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Nikolaos Matzakos, Nikolaos S. Papageorgiou

EXISTENCE OF PERIODIC SOLUTIONS FOR QUASILINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH DISCONTINUITIES

Abstract. We consider a quasilinear differential equation with discontinuous right hand side and periodic boundary conditions. To obtain an existence theory we pass to a relevant multivalued variant of the original problem, which we solve. Our approach is a mixture of the variational method (for nonsmooth locally Lipschitz functionals) and of the method of upper and lower solutions. The mixing of these two techniques is made possible by a nonresonance condition below the first nonzero eigenvalue of the one-dimensional p–Laplacian with periodic boundary conditions.

1. Introduction

In this paper we study the following periodic problem for quasilinear differential equations

$$(1)\qquad \left\{ \begin{array}{l} -(|x'(t)|^{p-2}x'(t))' = f(t,x(t)) \;\text{ a.e. on } T=[0,b] \\ x(0)=x(b),\; x'(0)=x'(b),\; a \le p < \infty \end{array} \right\}.$$

We do not assume $f(t,\cdot)$ to be continuous. So the problem (1) need not have a solution. In order to be able to develop a satisfactory existence theory, we have to pass to a multivalued problem, which is obtained by, roughly speaking, filling in the gaps at the discontinuity points of $f(t,\cdot)$. For this purpose we introduce the following two functions:

$$f_0(t,x) = \lim_{\varepsilon\downarrow 0} ess \inf_{|x-x'|\le\varepsilon} f(t,x')$$

$$\text{and}\quad f_1(t,x) = \lim_{\varepsilon\downarrow 0} ess \sup_{|x-x'|\le\varepsilon} f(t,x').$$

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Evidently $f_0(t,x) \leq f_1(t,x)$ for all $(t,x) \in T \times \mathbb{R}$. Then instead of (1), we consider the following multivalued version of it:

$$(2) \qquad \left\{ \begin{array}{ll} -\left(|x'(t)|^{p-2}x'(t)\right)' \in [f_0(t,x(t)), f_1(t,x(t))] & \text{a.e. on } T \\ x(0) = x(b),\ x'(0) = x'(b),\ a \leq p < \infty & \end{array} \right\}.$$

We prove the existence of a solution for problem (2) by combining the variational approach with the method of upper and lower solutions. The link between these two techniques is a nonresonance condition below the first nonzero eigenvalue of the one dimensional p–Laplacian with periodic boundary conditions.

Quasilinear ordinary differential equations with Dirichlet boundary conditions were studied by Boccardo–Drabek–Giachetti–Kucera [1], Drabek [7], DelPino–Elgueta–Manasevich [6] and De Coster [5]. Under homogeneous Dirichlet boundary conditions, the differential operator $Ax = -(|x'|^{p-2}x')'$ (the one dimensional p–Laplacian) is invertible (as a nonlinear map between appropriate spaces) and so classical Leray–Schauder degree techniques (like the Leray–Sçhauder principle) can be used. In contrast for the periodic problem, the corresponding operator has a nontrivial kernel and so a different aopproach is needed. This problem was examined by Guo [9], who considered a more general version of (1) by allowing f to depend also on x'. However he assumed f to be continuous in all three variables (including the time–variable). His approach was degree theoretic, based on Mawhin's coincidence degree theory. This forced him to introduce additional restrictive conditions on the function f (like growth condition, see hypothesis H_1, p. 710 in Guo [9]). Our method of proof here is completely different from that of Guo and is based on variational arguments mixed with ideas from the method of upper and lower solutions.

2. Preliminaries

Since the function $f(t,\cdot)$ is discontinuous, our variational technique will be based on the critical point theory for nonsmooth locally Lipschitz energy functionals. For easy reference we recall here the basic aspects of that theory. For details we refer to Chang [3] and Clarke [4].

Let X be a Banach space and $f : X \to \mathbb{R}$ a function. We say that $f(\cdot)$ is locally Lipschitz, if for every $x \in X$, there is a neighborhood U of x and a $k \geq 0$ depending on U such that

$$|f(y) - f(z)| \leq k\|y - z\|$$

for all $y, z \in U$. For every $h \in X$, we define the generalized directional derivative $f^o(x;h)$ by

$$f^o(x;h) = \overline{\lim_{\substack{x' \to x \\ \lambda \downarrow 0}}} \frac{f(x' + \lambda h) - f(x')}{\lambda}.$$

It is easy to check that $h \to f^o(x;h)$ is sublinear, $|f^o(x;h)| \leq k\|h\|$ and $h \to f^o(x;h)$ is continuous. So $f^o(x;\cdot)$ is the support function of a nonempty, convex and w^*–compact set

$$\partial f(x) = \{x^* \in X^* : (x^*, h) \leq f^o(x;h) \ \text{ for all } h \in X\}$$

known as the "generalized subdifferential of $f(\cdot)$ at x" (see Clarke [4]).

For every $x^* \in \partial f(x)$, we have $\|x^*\| \leq k$. Also if $f, g : X \to \mathbb{R}$ are locally Lipschitz functions, then $\partial(f + g)(x) \subseteq \partial f(x) + \partial g(x)$ and $\partial(\lambda f)(x) = \lambda \partial f(x)$ for all $\lambda \in \mathbb{R}$. If $f : X \to \mathbb{R}$ is convex, then it is well–known from convex analysis that $f(\cdot)$ is locally Lipschitz and its subdifferential in the sense of convex analysis coincides with the generalized subdifferential described above.

Let $f : X \to \mathbb{R}$ be locally Lipschitz on the Banach space x. A point X is said to be a "critical point" of $f(\cdot)$ if $0 \in \partial f(x)$. It is easy to see that if $x \in X$ is a local extremum of $f(\cdot)$, then x is a critical point of $f(\cdot)$. We say that $f(\cdot)$ satisfies the "(PS)–condition" (Palais–Smale condition), if any sequence $\{x_n\}_{n \geq 1} \subseteq X$ along which $\{f(x_n)\}_{n \geq 1}$ is bounded and $m(x_n) = \inf\{\|x^*\| : x^* \in \partial f(x_n)\} \to 0$ as $n \to \infty$, has a strongly convergent subsequence. Since for $f \in C^1(X, \mathbb{R})$, $\partial f(x) = \{f'(x)\}$, we see that for a smooth $f(\cdot)$, the above version of the (PS)–condition coincides with the classical one (see Rabinowitz [12]).

In our analysis of problem (2), we will need the following theorem, which is due to Chang [3] and extends to a nonsmooth setting the well–known mountain path theorem of Ambrosetti and Rabinowitz [12].

THEOREM 1. *Assume that:*

X is a reflexive Banach space, $X = X_1 \oplus X_2$ with $\dim X_1 < \infty$, $f : X \to \mathbb{R}$ is a locally Lipschitz function which satisfies the (PS)–condition and there exist constants $\beta_1 < \beta_2$ and a neighborhood U of 0 in X_1 such that $f|_{\partial U} \leq \beta_1$ and $f|_{X_2} \geq \beta_2$.

