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A FEW NOTES ON SUBALGEBRA LATTICES, PART I

Abstract. First, we apply results proved in [Pió1] and some results of graph theory to formulate and prove a necessary condition for partial (and thus also total) unary algebras to have isomorphic (strong) subalgebra lattices. Although this condition is not sufficient for arbitrary partial unary algebras, we can form, having this fact, a lot of new partial unary algebras with the same subalgebra lattices. Moreover, we use this result to characterize arbitrary two partial (thus in particular also total) monounary algebras with isomorphic (strong) subalgebra lattices. Having this result we can also describe all pairs (A, L), where A is a partial monounary algebra and L a lattice, such that the subalgebra lattice of A is isomorphic to L.

In the next part [Pió2] we apply the results of this paper to characterize connections between weak and strong subalgebra lattices of partial (thus also total) monounary algebras.

An important part of Universal Algebra and the theory of partial algebras is an investigation of connections between algebras and their lattices of subalgebras. A few such results concern also classical algebras. For example, D. Sachs in **[Sach]** proved that two Boolean algebras are isomorphic iff their lattices of subalgebras are isomorphic; E. Lukács and P.P. Pálfy showed in **[LuPa]** that the modularity of the subgroup lattice of the direct square $\mathbf{G} \times \mathbf{G}$ of any group $\mathbf{G}$ implies that $\mathbf{G}$ is commutative.

In the present part we investigate subalgebra lattices of unary and monounary algebras. But we do not restrict our attention to total algebras only, and we consider the more general case of partial algebras, because this approach is very fruitful to our investigation. More precisely, we use some results proved in **[Pió1]** and also several results of graph theory to prove one necessary condition for arbitrary two partial (and thus also total) unary algebras to have isomorphic (strong) subalgebra lattices (although in this part we consider only the ordinary kind of subalgebras, they will be sometimes called strong as opposed to the other kinds of partial subalgebras which will

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be considered in the second part). More precisely, we show that a contraction of a special subset of the carrier of an algebra to a point or an insertion of such a subset in the place of an element of an algebra preserves strong subalgebra lattices. Unfortunately, in this way we do not obtain a sufficient condition, i.e there are partial unary algebras with isomorphic strong subalgebra lattices and none of them is obtained from the other in this way. But first, having this fact we can form from any partial unary algebra a lot of new algebras with the same strong subdigraph lattices. Secondly, for partial monounary algebras this result forms also a sufficient condition. More precisely, we use this result to completely characterize arbitrary two partial (thus in particular also total) monounary algebras with isomorphic strong subalgebra lattices. Moreover, having this result we can also describe all pairs $(\mathbf{A}, \mathbf{L})$, where $\mathbf{A}$ is a partial monounary algebra and $\mathbf{L}$ a lattice, such that the subalgebra lattice of $\mathbf{A}$ is isomorphic to $\mathbf{L}$.

For basic notions and results concerning algebras (total and partial) see e.g. [**BRR**], [**Bur**] and [**Jón**], and concerning digraphs (i.e. directed graphs) see e.g. [**Ber**] and [**Ore**]. For any partial unary algebra $\mathbf{A} = (A, (k^A)_{k \in K})$ of unary type K (where K is a set of unary operation symbols), the complete and algebraic lattice of all strong subalgebras of $\mathbf{A}$ under (strong subalgebra) inclusion $\leq_s$ will be denoted by $\mathbf{S}_s(\mathbf{A})$. Further, for any digraph $\mathbf{D}$, by V^D and E^D we denote its sets of vertices and edges, respectively. In this paper we consider, in general, infinite digraphs (i.e. V^D and E^D may have arbitrary cardinality), because we use digraphs to represent partial unary algebras. Each partial unary algebra $\mathbf{A} = (A, (k^A)_{k \in K})$ can be represented by the digraph $\mathbf{D}(\mathbf{A})$ obtained from $\mathbf{A}$ by omitting the names of all operations (see [**Bar**] or [**Pió1**]). More formally, A is the set of all vertices of $\mathbf{D}(\mathbf{A})$, $\{(a, k, b) \in A \times K \times A : (a, b) \in k^A\}$ is the set of all (directed) edges of $\mathbf{D}(\mathbf{A})$, and for each edge (a, k, b), a is its initial vertex and b is its final vertex.

Note that this construction is a very particular case of the Grothendieck construction (see [**BaWe**] section 4.2 and 11.2), but applied to models of digraphs (in the category of sets and partial functions) rather than to functors. More precisely, a partial unary algebra $\mathbf{A}$ of type K can be obviously represented by a model of the type digraph $\mathbf{K}$ with exactly one vertex and unary operation symbols from K as edges (i.e. by a digraph homomorphism from $\mathbf{K}$ into the category of sets and partial functions). Next, by the Grothendieck construction applied to this model we get the digraph $\mathbf{D}(\mathbf{A})$ together with a homomorphism into $\mathbf{K}$. By forgetting this homomorphism we arrive at the above construction.

