N$, thus $Q_{|\mu|,\pi_1}$ is concentrated on $D_1 \cup \{0\}$, where

$$D_1 = \bigcap_{n_2,\dots,n_N\ge 0} \{x_1 : F_{n_2,\dots,n_N}(x_1) = 0\}.$$

By the induction assumption, if $x_1 \in D_1$, then $\mu_{x_1} = 0$. On the other hand, if $x_1 \notin D_1$, then we get

$$\mu_{x_1} = t_{x_1}\delta_{(x_1,\underbrace{0,\dots,0}_{N-1})}.$$

From this we infer that the measure $\mu = \int \mu_{x_1}dQ_{|\mu|,\pi_1}(x_1)$ is concentrated on the linear subspace $\mathbb{R}\times\underbrace{\{0\}\times\dots\times\{0\}}_{N-1}$. Repeating the above arguments to other projections π_k, where $2 \le k \le N$, there exists a scalar t such that $\mu = t\delta_{(0,\dots,0)}$.

If (ii) holds, then $t = m_{(0,\dots,0)} = 0$. Therefore, μ is the zero measure. It follows from the induction that (i) and (ii) hold for an arbitrary N.

(iii) As before, we begin the second step of the induction with the projection π_1. By Corollary 2 the measure $Q_{|\mu|,\pi_1}$ is concentrated on the set $D_1 \cup \{-1,0,1\}$. If $x_1 \notin D_1$, then it follows from the induction assumption that μ_{x_1} is concentrated on $\{-1,0,1\}^{N-1}$. If $x_1 \in D_1$, then simply $\mu_{x_1} = 0$. Therefore, for an arbitrary Borel set $A \subseteq \mathbb{R}^N$ we have

$$\begin{aligned}\mu(A) &= \int_{\mathbb{R}} \mu_{x_1}(A) dQ_{|\mu|,\pi_1}(x_1)\\ &= \int_{D_1\setminus\{-1,0,1\}} \mu_{x_1}(A) dQ_{|\mu|,\pi_1}(x_1) + \int_{\{-1,0,1\}} \mu_{x_1}(A) dQ_{|\mu|,\pi_1}(x_1)\\ &= \sum_{s\in\{-1,0,1\}} \mu_s(A \cap (\{s\} \times \mathbb{R}^{N-1})) Q_{|\mu|,\pi_1}(\{s\}).\end{aligned}$$

From the above it is easy to infer that μ is concentrated on

$$\{-1,0,1\} \times \mathbb{R}^{N-1}.$$

Applying the same arguments to other coordinates we obtain that μ is concentrated on the set $S_N = \{-1,0,1\}^N$. ∎

If in the above theorem our conditions on $m_{n_1,\ldots,n_N}$ are relaxed further, then we obtain the following proposition which applies to continuous measures. Namely, we have:

PROPOSITION 1. *Let X be a Borel subset of $(\mathbb{R}^N, \|\cdot\|_N)$ and μ be a signed and finite Borel measure on X. Assume that a Borel funtion $F(\underline{x}) \neq 0$, for all $\underline{x} \in X$, and that (20) holds. If there exist L and $j \in \{1,\ldots,n\}$ so that whenever $n_1,\ldots,n_{j-1},n_{j+1},\ldots,n_N \geq L$ then the sequence*

$$n \longrightarrow m_{n_1,\ldots,n_{j-1},n,n_{j+1},\ldots,n_N}$$

is eventually periodic, then μ is concentrated on the set

$$Z = \{-1,0,1\}^N \cup \bigcup_{l=1}^{N} \{\underline{x} \in \mathbb{R}^N \ : x_l = 0\}.$$

In particular, if $\mu \in M_{ac}(X)$, then $\mu = 0$.