Then $f(\cdot)$ has a critical point x and $f(x) = c \geq \beta_2$.

3. Auxiliary results

In this section we prove two auxiliary results which will allow us in Section 4 to use the method of upper and lower solutions together with theorem 1 in order to obtain a nontrivial solution for problem (2). In what follows $W^{1,p}_{per}(T) = \{x \in W^{1,p}(T) : x(0) = x(b)\}$. Recall that $W^{1,p}(T)$ is continuously embedded in $C(T)$ and so the pointwise evaluations at $t = 0$ and $t = b$ make sense.

First let us define what we mean by a solution of problem (2).

DEFINITION: By a solution of (2) we mean a function $x \in C^1(T)$ such that $|x'(\cdot)|^{p-2}x'(\cdot) \in W^{1,q}(T)(\frac{1}{p} + \frac{1}{q} = 1)$ and there exists $g \in L^q(T)$ such that $f_o(t, x(t)) \leq g(t) \leq f_1(t, x(t))$ a.e. on T, $-(|x'(t)|^{p-2}x'(t))' = g(t)$ a.e. on T and $x(0) = x(b)$, $x'(0) = x'(b)$.

Let $\lambda_2 > 0$ be the second eigenvalue of the one–dimensional p–Laplacian with periodic boundary conditions, i.e.

$$(3) \qquad \left\{ \begin{array}{l} -(|x'(t)|^{p-2}x'(t))' = \lambda_2 |x(t)|^{p-2}x(t) \quad \text{a.e. on } T \\ x(0) = x(b), \; x'(0) = x'(b) \end{array} \right\}.$$

Notice that $\lambda_1 = 0$ is the first eigenvalue. So $\lambda_2 > 0$ is the first nonzero eigenvalue of the p–Laplacian with periodic boundary conditions (see Showalter [13], Corollary 7.D, p. 78).

Now we introduce our hypothesis for the function $f(t, x)$. They will be in effect for the rest of this paper. Recall that a function $h : T \times \mathbb{R} \to \mathbb{R}$ is said to be "N–measurable", if for every $x : T \to \mathbb{R}$ which is Borel measurable, we have that $t \to h(t, x(t))$ is Borel measurable (superpositional measurability).
H(f): $f : T \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that:

(i) f_0, f_1 are N–measurable functions;

(ii) for every $M > 0$, there exists $\gamma_M \in L^q(T)$ such that for almost all $t \in T$ and all $|x| \leq M$, $|f(t, x)| \leq \gamma_M(t)$;

(iii) $\overline{\lim}_{|x| \to \infty} \frac{f(t,x)}{|x|^{p-2}x} < \lambda_2$ uniformly for almost all $t \in T$,

(iv) for almost all $t \in T$ and all $x \in \mathbb{R} \setminus \{0\}$ $f(t, x)x > 0$ (strict sign condition).

By virtue of hypothesis $H(f)$(iii), we see that there exists $\varepsilon > 0$ (with $0 < \lambda_2 - \varepsilon$) and $M > 0$ such that

$$(4) \qquad f(t, x) \leq (\lambda_2 - \varepsilon)|x|^{p-2}x \quad \text{for almost all } t \in T \text{ and all } x \geq M > 0$$

$$(5) \quad \text{and } f(t, x) \geq (\lambda_2 - \varepsilon)|x|^{p-2}x \quad \text{for almost all } t \in T \text{ and all } x \leq -M < 0.$$

Moreover, from hypothesis $H(f)$(ii) we have that

$$(6) \qquad |f(t, x)| \leq \gamma_M(t) \text{ for almost all } t \in T \text{ and all } |x| \leq M$$

Combining (3), (4) and (5) above, we infer that for some $\varepsilon > 0$ (with $0 < \lambda_2 - \varepsilon$) and $\gamma \in L^q(T)$ $\gamma \geq 0$, we have

$$f(t, x) \leq (\lambda_2 - \varepsilon)|x|^{p-2}x + \gamma(t) \qquad \text{for almost all } t \in T \text{ and all } x \geq 0$$
$$\text{and } f(t, x) \geq (\lambda_2 - \varepsilon)|x|^{p-2}x - \gamma(t)) \qquad \text{for almost all } t \in T \text{ and all } x \leq 0.$$

We consider the following two auxiliary periodic problems:

$$(7)\qquad \left\{\begin{array}{l} -(|\phi'(t)|^{p-2}\phi'(t))' = (\lambda_2-\varepsilon)|\phi(t)|^{p-2}\phi(t)+\gamma(t) \ \text{ a.e. on } T \\ \phi(0)=\phi(b),\ \phi'(0)=\phi'(b) \end{array}\right\}$$

and

$$(8)\qquad \left\{\begin{array}{l} -(|\psi'(t)|^{p-2}\psi'(t))' = (\lambda_2-\varepsilon)|\psi(t)|^{p-2}\psi(t)-\gamma(t) \ \text{ a.e. on } T \\ \psi(0)=\psi(b),\ \psi'(0)=\psi'(b). \end{array}\right\}.$$

By a solution of problem (6) (resp. of problem (7)), we mean a function $\phi \in C^1(T)$ (resp. $\psi \in C^1(T)$) with $|\phi'|^{p-2}\phi' \in W^{1,q}(T)$ (resp. $|\psi'|^{p-2}\psi' \in W^{1,q}(T)$), which satisfies (6) (resp. (7)).

In the next two propositions we prove that the two problems have solutions $\phi \geq 0$ and $\psi \leq 0$ respectively.

PROPOSITION 2. *Problem* (6) *has a solution* $\phi \geq 0$, $\phi \neq 0$.

Proof. Since $\gamma(t) \geq 0$ a.e. on T, from Theorem 9.5, p. 210 of Gilbarg–Trudinger [8], we know that if $\phi \in C^1(T)$ is a solution of (6), then $\phi(t) \geq 0$ for all $t \in T$.

So we have to show that a solution ϕ exists. To this end let $W^{1,p}_{per}(T) = Z \oplus Y$, where Z is the space of constant functions (i.e. $Z = \mathbb{R}$) and Y is the space of all functions in $W^{1,p}_{per}(T)$ with mean value zero (i.e. $Y = \{y \in W^{1,p}_{per}(T) : \int_0^b y(t)\,dt = 0\}$). Let $R : W^{1,p}_{per}(T) \to \mathbb{R}$ be defined by

$$R(x) = \frac{1}{p}\|x'\|_p^p - \frac{(\lambda_2-\varepsilon)}{p}\|x\|_p^p - \int_0^b \gamma(t)x(t)\,dt.$$

If $z \in Z = \mathbb{R}$, then we have

$$R(z) = -\frac{(\lambda_2-\varepsilon)}{p}\|z\|_p^p - z\int_0^b \gamma(t)\,dt$$

$$\Rightarrow R(z) = \left\{\begin{array}{ll} -\frac{(\lambda_2-\varepsilon)}{p}bz^p - z\|\gamma\|_1 & \text{if } z \geq 0 \\ -\frac{(\lambda_2-\varepsilon)}{p}b|z|^p + |z|\,\|\gamma\|_1 & \text{if } z \leq 0. \end{array}\right.$$

So we infer that $R(z) \to -\infty$ as $|z| \to \infty$.