In [**Pió1**] we defined a special kind of subdigraphs which correspond to strong subalgebras of partial unary algebras, and therefore they are also called strong. More precisely, let **D** and **H** be any digraphs. Then **H** is a strong subdigraph of **D** ($\mathbf{H} \leq_s \mathbf{D}$) iff **H** is an ordinary subdigraph of **D** and for each edge e of **D**, if the initial vertex of e belongs to **H**, then e belongs to **H** (in particular the final vertex of e also is in **H**). It is easy to see that for two strong subdigraphs **H** and **K** of **D**, they are equal (**H** is a strong subdigraph of **K**) iff $V^H = V^K$ ($V^H \subseteq V^K$). It is proved in [**Pió1**] that for each digraph **D**, its set of all strong subdigraphs forms a complete and algebraic lattice $\mathbf{S}_s(\mathbf{D})$ under (strong subdigraph) inclusion $\leq_s$.

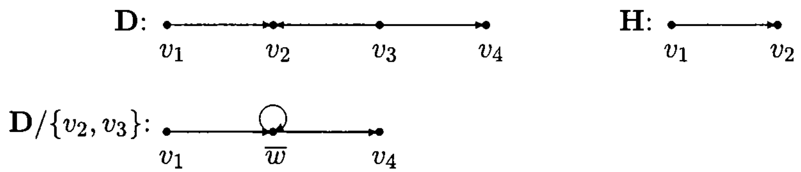
Obviously (see [**Pió1**]) for a partial unary algebra **A** and its strong subalgebra $\mathbf{B} \leq_s \mathbf{A}$, the digraph $\mathbf{D}(\mathbf{B})$ representing **B** is indeed a strong subdigraph of $\mathbf{D}(\mathbf{A})$. Moreover, this function (assigning to each strong subalgebra its digraph) forms a lattice isomorphism. Thus in [**Pió1**] we have obtained (by $\simeq$ we denote simultaneously isomorphism of lattices, algebras, digraphs etc.)

THEOREM 1. *For each partial unary algebra* **A**, $\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{S}_s(\mathbf{D}(\mathbf{A}))$.

Let **D** be a digraph and W a set of vertices, then the contraction of W (see e.g. [**Ber**]) is the operation defined by replacing W by a single point (which will be denoted often by $\overline{w}$) and replacing all directed edges with endpoints in W by a single loop in $\overline{w}$, and replacing each directed edge going into W (resp. out of W) by a directed edge with the same initial vertex (resp. final vertex) ending in $\overline{w}$ (resp. starting from $\overline{w}$). The digraph obtained from **D** by the contraction of W will be denoted by $\mathbf{D}/W$, we will also use the convention that $\mathbf{D}/\emptyset = \mathbf{D}$. Obviously if **D** is connected, then $\mathbf{D}/W$ is also connected. Further, by a simple verification we obtain that for each subdigraph **H** of **D**, $\mathbf{H}/(V^H \cap W)$ is a subdigraph of $\mathbf{D}/W$.

Of course the contraction of a set need not preserve the strong subdigraph lattice, because, for instance, from every non–empty digraph we can obtain a trivial digraph by contracting the set of all vertices.

Moreover, the example below shows that the operation of contraction of a vertex set does not preserve strong subdigraphs, in general.



Then **H** is a strong subdigraph of **D**, but $\mathbf{H}/(\{v_2, v_3\} \cap \{v_1, v_2\}) = \mathbf{H}$ is not a strong subdigraph of $\mathbf{D}/\{v_2, v_3\}$.

Now we show that for a special kind of sets of vertices, the operation of contraction preserves the strong subdigraph lattices. We start with several results describing when a strong subdigraph is preserved by this construction.

LEMMA 2. *Let* $\mathbf{D}$ *be a digraph,* $\mathbf{H}$ *a subdigraph of* $\mathbf{D}$ *and* $W \subseteq V^D \setminus V^H$. *Then* $\mathbf{H}$ *is a strong subdigraph of* $\mathbf{D}$ *iff* $\mathbf{H}$ *is a strong subdigraph of* $\mathbf{D}/W$.

LEMMA 3. *Let* $\mathbf{D}$ *be a digraph,* $\mathbf{H}$ *its strong subdigraph and* $W \subseteq V^H$. *Then* $\mathbf{H}/W$ *is a strong subdigraph of* $\mathbf{D}/W$.

For each digraph $\mathbf{D}$ and $W \subseteq V^D$, by $[W]_D$ we denote the subdigraph of $\mathbf{D}$ spanned on W, i.e. W is its set of vertices and all edges of $\mathbf{D}$ with endpoints in W forms its set of edges.

LEMMA 4. *Let* $\mathbf{D}$ *be a digraph,* $\mathbf{H}$ *its subdigraph,* $W \subseteq V^H$, *and let* $\mathbf{H}/W$ *be a strong subdigraph of* $\mathbf{D}/W$. *Then* $[V^H]_D$ *is a strong subdigraph of* $\mathbf{D}$.

The above three lemmas are obtained by a simple verification, and therefore their proofs are omitted.