Proof. Consider the function $\widetilde{F}(\underline{x}) = x_1^L \cdots x_N^L \cdot F((x_1,\ldots,x_N))$ instead of F. Similarly as in the proof of Theorem 4 we obtain that the measure $Q_{|\mu|,\pi_j}$ is concentrated on the set $\{-1,0,1\}$. If $x_j \notin \{-1,0,1\}$, then μ_{x_j} is concentrated on the set

$$\Big\{\underline{x} \in X : \int_X x_1^{n_1} \cdots x_{j-1}^{n_{j-1}} \cdot x_{j+1}^{n_{j+1}} \cdots x_N^{n_N} \cdot \widetilde{F}(\underline{x})\, d\mu(\underline{x}) = 0 \text{ for all } n_j \geq 0\Big\}.$$

Since $F(\underline{x}) \neq 0$ on X it follows that

$$\operatorname{supp}(\mu_{x_j}) \subseteq \{\underline{x} \in X : x_l = 0 \text{ for some } \ l \in \{1,\ldots,j-1,j+1,\ldots,N\}\}.$$

We have obtained that Z has Lebesgue measure 0. In particular, if μ is absolutely continuous then $\mu = 0$. ∎

Let $F(\underline{x}) \neq 0$ for all $\underline{x} \in X$ and μ be as in Theorem 4. Similarly as in Theorem 2, the system of moments $m_{n_1,n_2,\ldots,n_N}$, where $n_1, n_2, \ldots, n_N \geq 0$, completely describes μ. In order to restore μ we only need to know some of its moments. These are $m_{n_1,n_2,\ldots,n_N}$, where $0 \leq n_j \leq 2$. It remains to solve the system of equations:

$$\sum_{(s_1,s_2,\ldots,s_N)\in S_N} \prod_{j=1}^{N} s_j^{n_j} F((s_1,\ldots,s_N))\mu(\{(s_1,s_2,\ldots,s_N)\}) = m_{n_1,n_2,\ldots,n_N}$$

(according to the standard convention we assume that $(-1)^0 = 1$). We have:

COROLLARY 4. *Let X be a Borel subset of $\mathbb{R}^N$. Then for every function F satisfying* (20) *and such that $F(\underline{x}) \neq 0$ on X the family of functions*

$$\{x_1^{n_1} \cdot x_2^{n_2} \cdots x_N^{n_N} F((x_1, x_2, \ldots, x_N)) : 0 \leq n_j,\ 1 \leq j \leq N\} \tag{23}$$

is strictly $\underline{m}$-complete on $M_{ac}(X)$, where $m_{n_1,n_2,\ldots,n_N}$ is eventually constant or periodic.

It is worth emphasizing that if we consider only strictly positive F and nonnegative measures μ, then the moment problem becomes trivial and a smaller class than (23) is complete. This may be checked directly (i.e. without the use of Theorem 4) that if for every $1 \leq j \leq N$ the sequence

$$n \to \int \ldots \int x_j^n F(x_1, \ldots, x_N)\, d\mu((x_1, x_2, \ldots, x_N))$$

is eventually periodic, then since even moments $m_{2n}^{(j)}$ are separated from 0 and ∞, we obtain $\mu(\{\underline{x} \in X : x_j \notin \{-1, 0, 1\}\}) = 0$.

4. Infinite dimensional moment problem

Now let $\mathfrak{X}$ be a vector subspace of $\mathbb{R}^\infty$. As before, elements of $\mathfrak{X}$ are denoted by $\underline{x} = (x_1, x_2, \ldots)$. Given a strictly positive sequence $\lambda_j > 0$, we introduce on $\mathfrak{X}$ the functional $\|\underline{x}\|_\lambda = \sum_{j=1}^{\infty} |x_j|\lambda_j$. It will be always assumed that $\|\cdot\|_\lambda$ is finite on $\mathfrak{X}$; hence $(X, \|\cdot\|_\lambda)$ becomes a separable Banach space. The elements of its dual $\mathfrak{X}^*(= \ell^\infty)$ are denoted by $\underline{x}^* = (x_j^*)_{j=1}^\infty$. Clearly the dual action has the form $< \underline{x}, \underline{x}^* >= \sum_{j=1}^{\infty} x_j x_j^* \lambda_j$. The norm on $\mathfrak{X}^*$ is denoted by $\|\cdot\|^*$. The projection onto the first N coordinates is denoted by Π_N (i.e. $\mathfrak{X} \ni \underline{x} \to \Pi((x_1, x_2, \ldots)) = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$) while the projection on the k^{th} coordinate is denoted by π_k.