Next let $y \in Y$. Using the Poincare–Wirtinger inequality, we have

$$\begin{aligned} R(y) &= \frac{1}{p}\|y'\|_p^p - \frac{(\lambda_2-\varepsilon)}{p}\|y\|_p^p - \int_0^b \gamma(t)y(t)\,dt \\ &\geq \frac{1}{p}\left(1 - \frac{(\lambda_2-\varepsilon)}{p}\right)\|y'\|_p^p - \|\gamma\|_q \frac{1}{\lambda_2^{1/p}}\|y'\|_p \\ &\Rightarrow R(y) \to +\infty \text{ as } \|y\| \to \infty,\quad y \in Y, \end{aligned}$$

(because $\|y'\|_p$ is an equivalent norm on Y).

Finally we will show that $R(\cdot)$ satisfies (PS)–condition. To this end let $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{per}(T)$ such that $|R(x_n)| \leq M$ and $m(x_n) \to 0$ as $n \to \infty$. Pick $x_n^* \in \partial R(x_n)$ such that $m(x_n) = \|x_n^*\|$, $n \geq 1$. Such an element exists since $\partial R(x_n)$ is nonempty, w–compact in $W^{1,p}_{per}(T)^*$ and the norm of $W^{1,p}_{per}(T)^*$ is weakly lower semicontinuous. Let $J_1, J_2, G : W^{1,p}_{per}(T) \to \mathbb{R}$ be defined by

$$J_1(x) = \frac{1}{p}\|x'\|_p^p, \;\; J_2(x) = \frac{\lambda_2 - \varepsilon}{p}\|x\|_p^p \;\text{ and } G(x) = \int_0^b \gamma(t)x(t)\,dt.$$

We have $R(x) = J_1 - J_2(x) - G(x)$. Hence

$$\partial R(x) = \partial(J_1 - J_2 - G)(x) \subseteq \partial J_1(x) - \partial J_2(x) - \partial G(x) \;\text{ for all } x \in W^{1,p}_{per}(T).$$

Let $A : W^{1,p}_{per}(T) \to W^{1,p}_{per}(T)^*$ be a nonlinear operator defined by

$$< A(x), y >= \int_0^b |x'(t)|^{p-2}x'(t)y'(t)\,dt \;\text{ for all } x, y \in W^{1,p}(T).$$

Here by $< \cdot, \cdot >$ we denote the duality brackets for the pair $(W^{1,p}_{per}(T), W^{1,p}_{per}(T)^*)$. It is easy to verify that $A(\cdot)$ is monotone, demicontinuous and coercive (see for example Kourogenis–Papageorgiou [10]). Then $\partial J_1(x) = A(x)$. Also if $\widehat{J}_2 : L^p(T) \to \mathbb{R}$ is defined by $\widehat{J}_2(x) = \frac{1}{p}\|x\|_p^p$, then $J_2 = \widehat{J}_2|_{W^{1,p}_{per}(T)}$. Since $W^{1,p}_{per}(T)$ is embedded continuously and densely in $L^p(T)$, using Theorem 2.2 of Chang [3], we have that $\partial J_2(x) = (\lambda_2 - \varepsilon)|x(\cdot)|^{p-2}x(\cdot) \in L^q(T)$. So finally we have

$$x_n^* = A(x_n) - (\lambda_2 - \varepsilon)|x_n|^{p-2}x_n - \gamma.$$

We claim that the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{per}(T)$ is either uniformly (in $t \in T$) bounded from above or from below. Suppose not. Then we can find a subsequence, still denoted by $\{x_n\}_{n\geq 1}$ such that

$$L_n = \max_T x_n \to +\infty \;\text{ and }\; \ell_n = \min_T x_n \to -\infty \;\text{ as }\; n \to \infty.$$

Extending periodically $x_n(\cdot)$ on $[0, 2b]$, for all $n \geq 1$ large we can find $\alpha_n, \beta_n, \theta_n, \eta_n \in [0, 2b]$ such that

$$x_n(\alpha_n) = x_n(\beta_n) = 0, \;\; x_n(t) > 0 \;\text{ for } t \in (\alpha_n, \beta_n), \;\max[x_n(t) : t \in [\alpha_n, \beta_n]] = L_n$$

$$\text{and} \quad x_n(\theta_n) = x_n(\eta_n) = 0, \;\; x_n(t) < 0 \;\text{ for } t \in (\theta_n, \eta_n), \;\min[x_n(t) : t \in [\theta_n, \eta_n]] = \ell_n.$$

We will show that there exists $\xi_1 > 0$ or $\xi_2 > 0$ such that

$$\int_{\alpha_n}^{\beta_n} |x_n'(t)|^p\,dt - (\lambda_2 - \varepsilon)\int_{\alpha_n}^{\beta_n} |x_n(t)|^p\,dt \geq \xi^1 \int_{\alpha_n}^{\beta_n} |x_n'(t)|^p\,dt$$

$$\text{and}\quad \int_{\theta_n}^{\eta_n} |x_n'(t)|^p\,dt - (\lambda_2 - \varepsilon)\int_{\theta_n}^{\eta_n} |x_n(t)|^p\,dt \geq \xi^2 \int_{\theta_n}^{\eta_n} |x_n'(t)|^p\,dt.$$

Suppose not. Then at least for a subsequence, we will have

$$\int_{\alpha_n}^{\beta_n} |x_n'(t)|^p\,dt - (\lambda_2 - \varepsilon)\int_{\alpha_n}^{\beta_n} |x_n(t)|^p\,dt \leq \delta_n^1 \int_{\alpha_n}^{\beta_n} |x_n'(t)|^p\,dt$$

$$\text{and}\quad \int_{\theta_n}^{\eta_n} |x_n'(t)|^p\,dt - (\lambda_2 - \varepsilon)\int_{\theta_n}^{\eta_n} |x_n(t)|^p\,dt \leq \delta_n^2 \int_{\theta_n}^{\eta_n} |x_n'(t)|^p\,dt$$

with $\delta_n^1, \delta_n^2 \downarrow 0$ as $n \to \infty$. Also we may assume that $\alpha_n \to \alpha, \beta_n \to \beta, \theta_n \to \theta$ and $\eta_n \to \eta$

We will work with the first of the last two inequalities, since the arguments for the second are similar. Using Poincare's inequality on (α_n, β_n), we have

$$(\lambda_2 - \varepsilon)\int_{\alpha_n}^{\beta_n} |x_n(t)|^p\,dt \leq \frac{\lambda_2 - \varepsilon}{\overset{\circ}{\lambda}_1((\alpha_n, \beta_n))}\int_{\alpha_n}^{\beta_n} |x_n'(t)|^p\,dt.$$