LEMMA 5. *Let* $\mathbf{D}$ *be a digraph and* $W \subseteq V^D$, *and let* $\mathbf{K}$ *be a subdigraph (strong subdigraph) of* $\mathbf{D}/W$. *Then there is a subdigraph (strong subdigraph)* $\mathbf{H}$ *of* $\mathbf{D}$ *such that* $\mathbf{H}/(W \cap V^H) = \mathbf{K}$.

Proof. If $\overline{w}$ does not belong to $\mathbf{K}$ (where $\overline{w}$ is the vertex of $\mathbf{D}/W$ corresponding to the set W), then of course $\mathbf{K}$ is also a subdigraph of $\mathbf{D}$. Thus in this case we can take $\mathbf{H} = \mathbf{K}$. By L.2 we have that if $\mathbf{K}$ is a strong subdigraph of $\mathbf{D}/W$, then $\mathbf{H}$ is a strong subdigraph of $\mathbf{D}$.

Thus now we can assume that $\overline{w} \in V^K$. Let U be the set of all vertices of $\mathbf{K}$ without $\overline{w}$ and of all vertices of W (i.e. $U = (V^K \setminus \{\overline{w}\}) \cup W$) and let $\mathbf{H}$ be the subdigraph of $\mathbf{D}$ with the set of vertices U (observe that U is obviously a subset of $\mathbf{D}$) and with all edges of $\mathbf{D}$ such that their images (in $\mathbf{D}/W$) belong to $\mathbf{K}$. Then the definition of the contraction of W easy implies that $\mathbf{H}/W = \mathbf{K}$.

Now assume that $\mathbf{K}$ is a strong subdigraph of $\mathbf{D}/W$. Then $[U]_D$ is a strong subdigraph of $\mathbf{D}$ by L.4. Thus it is sufficient to prove $\mathbf{H} = [U]_D$. Of course each edge of $\mathbf{H}/W$ belongs to $[U]_D$. On the other hand, take an arbitrary edge e of $[U]_D$ and let $\overline{e}$ be its image in $\mathbf{D}/W$. Then it is easy to see that endpoints of $\overline{e}$ belong to $\mathbf{K}$, so $\overline{e}$ is also in $\mathbf{K}$ (because $\mathbf{K} \leq_s \mathbf{D}/W$). Hence, e is an edge of $\mathbf{H}$. This shows that the sets of edges of $\mathbf{H}$ and $[U]_D$ are equal. Thus $\mathbf{H} = [U]_D$, since their sets of vertices are equal. ∎

Before next results recall (see e.g. [**Ber**]) that a digraph $\mathbf{D}$ is strongly connected iff for any two distinct vertices v, w, there is a path from v to w. It is not difficult to prove (see also [**Ber**]) that a digraph $\mathbf{D}$ is strongly connected iff $\mathbf{D}$ is connected and every edge lies on a (directed) cycle. We assume that a path (cycle) does not encounter the same vertex twice (except the first and the last vertex).

LEMMA 6. *Let* $\mathbf{D}$ *be a digraph,* $\mathbf{H}$ *its strong subdigraph and let* $W \subseteq V^D$ *have common vertices with* $\mathbf{H}$ *and* $[W]_D$ *be a strongly connected digraph. Then* $W \subseteq V^H$.

Proof. Let $v \in V^H \cap W$ and take $w \in W$. Then there is a path $(e_1, \ldots, e_n)$ from v to w. Since the initial vertex of e_1 is equal v and $v \in V^H$ and $\mathbf{H} \leq_s \mathbf{D}$, we obtain by a simple induction that the final vertex of e_i belongs to $\mathbf{H}$ for $i = 1, \ldots, n$. Thus, in particular, w is in $\mathbf{H}$. ∎

Now we can formulate and prove the first important result.

THEOREM 7. *Let* $\mathbf{D}$ *be a digraph and let* $W \subseteq V^D$ *be a set such that* $[W]_D$ *is strongly connected. Then* $\mathbf{S}_s(\mathbf{D}) \simeq \mathbf{S}_s(\mathbf{D}/W)$.

Let φ be the function of the set of all strong subdigraphs of $\mathbf{D}$ into the set of all strong subdigraphs of $\mathbf{D}/W$ such that $\varphi(\mathbf{H}) = \mathbf{H}/(V^H \cap W)$.

Of course we show that φ is the desired lattice isomorphism. Observe first that φ is correctly defined. To this purpose take an arbitrary strong subdigraph $\mathbf{H}$ of $\mathbf{D}$. Then $W \subseteq V^H$ or $W \cap V^H = \emptyset$, by L.6. Hence, using L.2 and L.3, we obtain that $\varphi(\mathbf{H})$ is indeed a strong subdigraph of $\mathbf{D}/W$. In particular, $\varphi(\mathbf{H}) = \mathbf{H}$ or $\varphi(\mathbf{H}) = \mathbf{H}/W$.

