DEMONSTRATIO MATHEMATICA
Vol. XXXIII No 3 2000

Zbigniew Binderman

ON RIGHT INVERSES FOR FUNCTIONAL SHIFTS

In memory of Professor Siegfried Prössdorf

Abstract. The paper is concerned with functional shifts and functional R-shifts for a right invertible operator D (cf. [30], [15]). Conditions for different kinds of invertibility of these functional shifts on a set of all D-monomials are established.

0. Let X be a linear space over the field $\mathbb{C}$ of the complex numbers. Denote by $L(X)$ the set of all linear operators with domains and ranges in X and by $L_0(X)$ the set of those operators from $L(X)$ which are defined on the whole space X. By $R(X)$ we denote the set of all right invertible operators belonging to $L(X)$, by $\mathfrak{R}_D$ the set of all right inverses of a $D \in R(X)$, and by $\mathfrak{F}_D$ the set of all initial operators for D, i.e.

$$\mathfrak{R}_D := \{R \in L_0(X) : \; DR = I\},$$
$$\mathfrak{F}_D := \{F \in L_0(X) : \; F^2 = F, \; FX = \ker D \text{ and } \; \exists R \in \mathfrak{R}_D \;\; FR = 0\}.$$

In the sequel we shall assume that $\dim\ker D > 0$, i.e. D is right invertible but not invertible. If we know at least one right inverse R, we can determine the set $\mathfrak{R}_D$ of all right inverses and the set $\mathfrak{F}_D$ of all initial operators for a given $D \in R(X)$. The theory of right invertible operators and its applications S is presented in detail by D. Przeworska-Rolewicz in the books [24], [30].

Here and in the sequel we admit that $0^0 := 1$. We also write: $\mathbb{N}$ for the set of all positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

1991 *Mathematics Subject Classification*: 47B99, 47EO5, 47G99, 34A20.

Key words and phrases: right invertible operators, functional shifts, functional R-shifts, Pommiez operator, backward shift operator.

This paper has been presented at the International Conference "Algebraic Analysis and Related Topics", held in Banach Center, Warsaw, September 21–25, 1999.

For a given operator $D \in R(X)$, $R \in \mathfrak{R}_D$ we shall denote (cf. [24], [25], [32]):

• the space of smooth elements by

$$D_\infty := \bigcap_{k \in \mathbb{N}_0} D_k, \tag{0.1}$$

where $D_0 := X$, $D_k := \operatorname{dom} D^k (k \in \mathbb{N})$,

• the space of D-polynomials by

$$S := \bigcup_{i=1}^{\infty} \ker D^i = \operatorname{lin}\{R^k z : \; z \in \ker D, \quad k \in \mathbb{N}_0\}, \tag{0.2}$$

• the space of exponentials by

$$E := \bigcup_{\lambda \in \mathbb{C}} \ker(D - \lambda I), \tag{0.3}$$

• the space of D-analytic elements in a complete linear metric space X by

$$A_R(D) := \left\{x \in D_\infty : x = \sum_{n=0}^{\infty} R^n F D^n x\right\} = \left\{x : \lim_{n \to \infty} R^n D^n x = 0\right\}, \tag{0.4}$$

where F is an initial operator for D corresponding to an $R \in \mathfrak{R}_D$,

$$A(D) := \bigcup_{R \in \mathfrak{R}_D} A_R(D), \tag{0.5}$$

• in a complete linear metric space with a p-homogeneous F-norm $\| \; \|$ $(0 < p \le 1)$ by

$$X_1(D) := \left\{x \in D_\infty : \limsup_{n \to \infty} \sqrt[n]{\|D^n x\|} \le 1\right\}, \tag{0.6}$$

and

$$X_2(D) := \{x \in D_\infty : \{D^n x\} \text{ is bounded}\}. \tag{0.7}$$

In the sequel K will stand for a ring $K_r := \{h \in \mathbb{C} : 0 < |h| < r\}$, $0 < r \le +\infty$ $(K_\infty = \mathbb{C} \setminus \{0\})$. Denote by $H(\Omega)$ the class of all functions analytic on a set $\Omega \subseteq \mathbb{C}$. Suppose that a function $f \in H(K)$ has the following expansion

$$f(h) = \sum_{k=-n}^{\infty} a_k h^k \quad \text{for all} \quad h \in K, \quad n \in \mathbb{N}_0. \tag{0.8}$$

For an operator $D \in R(X)$ we define the set

$$S_f(D) := \left\{x \in D_\infty : \sum_{k=0}^{\infty} a_k h^k D^k x \;\text{ is convergent for all } h \in K\right\}. \tag{0.9}$$

DEFINITION 0.1 (of. [12], [15]). Suppose that $D \in R(X)$, $\ker D \neq \{0\}$ and $R \in \mathfrak{R}_D$ is arbitrarily fixed. A family $T_{f,K} = \{T_{f,h}\}_{h\in K} \subset L_0(X)$ is said to be a family of functional R-shifts for the operator D induced by the function f and R if

$$(0.10)\quad T_{f,h}x = \sum_{k=0}^{\infty} a_k h^k D^k x + \sum_{k=1}^{n} a_{-k} h^{-k} R^k x \quad \text{for all } h \in K;\ x \in S_f(D),$$

where f, $S_f(D)$ are determined by formulae (0.8), (0.9), respectively.