Here by $\overset{\circ}{\lambda}_1((\alpha_n, \beta_n)) > 0$ we denote the first (principal) eigenvalue of the one dimensional p–Laplacian on (α_n, β_n) with Dirichlet boundary conditions (i.e. of $(-\Delta_p, W_0^{1,p}(\alpha_n, \beta_n))$). From Otani [11] we know that

$$\overset{\circ}{\lambda}_1((\alpha_n, \beta_n)) = \frac{1}{(\beta_n - \alpha_n)^p}\frac{p-1}{p^p}\beta\left(\frac{1}{p}, \frac{1}{q}\right)^p$$

where $\beta(x, y)$ is the beta function defined by $\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$, $x, y > 0$. Note that $(\beta_n - \alpha_n) \leq |T|$ and so $\overset{\circ}{\lambda}_1(T) \leq \overset{\circ}{\lambda}_1((\alpha_n, \beta_n))$. Then we have

$$\int_{\alpha_n}^{\beta_n} |x_n'(t)|^p\,dt - \frac{\lambda_2 - \varepsilon}{\overset{\circ}{\lambda}_1((\alpha_n, \beta_n))}\int_{\alpha_n}^{\beta_n} |x_n'(t)|^p\,dt \leq \delta_n^1 \int_{\alpha_n}^{\beta_n} |x_n'(t)|^p\,dt.$$

Dividing both sides by $\int_{\alpha_n}^{\beta_n} |x_n'(t)|^p\,dt$, we obtain

$$1 - \frac{\lambda_2 - \varepsilon}{\overset{\circ}{\lambda}_1((\alpha_n, \beta_n))} \leq \delta_n^1.$$

Passing to the limit as $n \to \infty$, we have

$$1 \leq \frac{\lambda_2 - \varepsilon}{\overset{\circ}{\lambda}_1 (T)} < 1 \quad \left(\text{since } \overset{\circ}{\lambda}_1 (T) \leq \overset{\circ}{\lambda}_1 ((\alpha, \beta))\right)$$

a contradiction. Note that if $\alpha = 0, \beta = b$ then $\theta = \eta = b$ and vice versa. So we conclude that one of the inequalities (8) or (9) holds.

Suppose that (8) holds (the argument is similar if (9) holds). Define the functions $u_n : [0, 2b] \to \mathbb{R}$ as follows:

$$u_n(t) = \begin{cases} x_n(t) & \text{if } t \in [\alpha_n, \beta_n] \\ 0 & \text{if } t \in [0, 2b] \setminus [\alpha_n, \beta_n]. \end{cases}$$

Evidently $u_n \in W_0^{1,p}(0, 2b)$. Also let $w_n(t) = u_n(t)$ for all $t \in T$ if $\beta_n \leq b$, while if $\beta_n > b$ set $w_n(t) = u_n(t + b)$ for $t \in [0, \beta_n - b]$ and $w_n(t) = u_n(t)$ for $t \in [\beta_n - b, b]$. Clearly then $w_n \in W_{per}^{1,p}(T)$ and $\|w_n\|_{1,p,T} = \|u_n\|_{1,p,[0,2b]}$ (hence by $\|\cdot\|_{1,p,T}$ (resp. $\|\cdot\|_{1,p,[0,2b]}$) we denote the norm of the Sobolev space $W^{1,p}(T)$ (resp. of $W^{1,p}(0, 2b)$)). We have

$$| < x_n^*, y > | \leq \varepsilon_n \|y\| \quad \text{for all } y \in W_{per}^{1,p}(0, 2b).$$

Take $y = w_n$. We obtain

$$\int_{\alpha_n}^{\beta_n} |x_n'(t)|^p \, dt - (\lambda_2 - \varepsilon) \int_{\alpha_n}^{\beta_n} |x_n(t)|^p \, dt - \int_{\alpha_n}^{\beta_n} \gamma(t) x_n(t) \, dt \leq \varepsilon_n \|u_n\|_{1,p,[0,2b]}$$

$$\Rightarrow \xi_1 \int_{\alpha_n}^{\beta_n} |x_n'(t)|^p \, dt - \int_{\alpha_n}^{\beta_n} \gamma(t) x_n(t) \, dt \leq \varepsilon_n \|u_n\|_{1,p,[0,2b]}$$

$$\Rightarrow \xi_1 \int_{\alpha_n}^{\beta_n} |x_n'(t)|^p \, dt \leq c_1 \|x_n\|_{1,p,(\alpha_n,\beta_n)} \quad \text{for some } c_1 > 0$$

$$(\text{since } u_n|_{(\alpha_n,\beta_n)} = x_n).$$

Using Poincare's inequality in the space $W_0^{1,p}(\alpha_n, \beta_n)$ we obtain that $\|x_n\|_{1,p,(\alpha_n,\beta_n)} \leq c_2$ for some $c_2 > 0$ and all $n \geq 1$. So we have $\int_{\alpha_n}^{\beta_n} |x_n(t)| \, dt \leq c_3$ for some $c_3 > 0$ and all $n \geq 1$. Therefore

$$x_n(t) \leq \int_{\alpha_n}^{t} |x_n'(s)| \, ds \leq c_3 \quad \text{for all } t \in [\alpha_n, \beta_n]$$

$$\Rightarrow \max\{x_n(t) : t \in [\alpha_n, \beta_n]\} = L_n \leq c_3.$$

However, by hypothesis $L_n \to +\infty$ as $n \to \infty$ and so we have a contradiction. Thus we have proved that $\{x_n\}_{n \geq 1}$ is uniformly (in $t \in T$) bounded from above or below. To fix things we will assume that it is bounded from above. So $x_n(t) \leq c_4$ for all $t \in T$ and all $n \geq 1$. Recall that

$$| < x_n^*, y > | \leq \varepsilon_n \|y\| \quad \text{for all } y \in W^{1,p}_{per}(T), \quad \text{with } \varepsilon_n \downarrow 0.$$

Take $y = 1$. We obtain

$$\left|(\lambda_2 - \varepsilon)\int_0^b |x_n(t)|^{p-2}x_n(t)\,dt\right| \leq c_5 \quad \text{for some } c_5 > 0 \quad \text{and all} \quad n \geq 1$$

$$\Rightarrow \left| \int_{\{x_n \geq 0\}} |x_n(t)|^{p-2}x_n(t)\,dt + \int_{\{x_n \leq 0\}} |x_n(t)|^{p-2}x_n(t)\,dt \right|$$

$$\leq c_6 \quad \left(c_6 = \frac{1}{\lambda_2 - \varepsilon}c_5\right).$$

Since $x_n(t) \leq c_4$ for all $t \in T$ and all $n \geq 1$, then we see that

$$\left| \int_{\{x_n \leq 0\}} |x_n(t)|^{p-2}x_n(t)\,dt \right| \leq c_7 \quad (c_7 = c_6 + c_4^{p-1}b)$$

$$\Rightarrow \int_{\{x_n \leq 0\}} |x_n(t)|^{p-2}(-x_n(t))\,dt \leq c_7$$

$$\Rightarrow \sup_{n \geq 1} \int_{\{x_n \leq 0\}} |x_n(t)|^{p-1}\,dt \leq c_7$$

$$\Rightarrow \sup_{n \geq 1} \int_0^b |x_n(t)|^{p-1}\,dt \leq c_8 \quad (c_8 = c_6 + 2c_4^{p-1}b); \quad \text{i.e. } \sup_{n \geq 1} \|x_n\|_p^{p-1} \leq c_8.$$