Now take two strong subdigraphs $\mathbf{H}, \mathbf{K}$ of $\mathbf{D}$ and assume that $\varphi(\mathbf{H}) = \varphi(\mathbf{K})$. If $\overline{w}$ (where $\overline{w}$ is the vertex of $\mathbf{D}/W$ corresponding to the set W) does not belong to $\varphi(\mathbf{H})$, then W and $\mathbf{H}$, $\mathbf{K}$ are disjoint, so $\mathbf{H} = \varphi(\mathbf{H}) = \varphi(\mathbf{K}) = \mathbf{K}$. Thus we can assume that $\overline{w}$ belongs to $\varphi(\mathbf{H})$. Then W and $\mathbf{H}$, and also W and $\mathbf{K}$ have common vertices, so W is contained in $\mathbf{H}$ and $\mathbf{K}$, by L.6. Hence, since other vertices of $\varphi(\mathbf{H})$ and $\varphi(\mathbf{K})$ (i.e. different from $\overline{w}$) are the same as in $\mathbf{H}$ and $\mathbf{K}$, respectively, we deduce that the sets of vertices of $\mathbf{H}$ and $\mathbf{K}$ are equal. This implies that $\mathbf{H} = \mathbf{K}$, because they are strong subdigraphs of $\mathbf{D}$.

Thus we have shown that φ is injective, so by L.5 φ is a bijection of the set of all strong subdigraphs of $\mathbf{D}$ onto the set of all strong subdigraphs of $\mathbf{D}/W$.

Now we must only prove that φ and its inverse φ^{-1} preserve (strong subdigraph) inclusion $\leq_s$. Take two arbitrary strong subdigraph $\mathbf{H}$ and $\mathbf{K}$ of $\mathbf{D}$, and observe that $\mathbf{H} \leq_s \mathbf{K}$ iff $V^H \subseteq V^K$; and analogously $\varphi(\mathbf{H}) \leq_s \varphi(\mathbf{K})$ iff $V^{\varphi(H)} \subseteq V^{\varphi(K)}$. Thus it is sufficient to show that the vertex set of $\mathbf{H}$ is contained in $\mathbf{K}$ iff $\varphi(\mathbf{K})$ contains the vertex set of $\varphi(\mathbf{H})$. But this fact easily follows from the definition of the contraction of W (in a similar way as in the proof that φ is injective), because $\mathbf{H}$ ($\mathbf{K}$) contains W or $\mathbf{H}$ ($\mathbf{K}$) and W is disjoint. This completes the proof. ∎

Now, using the above graph theorem, we can formulate and prove the first algebraic result of this paper concerning unary partial algebras. More precisely

THEOREM 8. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be partial unary algebras (which can be of different types) satisfying the following condition:* $\mathbf{D}(\mathbf{B}) \simeq \mathbf{D}(\mathbf{A})/W$ *for some subset* W *of* A *such that* $[W]_{D(A)}$ *is strongly connected. Then* $\mathbf{S}_s(\mathbf{B}) \simeq \mathbf{S}_s(\mathbf{A})$.

Proof. follows directly from Th.1 and Th.7. ∎

Now take an arbitrary digraph $\mathbf{D}$ and observe that we can apply the operation of the contraction of a vertex set to each of its connected components (i.e. maximal connected subdigraphs) separately. More formally, let $\{\mathbf{D}_i\}_{i\in I}$ be the family of all the connected components of $\mathbf{D}$ and let $\{W_i\}_{i\in I}$ be an arbitrary family of subsets of the vertex set of $\mathbf{D}$ such that W_i is contained in $\mathbf{D}_i$ for each $i \in I$. Then we can take the family $\{\mathbf{D}_i/W_i\}_{i\in I}$ of digraphs, and next we can take the disjoint union of this family. The digraph so obtained will be denoted by $\mathbf{D}/\{W_i\}_{i\in I}$. Note that if W is a subset of $\mathbf{D}_j$ for some $j \in I$, then $\mathbf{D}/W = \mathbf{D}/\{W_i\}_{i\in I}$, where $W_j = W$ and $W_i = \emptyset$ for each $i \neq j$.

Now we prove a result analogous to Th.7 for this generalized construction. Observe first that the following fact holds:

PROPOSITION 9. *Let* $\mathbf{D}$ *be a digraph and* $\{\mathbf{D}_i\}_{i\in I}$ *a family of its connected components. Then* $\mathbf{S}_s(\mathbf{D}) \simeq \prod_{i\in I} \mathbf{S}_s(\mathbf{D}_i)$.

Proof. This isomorphism φ is given by a function assigning to each strong subdigraph $\mathbf{H}$ of $\mathbf{D}$ the sequence of all its connected components $(\mathbf{H}_i)_{i\in I}$ in such a way that $\mathbf{H}_i$ is a subdigraph (perhaps empty) of $\mathbf{D}_i$ for all $i \in I$, i.e. $\mathbf{H}_i = \mathbf{H} \cap \mathbf{D}_i$.