If $F : X \to \mathbb{R}$ is Borel, then condition (20) is replaced by

$$|F(\underline{x})| \leq Ae^{-\sum_{j=1}^{\infty} \lambda_j |x_j|}, \tag{24}$$

where $X \subseteq \mathfrak{X}$ is Borel and $A, \varepsilon > 0$ are some constants. The subspaces $\mathfrak{X}_N = \{(x_1, x_2, \ldots, x_N, 0, 0, \ldots) : x_j \in \mathbb{R}\}$ are isomorphic to $\mathbb{R}^N$. Similarly as in the finite dimensional case, functions defined by the integrals

$$F_{n_{N+1},\ldots,n_{N+K}}(x_1, \ldots, x_N) = \int x_{N+1}^{n_{N+1}} \cdots x_{N+K}^{n_{N+K}} F(\underline{x})\, d\mu_{x_1,\ldots,x_N}(\underline{x}),$$

satisfy condition (20) on $\mathfrak{X}_N = \mathbb{R}^N$. Constants A and ε must be however adjusted. The moments are again defined as

$$m_{n_1,n_2,\ldots,n_N} = \int_{\mathfrak{X}} x_1^{n_1} \cdot x_2^{n_2} \cdots x_N^{n_N} F(\underline{x})\, d\mu(\underline{x}),$$

where μ is a finite signed Borel measure on X.

LEMMA 2. *Let μ be a signed measure on $X \subseteq \mathfrak{X}$. Suppose that $F : X \to \mathbb{R}$ satisfies* (24). *If $m_{n_1,n_2,\ldots,n_N} = 0$ for all $n_1, n_2, \ldots, n_N$, then supp(μ) $\subseteq \{\underline{x} : F(\underline{x}) = 0\}$.*

Proof. We define the characteristic function

$$\Psi(x^*) = \int_{\mathfrak{X}} e^{i<\underline{x},\underline{x}^*>} F(\underline{x})\, d\mu(\underline{x}).$$

Applying the Lebesgue dominated convergence theorem we obtain

$$\int_{\mathfrak{X}} < \underline{x}, \underline{x}^* >^n F(\underline{x})\, d\mu(\underline{x}) = 0$$

where $n \geq 0$ and $\|\underline{x}^*\|^* < 1$. This implies that $\Psi(\underline{x}^*) = 0$ for all such $\underline{x}^*$. In particular, if we consider the measure (on Borel sets $B \subseteq \mathbb{R}$)

$$\nu_{\underline{x}^*}(B) = \int_{\underline{x}^{*-1}(B)} F(\underline{x})\, d\mu(\underline{x}),$$

then $\int s^n d\nu_{\underline{x}^*}(s) = 0$ for all $n \geq 0$, where $\|\underline{x}^*\|^* < 1$. It is easy to verify that $\int e^{\frac{1}{2}|s|}\, d|\nu_{\underline{x}^*}|(s) < \infty$. Hence by Lemma 1 $\nu_{\underline{x}^*}$ is the zero measure on $\mathbb{R}$. This gives

$$\int_{\mathbb{R}} e^{iLs}\, d\nu_{\underline{x}^*}(s) = 0$$

for all $L > 0$. We get

$$0 = \int_{\mathfrak{X}} e^{iL<\underline{x},\underline{x}^*>} F(\underline{x})\, d\mu(\underline{x}) = \int_{\mathfrak{X}} e^{i<\underline{x},L\underline{x}^*>} F(\underline{x})\, d\mu(\underline{x}) = \Psi(L\underline{x}^*)$$

for all $\|\underline{x}^*\|^* < 1$ and $L > 0$. In particular, the characteristic function of $F(\underline{x})d\mu(\underline{x})$ is zero. Hence supp(μ) $\subseteq \{\underline{x} : F(\underline{x}) = 0\}$. ∎

Using Lemma 2 the following result can be proved similarly as Theorem 4. Therefore its proof is limited to a short sketch.