Now let $x_n = z_n + y_n$, $z_n \in Z = \mathbb{R}$ and $y_n \in Y$, $n \geq 1$. We have

(11) $\quad < x_n^*, y_n > \leq \varepsilon_n \|y_n\|$

$$\Rightarrow \int_0^b |y_n'(t)|^p\,dt - (\lambda_2 - \varepsilon)\int_0^b |x_n(t)|^{p-1}y_n(t)\,dt - \int_0^b \gamma(t)y_n(t)\,dt \leq c_1\|y_n\|.$$

From the Poincare–Wirtinger inequality we know that

$$\|y_n\|_p^p \leq \frac{1}{\lambda_2}\|y_n'\|_p^p.$$

Using this fact in (10) we obtain

$$\text{(12)} \quad \|y_n'\|_p^p \leq (\lambda_2 - \varepsilon)\|x_n\|_p^{p-1}\|y_n\|_p + \|\gamma\|_q\|y_n\|_p + c_9\|y_n'\|_p \quad \text{for some } c_9 > 0$$

$$\leq \left[(\lambda_2 - \varepsilon)c_8\frac{1}{\lambda_2^{1/p}} + \|\gamma\|_q\frac{1}{\lambda_2^{1/q}} + c_9\right]\|y_n'\|_p$$

$$\Rightarrow \{y_n' = x_n'\}_{n \geq 1} \subseteq L^p(T) \quad \text{is bounded.}$$

Finally recall that $|R(x_n)| \leq M$ for all $n \geq 1$. So using the boundedness of $\{x_n'\}_{n\geq 1} \subseteq L^p(T)$ we have

$$\begin{aligned} -\frac{1}{p}\|x_n'\|_p^p + \frac{\lambda_2-\varepsilon}{p}\|x_n\|_p^p &\leq M + \|\gamma\|_q\|x_n\|_p \\ \Rightarrow \|x_n\|_p^p \leq c_{10}(1+\|\gamma\|_q\|x_n\|_p) & \text{ for some } c_{10}>0 \\ \Rightarrow \{x_n\}_{n\geq 1} \subseteq L^p(T) & \text{ is bounded.} \end{aligned}$$

So by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}_{per}(T)$ and $x_n \to x$ in $L^p(T)$ as $n \to \infty$. Therefore we have

$$\begin{aligned} < x_n^*, x_n - x >= &< A(x_n), x_n - x > \\ &- (\lambda_2-\varepsilon)\int_0^b |x_n(t)|^{p-2}x_n(t)(x_n(t)-x(t))\,dt \\ &- \int_0^b \gamma(t)(x_n(t)-x(t))\,dt \\ &\leq \varepsilon_n\|x_n - x\|. \end{aligned}$$

But $\int_0^b |x_n(t)|^{p-2}x_n(t)(x_n(t)-x(t))\,dt \to 0$ and $\int_0^b \gamma(t)(x_n(t)-x(t))\,dt \to 0$ as $n \to \infty$. Thus we obtain that

$$\overline{\lim} < A(x_n), x_n - x > \leq 0.$$

Since $A(\cdot)$ is maximal monotone (being monotone, democontinuous, everywhere defined, see Zeidler [14]) it has the generalized pseudomonotone property (see Browder–Hess [2]) and so

$$\begin{aligned} &< A(x_n), x_n > \to < A(x), x > \quad \text{as } n \to \infty \\ &\Rightarrow \|x_n'\|_p \to \|x'\|_p \quad \text{as } n \to \infty. \end{aligned}$$

Because $x_n' \xrightarrow{w} x'$ in $L^p(T)$ and the latter space is uniformly convex (thus has the Kadec–Klee property), we conclude that $x_n \to x$ in $W^{1,p}_{per}(T)$ as $n \to \infty$. Therefore $R(\cdot)$ satisfies the (PS)–condition.

Apply Theorem 1 to obtain $\phi \in W^{1,p}_{per}(T)$, $\phi \neq 0$ such that $0 \in \partial R(\phi) = A(\phi) - (\lambda_2 - \varepsilon)|\phi|^{p-2}\phi - \gamma$. Then for all $\xi \in C_0^\infty(0,b)$ we have

$$\begin{aligned} (13)\quad (\lambda_2-\varepsilon)\int_0^b |\phi(t)|^{p-2}\phi(t)\xi(t)\,dt + \int_0^b \gamma(t)\xi(t)\,dt &=< A\phi), \xi > \\ &= \int_0^b |\phi'(t)|^{p-2}\phi'(t)\xi'(t)\,dt. \end{aligned}$$

Since $|\phi(\cdot)|^{p-2}\phi(\cdot)+\gamma(\cdot)\in L^q(T)$, then from the definition of the distributional derivative, it follows that $|\phi'(\cdot)|^{p-2}\phi'(\cdot)\in W^{1,q}(T)$ and

$$(14)\qquad \left\{\begin{array}{l} -\left(|\phi'(t)|^{p-2}\phi'(t)\right)'=(\lambda_2-\varepsilon)|\phi(t)|^{p-2}\phi(t)+\gamma(t) \;\text{ a.e. on } T \\ \phi(0)=\phi(b) \end{array}\right\}.$$

Also from Green's formula (integration by parts), we have

$$\int_0^b (|\phi'(t)|^{p-2}\phi'(t))'y(t)\,dt =$$

$$|\phi'(b)|^{p-2}\phi'(b)y(b)-|\phi'(0)|^{p-2}\phi'(0)y(0)-\int_0^b|\phi'(t)|^{p-2}\phi'(t)y'(t)\,dt$$

for all $y\in W^{1,p}_{per}(T)$.

Using (12) and (13) we obtain

$$|\phi'(0)|^{p-2}\phi'(0)y(0)=|\phi'(b)|^{p-2}\phi'(b)y(b).$$

Let $y\in W^{1,p}_{per}(T)$ be such that $y(0)=y(b)=1$. We have

$$|\phi'(0)|^{p-2}\phi'(0)=|\phi'(b)|^{p-2}\phi'(b).$$

But the map $r\longrightarrow \frac{1}{p}|r|^p$ is strictly convex and differentiable on $\mathbb{R}$. So its derivative $r\longrightarrow |r|^{p-2}r$ is strictly monotone. Hence $\phi'(0)=\phi'(b)$, i.e. $\phi\in C^1(T)$ is a nontrivial solution of (6) and from the beginning of the proof we know that $\phi\geq 0$.

In a similar way we can prove the next proposition.

PROPOSITION 3. *Problem* (7) *has a solution* $\psi\leq 0$, $\psi\neq 0$.

4. Main result

We start this section by introducing the notions of upper and lower solutions for problem (2).