Obviously φ is correctly defined, since if $\mathbf{H}$ is a strong subdigraph of $\mathbf{D}$, then each of its connected components is also a strong subdigraph of $\mathbf{D}$, so in particular also of some corresponding connected component of $\mathbf{D}$. φ is surjective, because if $(\mathbf{H}_i)_{i\in I}$ is a sequence of strong subdigraphs (i.e. $\mathbf{H}_i \leq_s \mathbf{D}_i$ for $i \in I$), then the disjoint sum $\mathbf{H}$ of the family $\{\mathbf{H}_i\}_{i\in I}$ is a strong subdigraph of $\mathbf{D}$ and of course $\varphi(\mathbf{H}) = (\mathbf{H}_i)_{i\in I}$. It is trivial that φ is an injection, since each digraph has exactly one decomposition onto connected components. Moreover, φ and its inverse φ^{-1} preserve the relation $\leq_s$, since for each strong subdigraphs $\mathbf{H}$ and $\mathbf{K}$ of $\mathbf{D}$, $\mathbf{H} \leq_s \mathbf{K}$ iff $V^H \subseteq V^K$ iff $V^{H_i} \subseteq V^{K_i}$ for all $i \in I$ iff $\mathbf{H}_i \leq_s \mathbf{K}_i$ for $i \in I$, where $\{\mathbf{H}_i\}_{i\in I}$ and $\{\mathbf{K}_i\}_{i\in I}$ are the families of connected components of $\mathbf{H}$ and $\mathbf{K}$, respectively. ∎

Before the next result observe that for any digraph $\mathbf{D}$, its connected component $\mathbf{H}$ and subset $W \subseteq V^H$, $[W]_H$ and $[W]_D$ are equal.

THEOREM 10. *Let* $\mathbf{D}$ *be a digraph,* $\{\mathbf{D}_i\}_{i\in I}$ *the family of all its connected components and let* $\{W_i\}_{i\in I}$ *be a family of subsets of* V^D *such that for each* $i \in I$, W_i *is contained in* $\mathbf{D}_i$ *and* $[W_i]_D$ *is strongly connected. Then* $\mathbf{S}_s(\mathbf{D}) \simeq \mathbf{S}_s(\mathbf{D}/\{W_i\}_{i\in I})$.

Proof. Th.8 implies that $\mathbf{S}_s(\mathbf{D}_i)$ and $\mathbf{S}_s(\mathbf{D}_i/W_i)$ are isomorphic for $i \in I$. Thus the direct products $\prod_{i\in I}\mathbf{S}_s(\mathbf{D}_i)$ and $\prod_{i\in I}\mathbf{S}_s(\mathbf{D}_i/W_i)$ are isomorphic, so $\mathbf{S}_s(\mathbf{D}) \simeq \mathbf{S}_s(\mathbf{D}/\{W_i\}_{i\in I})$, by P.9. ■

Now we can formulate our second algebraic result on partial unary algebras.

THEOREM 11. *Let partial unary algebras* $\mathbf{A}$ *and* $\mathbf{B}$ *(which can be of different types) satisfy the following condition:* $\mathbf{D}(\mathbf{B}) \simeq \mathbf{D}(\mathbf{A})/\{W_i\}_{i\in I}$ *for some family* $\{W_i\}_{i\in I}$ *of subsets of* A *such that for each* $i \in I$, W_i *is contained in* $\mathbf{D}_i(\mathbf{A})$ *(where* $\{\mathbf{D}_i(\mathbf{A})\}_{i\in I}$ *is the family of all connected components of* $\mathbf{D}(\mathbf{A})$*) and* $[W_i]_{D(A)}$ *is strongly connected. Then* $\mathbf{S}_s(\mathbf{B}) \simeq \mathbf{S}_s(\mathbf{A})$.

Proof. follows directly from Th.1 and Th.10. ■

It is easy to see that the necessary condition in the above theorem is not sufficient, i.e. there are partial unary algebras with isomorphic strong subalgebra lattices and there is no family $\{W_i\}_{i\in I}$ of sets as in the theorem such that $\mathbf{D}(\mathbf{B})$ is isomorphic to $\mathbf{D}(\mathbf{A})/\{W_i\}_{i\in I}$. But, having Th.8 and this theorem we can construct from a given partial unary algebra $\mathbf{A}$ a lot of new partial unary algebras with strong subalgebra lattices isomorphic to $\mathbf{S}_s(\mathbf{A})$. To this purpose we must only contract any subset of $\mathbf{A}$ such that each two of its elements generate the same strong subalgebra (note that this set need not form itself a strong subalgebra). Of course we can apply this construction to each connected component of $\mathbf{A}$ separately. Conversely, we can also insert such a subset (i.e. satisfying the above condition) in the place of an element of $\mathbf{A}$, and again we can blow up in this way each connected component of $\mathbf{A}$ separately. Obviously these two constructions do not preserve types of algebras, in general.

Moreover,we now show that for partial monounary algebras the above theorem forms also a sufficient condition. Let $\mathbf{A}$ be a partial monounary algebra (i.e. a partial algebra with one partial unary operation). Then its digraph $\mathbf{D}(\mathbf{A})$ is a functional digraph (i.e. at most one edge starts from any vertex) and of course if $\mathbf{A}$ is total, then $\mathbf{D}(\mathbf{A})$ is total (i.e. exactly one edge starts from each vertex). Observe that the inverse fact is also true. More precisely, for every functional digraph $\mathbf{D}$, there is a partial monounary algebra $\mathbf{A}$ corresponding to $\mathbf{D}$, i.e. $\mathbf{D}(\mathbf{A})$ is isomorphic to $\mathbf{D}$.