THEOREM 5. *Let X be a Borel subset of $\mathfrak{X}$, F satisfies (24) on X, and $\mu \in M(X)$. Then*

(i) *If $m_{n_1,n_2,\ldots,n_N} \equiv 0$ for all $n_1, n_2, \ldots, n_N \geq 0$, and $F(\underline{x}) \neq 0$ for all $\underline{x} \in X$, then $\mu = 0$.*

(ii) *If for every j there exists J_j such that for all $n_1, \ldots, n_{j-1}, n_{j+1}, \ldots$ $\ldots, n_N \geq 0$, and $k \geq J_j$ we have*

$$m_{n_1,n_2,\ldots,n_{j-1},k,n_{j+1},\ldots,n_N} = 0,$$

where $F(\underline{x}) \neq 0$ for all $\underline{x} \in X$, then $\mu = t\delta_{\underline{0}}$ for some scalar t.

(iii) *If for every fixed $n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_N \geq 0$ the sequence*

$$k \to m_{n_1,n_2,\ldots,n_{j-1},k,n_{j+1},\ldots,n_N}$$

is eventually periodic, then μ is concentrated on the set

$$S = \prod_{j=1}^{\infty} \{-1, 0, 1\}.$$

The measure μ is determined by low level moments $m_{n_1,n_2,\ldots,n_N}$, where $n_1, n_2, \ldots, n_N \in \{0, 1, 2\}$. If in addition, for some projection Π_N, the corresponding measure $Q_{|\mu|,\Pi_N}$ is continuous, then $\mu = 0$.

Proof. (i) follows directly from Lemma 2. In order to obtain (ii) we apply Theorem 4 (i) and get

$$\operatorname{supp}(\mu) \subseteq \bigcap_{N=1}^{\infty} \{\underline{x} \ : x_1 = 0 \ldots x_N = 0\} = \{\underline{0}\}.$$

The proof of (iii) is essentially the same as Theorem 4 (iii). We simply consider all projections π_j and the corresponding desintegrations $\mu = \int \mu_{x_j}\, dQ_{|\mu|,\pi_j}(x_j)$. It follows that $\mu_{x_j} = 0$ for all $x_j \notin \{-1, 0, 1\}$. As a result $\operatorname{supp}(\mu) \subseteq S$. Now, if we assume that for some natural N the measure $Q_{|\mu|,\Pi_N}$ is continuous, then $\mu = \int \mu_{x_1,\ldots,x_N} dQ_{|\mu|,\Pi_N}((x_1, \ldots, x_N)) = 0$. ■

The next result follows directly from Theorem 5.

PROPOSITION 2. *Let $\eta = (\eta_1, \eta_2, \ldots)$ and $\xi = (\xi_1, \xi_2, \ldots)$ be random vectors on $\mathfrak{X} \subseteq \mathbb{R}^\infty$. If there exists a strictly positive vector $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots)$ such that for all $n_1, n_2, \ldots, n_N \geq 0$ we have*

$$E\eta_1^{n_1} \cdots \eta_N^{n_N} e^{-\sum_{j\geq 1} \lambda_j |\eta_j|} = E\xi_1^{n_1} \cdots \xi_N^{n_N} e^{-\sum_{j\geq 1} \lambda_j |\xi_j|} < \infty,$$

then η and ξ have the same distributions.

Proof. Let $\mu = \mu_\eta - \mu_\xi$, where μ_η and μ_ξ denote the distributions of η and ξ respectively. We get $m_{n_1,n_2,\ldots,n_N} = 0$ for all $n_1, n_2, \ldots, n_N \geq 0$. We easily check that the function

$$\mathfrak{X} \ni \underline{x} \to F(\underline{x}) = e^{-\sum_{j\geq 1} \lambda_j |x_j|}$$

is strictly positive and satisfies (24). Now it remains to apply Theorem 5. ∎

The last result of the paper is another generalization of Theorems 4 and 5 from [LT]. It is a direct combination of Theorem 5 (iii) and Proposition 2. Namely we have:

COROLLARY 5. *If in the above Proposition 2, for some N we have .*

$$Q_{|\mu_\eta - \mu_\xi|, \Pi_N} \in M_{ac}(\mathbb{R}^N),$$

then η and ξ have the same distributions if an only if the sequence

$$k \to m_{n_1, n_2, \ldots, n_{j-1}, k, n_{j+1}, \ldots, n_N}$$

is eventually periodic, when $n_1, n_2, \ldots, n_{j-1}, n_{j+1}, \ldots, n_N$ are large enough.

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