DEFINITION:

(a) A function $u\in C^1(T)$ is said to be an "upper solution" for problem (2) if $|u'|^{p-2}u'\in W^{1,q}(T)$ and $-(|u'(t)|^{p-2}u'(t))'\geq f_1(t,u(t))$ a.e. on T, $u(0)=u(b)$, $u'(0)\leq u'(b)$.

(b) A function $v\in C^1(T)$ is said to be an "lower solution" for problem (2) if $|v'|^{p-2}v'\in W^{1,q}(T)$ and $-(|v'(t)|^{p-2}v'(t))'\leq f_0(t,v(t))$ a.e. on T, $v(0)=v(b)$, $v'(0)\geq v'(b)$.

Evidently the functions ϕ and ψ obtained in Propositions 2 and 3 are upper and lower solutions respectively for problem (2).

Now let $\tau : T \times \mathbb{R} \to \mathbb{R}$ and $\beta : T \times \mathbb{R} \to \mathbb{R}$ be respectively the truncation and penalty functions defined by

$$\tau(t,x) = \begin{cases} \phi(t) & \text{if } \phi(t) \le x, \\ x & \text{if } \psi(t) \le x \le \phi(t) \\ \psi(t) & \text{if } x \le \psi(t) \end{cases} \quad \text{and} \quad \beta(t,x) = \begin{cases} 1 & \text{if } \phi(t) < x \\ 0 & \text{if } \psi(t) \le x \le \phi(t) \\ -1 & \text{if } x < \psi(t). \end{cases}$$

Because $\phi \ge 0$, $\phi \ne 0$ and by virtue of hypothesis $H(f)$(iv) we see that $f(t,\phi(t)) \ge 0$ a.e. on T and the inequality is strict on a set of positive Lebesque measure. So $\int_0^b f(t,\phi(t))\,dt > 0$. Similarly we obtain that $\int_0^b f(t,\psi(t))\,dt < 0$. Thus we can find $\lambda > 0$ small enough so that $\int_0^b f(t,\phi(t))\,dt > \lambda b$ and $\int_0^b f(t,\psi(t))\,dt < -\lambda b$. With $\lambda > 0$ chosen this way, we define $h(t,x) = g(t,x) - \lambda\beta(t,x)$ with $g(t,x) = f(t,\tau(t,x))$. Clearly $h(\cdot,\cdot)$ is Borel measurable and for almost all $t \in T$ and all $x \in \mathbb{R}$, we have $|h(t,x)| \le \gamma(t)$ with $\gamma \in L^q(T)$ (for example we can take $\gamma = \gamma_M + \lambda$ with $M = \max\{\|\phi\|_\infty, \|\psi\|_\infty\}$).

Let $J, H : W^{1,p}_{per}(T) \to \mathbb{R}$ be the fuctionals defined by

$$J(x) = \frac{1}{p}\|x'\|_p^p \quad \text{and} \quad H(x) = \int_0^b \int_0^{x(t)} h(t,r)\,dr\,dt.$$

Note that $J \in C^1(W^{1,p}_{per}(T))$ with $\partial J(x) = J'(x) = A(x)$ (see the proof of Proposition 2) and H is locally Lipschitz (see Chang [3]). So if we set $R(x) = J(x) - H(x)$, then $R : W^{1,p}_{per}(T) \to \mathbb{R}$ is locally Lipschitz. In the next proposition we establish the existence of nontrivial critical points for the functional $R(\cdot)$.

PROPOSITION 4. *If hypotheses $H(f)$ hold, then $R(\cdot)$ has a nontrivial critical point $x(\cdot)$.*

Proof. Again consider the decomposition $W^{1,p}_{per}(T) = Z \oplus Y$ with $Z = \mathbb{R}$ and $Y = \{y \in W^{1,p}_{per}(T) : \int_0^b y(t)\,dt = 0\}$. For $z \in Z = \mathbb{R}$ we have

$$R(z) = -\int_0^b\int_0^z h(t,r)\,dr\,dt = -\int_0^b\int_0^z g(t,r)\,dr\,dt + \lambda\int_0^b\int_0^z \beta(t,r)\,dr\,dt.$$

First suppose that $z \ge \|\phi\|_\infty > 0$. From the definitions of $g(t,r)$ and $\beta(t,r)$ we have

$$R(z) = -\int_0^b \int_0^{\phi(t)} f(t,r)\,dr\,dt - \int_0^b \int_{\phi(t)}^z f(t,\phi(t))\,dt + \lambda \int_0^b \int_{\phi(t)}^z dr\,dt$$
$$= -\int_0^b F(t,\phi(t))\,dt - \int_0^b (f(t,\phi(t)) - \lambda)(z - \phi(t))\,dt.$$

Note that $|\int_0^b F(t,\phi(t))\,dt| \le \|\gamma\|_1 \|\phi\|_\infty$ and $|\int_0^b (f(t,\phi(t)) - \lambda)\phi(t)\,dt| \le (\|\gamma\|_1 + \lambda b)\|\phi\|_\infty$. Also $\int_0^b (f(t,\phi(t)) - \lambda) z\,dt = z(\int_0^b f(t,\phi(t))\,dt - \lambda b) > 0$ (recall the choice of $\lambda > 0$). Thus we have

$$R(z) \to -\infty \quad \text{as} \quad z \to +\infty.$$

If $z \le -\|\psi\|_\infty < 0$, we have

$$R(z) = -\int_0^b \int_0^{\psi(t)} f(t,r)\,dr\,dt - \int_0^b \int_{\psi(t)}^z f(t,\psi(t))\,dr\,dt - \lambda \int_0^b \int_{\psi(t)}^z dr\,dt$$
$$= -\int_0^b F(t,\psi(t))\,dt - \int_0^b (f(t,\psi(t)) + \lambda)(z - \psi(t))\,dt.$$

Again $|\int_0^b F(t,\psi(t))\,dt| \le \|\gamma\|_1 \|\psi\|_\infty$ and $|\int_0^b (f(t,\psi(t)) + \lambda)\psi(t)\,dt| \le (\|\gamma\|_1 + \lambda b)\|\psi\|_\infty$. Also our choice of $\lambda > 0$ implies that $\int_0^b (f(t,\psi(t)) + \lambda) z\,dt = z(\int_0^b f(t,\psi(t))\,dt + \lambda b) < 0$. Thus we have

$$R(z) \to +\infty \quad \text{as} \quad z \to -\infty.$$

Therefore we can say that $R(z) \to -\infty$ as $|z| \to \infty$.