It is obvious and well–known that for every functional digraph $\mathbf{D}$, each of its connected components contains at most one directed cycle. Thus we can contract each non–trivial cycle (i.e. with at least two vertices), and the digraph so obtained will be denoted by $\mathbf{T}^d(\mathbf{D})$. Moreover, $\mathbf{Ts}^d(\mathbf{D})$ is a digraph obtained from $\mathbf{T}^d(\mathbf{D})$ by omitting all loops. Observe also that $\mathbf{T}^d(\mathbf{D})$ has no non–trivial directed cycles, so $\mathbf{Ts}^d(\mathbf{D})$ has no directed cycles (including loops).

THEOREM 12. *Let* $\mathbf{D}$ *be a functional digraph. Then*

$$\mathbf{S}_s(\mathbf{D}) \simeq \mathbf{S}_s(\mathbf{T}^d(\mathbf{D})) \simeq \mathbf{S}_s(\mathbf{Ts}^d(\mathbf{D})).$$

Proof. Observe that each non–trivial directed cycle of $\mathbf{D}$ forms a strongly connected digraph and any two different directed cycles belong to two distinct connected components. Thus the first isomorphism is implied by Th.10. Moreover, the second isomorphism is easy to see, since $\mathbf{Ts}^d(\mathbf{D})$ is obtained from $\mathbf{T}^d(\mathbf{D})$ by omitting only loops. ∎

Now take a partial monounary algebra $\mathbf{A}$ and let

$$\mathbf{T}^d(\mathbf{A}) := \mathbf{T}^d(\mathbf{D}(\mathbf{A})) \quad \text{and} \quad \mathbf{Ts}^d(\mathbf{A}) := \mathbf{Ts}^d(\mathbf{D}(\mathbf{A}))$$

Then the above results for functional digraphs imply that $\mathbf{T}^d(\mathbf{A})$ and $\mathbf{Ts}^d(\mathbf{A})$ are functional digraphs without non–trivial directed cycles, and $\mathbf{Ts}^d(\mathbf{A})$ has no loops either. Moreover, Th.1 and Th.12 imply

THEOREM 13. *Let* $\mathbf{A}$ *be a partial monounary algebra. Then*

$$\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{S}_s(\mathbf{T}^d(\mathbf{A})) \simeq \mathbf{S}_s(\mathbf{Ts}^d(\mathbf{A})).$$

Now we show that $\mathbf{Ts}^d(\mathbf{A})$ uniquely determines the strong subalgebra lattice $\mathbf{S}_s(\mathbf{A})$ for any partial monounary algebra $\mathbf{A}$. More formally, we prove that two monounary partial algebras $\mathbf{A}$ and $\mathbf{B}$ have isomorphic strong subalgebra lattices if their digraphs $\mathbf{Ts}^d(\mathbf{A})$ and $\mathbf{Ts}^d(\mathbf{B})$ are isomorphic. To this purpose we start with the following result from [**JoSe**] (see also [**Jón**]) characterizing the strong subalgebra lattice of a partial monounary algebra.

THEOREM 14. *A complete lattice* $\mathbf{L} = (L, \leq_L)$ *is isomorphic to the strong subalgebra lattice* $\mathbf{S}_s(\mathbf{A})$ *for some partial monounary algebra* $\mathbf{A}$ *iff* $\mathbf{L}$ *is algebraic, distributive and*

(1) *every element of* $\mathbf{L}$ *is a join of completely join–irreducible elements,*

(2) *for each completely join–irreducible element* i*, the set of all completely join–irreducible elements which are less or equal than* i *(with respect to the lattice ordering* $\leq_L$*) is totally ordered by* $\leq_L$ *and is finite or isomorphic to the set of all non–positive integers with the natural less–or–equal order.*

Recall that an element l of $\mathbf{L}$ is completely join–irreducible iff for any subset K of $\mathbf{L}$, $l = \bigvee K$ (i.e. l is the supremum of K) implies $l \in K$. Recall also that for any two elements i, j of $\mathbf{L}$, j is covered by i (or i covers j) iff $j \leq_L i$ and there is no element k of $\mathbf{L}$ distinct from i and j and $j \leq_L k \leq_L i$.

Secondly, for an algebraic and distributive lattice $\mathbf{L}$ satisfying (1), (2) of Th.14 a partial monounary algebra $\mathbf{A} = (A, (k^A))$ such that $\mathbf{S}_s(\mathbf{A})$ is isomorphic to $\mathbf{L}$ can be constructed as follows: A is the set of all completely join–irreducible elements of $\mathbf{L}$ and for every $a \in A$, if a is minimal in A (with respect to $\leq_L$), then the unary operation k^A on a is not defined; if a is not minimal, then $k^A(a)$ is the unique element of A covered by a (observe that (2) of Th.14 implies that every element $i \in A$ is either minimal or covers a unique element of A). Note that $\mathbf{D}(\mathbf{A})$ is a functional digraph without directed cycles (nor loops).

Of course $\mathbf{A}$ can be completed to a total monounary algebra $\overline{\mathbf{A}}$ (for every minimal element $a \in A$ we set $k^{\overline{A}}(a)$ equal to a), but then the functional digraph corresponding to this total monounary algebra has loops.

Observe that with every algebraic and distributive lattice $\mathbf{L} = (L, \leq_L)$ satisfying (1), (2) of Th.14 we can associate a functional digraph $\mathbf{Ds}(\mathbf{L})$ in the following way: we first consider the partial monounary algebra $\mathbf{A}$ defined above, and next we set $\mathbf{Ds}(\mathbf{L}) = \mathbf{D}(\mathbf{A})$. In other words, $\mathbf{Ds}(\mathbf{L})$ is a digraph such that the set of all completely join–irreducible elements of $\mathbf{L}$ is its set of all vertices, the set of pairs (p, q), where p and q are completely join–irreducible elements and p covers q, is its set of all (directed) edges and for every edge (p, q), p is its initial vertex and q is its final vertex. Note that by Th.1, since $\mathbf{S}_s(\mathbf{A})$ is isomorphic to $\mathbf{L}$, we have that $\mathbf{S}_s(\mathbf{Ds}(\mathbf{L}))$ is also isomorphic to $\mathbf{L}$.

Note also that $\mathbf{Ds}(\mathbf{L})$ can be easily completed to a total functional digraph $\mathbf{D}(\mathbf{L})$ by adding a loop to each vertex without a starting edge. More precisely, to the edge set of $\mathbf{Ds}(\mathbf{L})$ we add all pairs (p, p), where p is completely join–irreducible and is minimal in the set of all completely join–irreducible elements. Obviously this digraph is equal to the digraph representing the total monounary algebra $\overline{\mathbf{A}}$ corresponding to $\mathbf{L}$ (defined above).

LEMMA 15. *Let* $\mathbf{D}$ *be a functional digraph without directed cycles (nor loops). Then* $\mathbf{Ds}(\mathbf{S}_s(\mathbf{D})) \simeq \mathbf{D}$*, i.e. the digraph obtained from the strong subdigraph lattice of* $\mathbf{D}$ *is isomorphic to* $\mathbf{D}$.

Proof. First, for each vertex v of $\mathbf{D}$, we denote by $\langle v \rangle_D$ the least strong subdigraph containing v.

Secondly, in **[Pió1]** we proved that a vertex u belongs to $\langle v \rangle_D$ iff $u = v$ or there is a path from v to u. (This is a graph–theoretical generalization of the classical result on the generation of (strong) subalgebras and its proof is similar.) Hence, if there is a path from v to u, then the vertex set of $\langle u\rangle_D$ is contained in $\langle v\rangle_D$. So $\langle u\rangle_D \leq_s \langle v\rangle_D$, since they are strong subdigraphs.

Thirdly, in the same way as for unary (total) algebras (see e.g. [**Jón**]) we obtain that a strong subdigraph $\mathbf{H}$ of $\mathbf{D}$ is a completely join–irreducible element of $\mathbf{S}_s(\mathbf{D})$ iff $\mathbf{H} = \langle v\rangle_D$ for some vertex of $\mathbf{D}$.

Fourthly, it is obvious and well–known that for every functional digraph $\mathbf{G}$, each of its regular edges is an isthmus. Recall (see e.g. [**Ber**]) that e is an isthmus iff e is regular (i.e. is not a loop) and e is the only directed path from its initial vertex to its final vertex.

Observe that in our case, each edge of $\mathbf{D}$ is an isthmus, because $\mathbf{D}$ has no loops.

Now we use the above facts to prove several connections between $\mathbf{D}$ and its strong subdigraph lattice $\mathbf{S}_s(\mathbf{D})$.

Take two vertices v, u of $\mathbf{D}$ and assume $\langle v\rangle_D = \langle u\rangle_D$. Then there are paths from v to u and from u to v. But $\mathbf{D}$ has no directed cycles, so $v = u$

Let v and u be vertices of $\mathbf{D}$ such that there is an edge from v to u. Then $\langle u\rangle_D \leq_s \langle v\rangle_D$, because this edge forms a path from v to u. Moreover, we show that $\langle u\rangle_D$ is covered by $\langle v\rangle_D$. To see this take a strong subdigraph $\mathbf{H}$ of $\mathbf{D}$ such that $\mathbf{H}$ is a completely join–irreducible element in $\mathbf{S}_s(\mathbf{D})$ and $\langle u\rangle_D \leq_s \mathbf{H} \leq_s \langle v\rangle_D$. Then $\mathbf{H} = \langle w\rangle_D$ for some vertex w. Assume that w is different from u, v. Then there is a path from v to w and a path from w to u. Since $\mathbf{D}$ has no directed cycles, these two paths form a path (with at least three vertices) from v to u. But this is impossible, since every edge of $\mathbf{D}$ is an isthmus. Thus $w = v$ or $w = u$, so $\mathbf{H} = \langle v\rangle_D$ or $\mathbf{H} = \langle u\rangle_D$.