Next let $y \in Y$. We have

$$R(y) = \frac{1}{p}\|y'\|_p^p - \int_0^b \int_0^{y(t)} h(t,r)\,dr\,dt \ge \frac{1}{p}\|y'\|_p^p - \|\gamma\|_1 \|y\|_\infty.$$

Since $W^{1,p}_{per}(T)$ is embedded continuously in $C(T)$ and by using the Poincare–Wirtinger inequality, we obtain

$$R(y) \ge \frac{1}{p}\|y'\|_p^p - c_1 \|y'\|_p \quad \text{for some } c_1 > 0$$
$$\Rightarrow R(y) \to +\infty \quad \text{as } \|y\| \to \infty, \; y \in Y.$$

Finally let us check that $R(\cdot)$ satisfies the (PS)–condition. To this end let $\{x_n\}_{n \ge 1} \subseteq W^{1,p}_{per}(T)$ such that $|R(x_n)| \le M$ and $m(x_n) \to 0$ as $n \ge \infty$. Let $x_n^* \in \partial R(x_n)$ such that $m(x_n) = \|x_n^*\|$, $n \ge 1$. We know that $\partial R(x_n) \subseteq \partial J(x_n) - \partial H(x_n)$ (see Section 2) and $\partial J(x_n) = A(x_n)$. Hence $x_n^* = A(x_n) - \theta_n$ with $\theta_n \in \partial H(x_n)$. Let $\widehat{H} : L^p(T) \to \mathbb{R}$ be defined by

$\widehat{H}(x) = \int_0^b \int_0^{x(t)} h(t,r)\, dr\, dt$. Evidently $H = \widehat{H}|_{W^{1,p}_{per}(T)}$. Using Theorem 2.2 of Chang [3], we have $\partial H(x_n) \subseteq \partial \widehat{H}(x_n) \subseteq L^q(T)$. Moreover, by definition (see Section 2)

$$\partial \widehat{H}(x_n) = \{\theta \in L^q(T) : \int_0^b \theta(t) y(t)\, dt \leq \widehat{H}^0(x_n; y) \text{ for all } y \in L^p(T)\},$$

where

$$\widehat{H}^0(x_n; y) = \overline{\lim_{\substack{a \to 0 \\ \lambda \downarrow 0}}} \frac{1}{\lambda}\left[\widehat{H}(x_n + a + \lambda y) - \widehat{H}(x_n + a)\right]$$

$$= \overline{\lim_{\substack{a \to 0 \\ \lambda \downarrow 0}}} \frac{1}{\lambda} \int_0^b \int_{(x_n+a)(t)}^{(x_n+a+\lambda y)(t)} h(t,r)\, dr\, dt.$$

Performing a change of variables according to $r(\eta) = x_n(t) + a(t) + \eta\lambda y(t)$ and using Fatou's lemma, we obtain

$$\widehat{H}^0(x_n; y) \leq \int_0^b \overline{\lim_{\substack{a \to 0 \\ \lambda \downarrow 0}}} \int_0^1 h(t, x_n(t) + a(t) + \eta\lambda y(t))\, d\eta\, dt$$

$$\leq \int_{\{y>0\}} h_1(t, x_n(t)) y(t)\, dt + \int_{\{y<0\}} h_0(t, x_n(t)) y(t)\, dt$$

$$\Rightarrow \int_0^b \theta_n(t) y(t)\, dt \leq \int_{\{y>0\}} h_1(t, x_n(t)) y(t)\, dt$$

$$+ \int_{\{y<0\}} h_0(t, x_n(t)) y(t)\, dt \quad \text{for all } y \in L^p(T)$$

$$\Rightarrow h_0(t, x_n(t)) \leq \theta_n(t) \leq h_1(t, x_n(t)) \quad \text{a.e. on } T$$

$$\Rightarrow \{\theta_n\}_{n \geq 1} \subseteq L^q(T) \text{ is bounded.}$$

Let $x_n = z_n + y_n$ with $z_n \in Z = \mathbb{R}$ and $y_n \in Y$, $n \geq 1$. We have

$$| < x_n^*, y_n > | = \left| \|y_n'\|_p^p - \int_0^b \theta_n(t) y_n(t)\, dt \right| \leq \varepsilon_n \|y_n\| \quad (\varepsilon_n \downarrow 0)$$

$$\Rightarrow \|y_n'\|_p^p \leq (\varepsilon_n + \|\theta\|_q) c_2 \|y_n'\|_p \quad \text{for some } c_2 > 0$$

(here we have used the Poincare–Wirtinger inequality). Therefore $\{y_n' = x_n'\}_{n \geq 1} \subseteq L^p(T)$ is bounded. We claim that $\{x_n\}_{n \geq 1} \subseteq C(T)$ is bounded. Suppose not. Then because $\{x_n'\}_{n \geq 1} \subseteq L^p(T)$ is bounded, we must have $L_n = \max_T x_n \to -\infty$ or $\ell_n = \min_T x_n \to +\infty$ as $n \to \infty$. Suppose the first is true (the analysis being similar if the second holds). Then we have

$$|R(x_n)| \le M \quad \text{for all } n \ge 1$$
$$\Rightarrow -\frac{1}{p}\|x_n'\|_p^p + \int_0^b \int_0^{x_n(t)} h(t,r)\,dr\,dt \le M$$
$$\Rightarrow -\frac{1}{p}\|x_n'\|_p^p + \int_0^b \int_0^{L_n} h(t,r)\,dr\,dt - \int_0^b \int_{x_n(t)}^{L_n} h(t,r)\,dr\,dt \le M.$$

Note that

$$\int_0^b \int_{x_n(t)}^{L_n} h(t,r)\,dr\,dt \le \int_0^b \gamma(t)(L_n - x_n(t))\,dt \le \|\gamma\|_1\|L_n - x_n\|_\infty$$
$$\Rightarrow -\int_0^b \int_{x_n(t)}^{L_n} h(t,r)\,dr\,dt \ge -\|\gamma\|_1\|L_n - x_n\|_\infty.$$

Choose $t_n \in T$ such that $L_n = x_n(t_n)$, $n \ge 1$. We have

$$x_n(t_n) - x_n(t) = \int_t^{t_n} x_n'(s)\,ds, \quad t \in T$$
$$\Rightarrow \|x_n(t_n) - x_n(t)\| = \|L_n - x_n(t)\| \le \sup_{n\ge1}\|x_n'\|_1 \le c_3$$
$$\text{for some } c_3 > 0 \text{ and all } t \in T$$
$$\Rightarrow \|L_n - x_n\|_\infty \le c_3.$$

Thus it follows that for $n \ge 1$ large enough (so that $L_n \le -\|\psi\|_\infty$), we have

$$\int_0^b \int_0^{L_n} h(t,r)\,dr\,dt \le M + \sup_{n\ge1}\frac{1}{p}\|x_n'\|_p^p + \|\gamma\|_1 c_3 = c_4 < +\infty \tag{15}$$
$$\Rightarrow \int_0^b \int_0^{L_n} h(t,r)\,dr\,dt = \int_0^b \int_0^{\psi(t)} g(t,r)\,dr\,dt + \int_0^b \int_{\psi(t)}^{L_n} g(t,r)\,dr\,dt + \lambda\int_0^b \int_{\psi(t)}^{L_n} dr\,dt$$
$$= \int_0^b F(t,\psi(t))\,dt + \int_0^b f(t,\psi(t))(L_n - \psi(t))\,dt + \lambda\int_0^b (L_n - \psi(t))\,dt$$
$$\le c_4.$$