Now take two completely join–irreducible elements $\langle u\rangle_D$, $\langle v\rangle_D$ of $\mathbf{S}_s(\mathbf{D})$ such that $\langle u\rangle_D$ is covered by $\langle v\rangle_D$. In particular, u belongs to $\langle v\rangle_D$, so there is a path from v to u. Assume that this path has a vertex w different from v and u. Then there are paths from v to w and from w to u, so $\langle u\rangle_D \leq_s \langle w\rangle_D \leq_s \langle v\rangle_D$ and these three strong subdigraphs of $\mathbf{D}$ are pairwise different. This contradiction proves that this path is an edge from v to u.

Summarizing, we have shown that the function φ from the vertex set of $\mathbf{D}$ into the set of all completely join–irreducible elements of $\mathbf{S}_s(\mathbf{D})$ assigning to each vertex v the strong subdigraph $\langle v\rangle_D$ is a bijection and for any two vertices v, u, there is an edge in $\mathbf{D}$ from v to u iff there is an edge in $\mathbf{Ds}(\mathbf{S}_s(\mathbf{D}))$ from $\langle v\rangle_D$ to $\langle u\rangle_D$ (i.e. $\langle v\rangle_D$ covers $\langle u\rangle_D$). This implies that $\mathbf{D}$ and $\mathbf{Ds}(\mathbf{S}_s(\mathbf{D}))$ are isomorphic, since these digraphs are functional (in particular, for any two vertices there is at most one edge from the first to the other). ∎

Now take two complete lattices **L** and **K** and assume that they are isomorphic and that φ is this lattice isomorphism. Then φ restricted to the set of all completely join–irreducible elements of **L** is a bijection of this set to the set of all completely join–irreducible elements of **K**. Moreover, φ preserves the covering relation. These two facts easily imply that φ induces also an isomorphism of the digraphs $\mathbf{Ds}(\mathbf{L})$ and $\mathbf{Ds}(\mathbf{K})$ (and also of $\mathbf{D}(\mathbf{L})$ and $\mathbf{D}(\mathbf{K})$.

THEOREM 16. *Let* **D** *be a functional digraph. Then* $\mathbf{Ds}(\mathbf{S}_s(\mathbf{D})) \simeq \mathbf{Ts}^d(\mathbf{D})$.

Proof. By Th.12 $\mathbf{S}_s(\mathbf{D})$ and $\mathbf{S}_s(\mathbf{Ts}^d(\mathbf{D}))$ are isomorphic, so $\mathbf{Ds}(\mathbf{S}_s(\mathbf{D}))$ and $\mathbf{Ds}(\mathbf{S}_s(\mathbf{Ts}^d(\mathbf{D})))$ are isomorphic. Hence, using L.15, we obtain our thesis. ∎

Th.1 and Th.16 imply the following algebraic result:

THEOREM 17. *Let* **A** *be a partial monounary algebra. Then*

$$\mathbf{Ds}(\mathbf{S}_s(\mathbf{A})) \simeq \mathbf{Ts}^d(\mathbf{A}).$$

Now we can formulate and prove the two main results of this paper.

THEOREM 18. *Let* **A** *and* **B** *be partial monounary algebras. Then*

$$\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{S}_s(\mathbf{B}) \quad \textit{iff} \quad \mathbf{Ts}^d(\mathbf{A}) \simeq \mathbf{Ts}^d(\mathbf{B}).$$

Proof. Of course two isomorphic digraphs have isomorphic strong subdigraph lattices. Hence, using Th.1 and Th.16, we obtain the implication $\Leftarrow$. On the other hand, if the strong subalgebra lattices of **A** and **B** are isomorphic, then the digraphs corresponding to these lattices $\mathbf{Ds}(\mathbf{S}_s(\mathbf{A}))$ and $\mathbf{Ds}(\mathbf{S}_s(\mathbf{B}))$ are isomorphic. Thus by Th.17 we deduce the implication $\Rightarrow$. ∎

THEOREM 19. *Let* **A** *be a partial monounary algebra and let an algebraic and distributive lattice* **L** *satisfy* (1) *and* (2) *of Th.*14. *Then*

$$\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{L} \quad \textit{iff} \quad \mathbf{Ts}^d(\mathbf{A}) \simeq \mathbf{Ds}(\mathbf{L}).$$

Proof. Since the strong subdigraph lattice of $\mathbf{Ds}(\mathbf{L})$ is isomorphic to **L**, and moreover, two isomorphic digraphs have isomorphic strong subdigraph lattices, the implication $\Leftarrow$ is implied by Th.13. On the other hand, if $\mathbf{S}_s(\mathbf{A})$ and **L** are isomorphic, then the digraphs $\mathbf{Ds}(\mathbf{S}_s(\mathbf{A}))$ and $\mathbf{Ds}(\mathbf{L})$ are isomorphic. Thus by Th.17 we infer the implication $\Rightarrow$. ∎

Remark. Obviously we can also formulate and prove (in the same way) analogous results for digraphs and their strong subdigraph lattices.

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