Note that from the choice of $\lambda > 0$, we have $\int_0^b (f(t,\psi(t)) + \lambda)\,dt < 0$. Also $|\int_0^b F(t,\psi(t))\,dt|, |\int_0^b f(t,\psi(t))\psi(t)\,dt| \le c_5$ for some $c_5 > 0$. since $L_n \to -\infty$, we have $L_n\int_0^b (f(t,\psi(t))+\lambda)\,dt \to +\infty$ as $n \to \infty$ and we have a contradiction to (14). So indeed $\{x_n\}_{n\geq 1} \subseteq C(T)$ is bounded and this together with the boundedness of $\{x_n' = y_n'\}_{n\geq 1} \subseteq L^p(T)$ implies that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(T)$ is bounded. Arguing as in the proof of proposition 2, we can extract a strongly convergent subsequence of $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{per}(T)$. Thus $R(\cdot)$ satisfies the (PS)–condition. Applying Theorem 1 one can obtain $x \in W^{1,p}_{per}(T)$, $x \neq 0$ such that $0 \in \partial R(x)$. ■

Using this proposition, we can have an existence theorem for problem (2).

THEOREM 5. *If hypotheses $H(f)$ hold, then problem* (2) *has a nontrivial solution $x(\cdot)$.*

Proof. Let $x \in W^{1,p}_{per}(T)$ be a nontrivial critical point of the functional $R(\cdot)$. It exists by Proposition 4. We have $0 \in \partial R(x) \subseteq A(x) - \partial H(x)$, hence $A(x) = \theta$ for $\theta \in \partial H(x)$. Arguing as at the end of the proof of Proposition 2, we can have that

$$\left\{ \begin{array}{l} -(|x'(t)|^{p-2}x'(t))' = \theta(t) \quad \text{a.e. on } T \\ x(0) = x(b), \quad x'(0) = x'(b). \end{array} \right\}.$$

Also let $V, B : W^{1,p}_{per}(T) \to \mathbb{R}$ be the locally Lipschitz functionals defined by $V(x) = \int_0^b \int_0^{x(t)} g(t,r)\,dr\,dt$ and $B(x) = \int_0^b \int_0^{x(t)} \beta(t,r)\,dr\,dt$ (see Chang [3]). We see that $H(x) = V(x) - \lambda B(x)$. Hence we have that $\theta \in \partial H(x) \subseteq \partial V(x) - \lambda \partial B(x)$ (see Section 2) and so $\theta = v - \lambda w$ with $v \in \partial V(x)$ and $w \in \partial B(x)$. Since ψ is a lower solution of problem (2), we have

$$\begin{aligned} (16) \qquad & \int_0^b (|x'(t)|^{p-2}x'(t)y'(t) - |\psi'(t)|^{p-2}\psi'(t)y'(t))\,dt \\ & \geq \int_0^b (v(t) - \lambda w(t) - f_0(t,\psi(t)))y(t)\,dt \quad \text{for all } y \in W^{1,p}_{per}(T) \cap L^p(T)_+ . \end{aligned}$$

Let $y = (\psi - x)_+ \in W^{1,p}_{per}(T) \cap L^p(T)_+$ and recall that

$$(\psi - x)'_+(t) = \begin{cases} (\psi - x)'(t) & \text{if } (\psi - x)(t) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(see Gilbarg–Trudinger [8]). We have

$$(17) \qquad \int_0^b (|x'(t)|^{p-2}x'(t) - |\psi'(t)|^{p-2}\psi'(t))(\psi - x)'_+(t)\,dt$$

$$= \int_{\{\psi>x\}} (|x'(t)|^{p-2}x'(t) - |\psi'(t)|^{p-2}\psi'(t))(\psi'(t) - x'(t))\,dt$$

$$\le -2^{2-p} \int_{\{\psi>x\}} |\psi'(t) - x'(t)|^p\,dt = -2^{2-p}\int_0^b |(\psi - x)'_+(t)|^p\,dt \le 0.$$

Also we have that

$$\int_0^b (v(t) - f_0(t,\psi(t)))(\psi - x)_+(t)\,dt = \int_{\{\psi>x\}} (v(t) - f_0(t,\psi(t)))(\psi - x)(t)\,dt.$$

Recall that since $v \in \partial H(x)$, that $v(t) \ge g_0(t,x(t))$ a.e. on T. Also from the definition of $g(t,r)$ we see that for almost all $t \in \{x < \psi\}$, we have $f_0(t,\psi(t)) = g_0(t,x(t))$. So $f_0(t,\psi(t)) \le v(t)$ a.e. on $\{x < \psi\}$. Hence we have

$$\int_0^b (v(t) - f_0(t,\psi(t)))(\psi - x)_+(t)\,dt \ge 0. \tag{18}$$

Finally note that since $w \in \partial B(x)$, we have $\beta_0(t,x(t)) \le w(t) \le \beta_1(t,x(t))$ a.e. on T. But from the definition of the penalty function $\beta(t,r)$, we see that for almost all $t \in \{x < \psi\}$ we have $\beta_0(t,x(t)) = \beta_1(t,x(t)) = -1$, hence $w(t) = -1$ a.e. on $\{x < \psi\}$. Thus we have

$$-\lambda\int_0^b w(t)(\psi - x)_+(t)\,dt = -\lambda \int_{\{\psi>x\}} (-1)(\psi-x)(t)\,dt = \lambda\int_0^b (\psi-x)_+(t)\,dt. \tag{19}$$

Using (17), (18) and (19) in (16), we have

$$\lambda\int_0^b (\psi - x)_+(t)\,dt = 0\ ,\quad \lambda > 0,$$

$$\Rightarrow \psi(t) \le x(t)\ \text{ for all } t \in T.$$

Similarly we obtain that $(x(t) \le \phi(t)$ for all $t \in T$. Therefore we see that for all $t \in T$ $\psi(t) \le x(t) \le \phi(t)$. Recall that $-(|x'(t)|^{p-2}x'(t))' = \theta(t)$ a.e. on T with $h_0(t,x(t)) \le \theta(t) \le h_1(t,x(t))$ a.e. on T. Using the definitions of h, h_0 and h_1 and the fact that $\psi(t) \le x(t) \le \phi(t)$ for all $t \in T$, we can check that $h_0(t,x(t)) = f_0(t,x(t))$ and $h_1(t,x(t)) = f_1(t,x(t))$ a.e. on T. Thus we conclude that

$$\left\{ \begin{array}{l} -(|x'(t)|^{p-2}x'(t))' \in [f_0(t,x(t)), f_1(t,x(t))]\ \text{ a.e. on } T \\ x(0) = x(b),\quad x'(0) = x'(b). \end{array} \right\}$$

$\Rightarrow x(\cdot)$ is a nontrivial solution of problem (2). ∎

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NATIONAL TECHNICAL UNIVERSITY
DEPARTMENT OF MATHEMATICS
ZOGRAFOU CAMPUS
ATHENS 157 80, GREECE
E-mail: npapg@math.ntua.gr

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