Functional R-shifts are induced by functions having a removable singularities at $h = 0$ and they have been called functional shifts (cf. [8], [30]). Obviously, it this case $f(h) = \sum_{k=0}^{\infty} a_k h^k$ for all $h \in K$ and formula (0.10) has the form

$$(0.11)\qquad T_{f,h}x = \sum_{k=0}^{\infty} a_k h^k D^k x \quad \text{for all } h \in K;\ x \in S_f(D).$$

The definition of functional shifts is independent on $R \in \mathfrak{R}_D$. The theory of functional and sequential shifts induced by a right invertible is presented in detail in the author's works [1]–[11], [14], [30]. The theory of R-functional shifts induced by an operator $D \in R(X)$, $R \in \mathfrak{R}_D$ and a function $f \in H(K)$ having an isolated singularity at the point $h = 0$ can be found in [12], [13], [15], [16]. For the entire collection of shifts for right invertible operators see D. Przeworska-Rolewicz [23]-[31] and [17], [18].

THEOREM 0.1 (cf. [30]). *Suppose that $D \in R(X)$, $\dim\ker D > 0$ and $T_{f,K} = \{T_{f,K}\}_{h\in K}$ is a family R-functional shifts for D induced by $f \in H(K)$. Then*

(i) $S \subset S_f(D)$,

$$\ker(D - \lambda I) \subset S_f(D) \textit{ for all } \lambda : \lambda K \subset K,$$
$$E \subset S_f(D) \textit{ for } K = K_\infty,$$

where S, E are defined by formulae (0.2), (0.3), *respectively;*

(ii) *if X is a complete linear metric space, $T_{f,K}$ is a family of continuous functional shifts, then*

$$A(D) \subset S_f(D) \quad \textit{if } D \textit{ is closed},$$
$$A_{R_1}(D) \subset S_f(D) \quad \textit{if } R_1 \in R_D \textit{ is continuous},$$

where $A(D)$, $\mathbf{A}_{R_1}(D)$ are defined by formulae (0.4), (0.5), *respectively;*

(iii) *if X is a complete linear metric space with p-homogeneous F-norm* $(0 < p \leq 1)$ *then*

$$X_1(D) \subset S_f(D) \textit{ for } K = K_1,$$
$$X_2(D) \subset S_f(D) \textit{ for } K = K_\infty,$$

where $X_1(D)$, $X_2(D)$ *are defined by formulae* 0.6, 0.7, *respectively,* $S_f(D)$ *is determined by formula* 0.9.

THEOREM 0.2 (cf. [30]). *Let* $D \in R(X)$, $\ker D \neq \{0\}$ *and let either* $K' = K_1 \cup \{0\}$ *or* $K' = \mathbb{C}$. *Let* $T(K')$ *be the set of all families of functional shifts for* D *induced by functions analytic on the set* K', *i.e.* $T(K') = \{T_{g,K'} : g \in H(K')\}$, *where* $T_{g,K'} = \{T_{g,h}\}_{h\in K'}$, $T_{g,0} := g(0)I$. *Write*

$$T_Y(K') = T(K')|_Y = \{T_{g,K'}|_Y : g \in H(K')\} \text{ for } Y \subset \bigcap_{g\in H(K')} S_g(D).$$

If Y *is the set* S *of all* D*-monomials then*

(i) *the set* $T_S(K')$ *is a commutative algebra with the operations*

$$T_{f+g,K'} = T_{f,K'} + T_{g,K'}, \quad T_{\alpha f,K'} = \alpha T_{f,K'}, \quad T_{f,K'}T_{g,K'} = T_{fg,K'},$$

where $f, g \in H(K'), \alpha \in \mathbb{C}$;

(ii) *the algebras* $H(K')$ *and* $T_S(K')$ *are isomorphic and* $T : f \to T_{f,K'}|_S$ *is an algebraic isomorphism from* $H(K')$ *onto* $T_S(K')$.

REMARK 0.1 (cf. [30]). Suppose that all assumptions of Theorem 0.2 are satisfied. Results similar to Theorem 0.2 for the set $T_S(K')$ hold for sets $T_Y(K')$, where

$$Y = \begin{cases} \ker(D - \lambda I) \neq 0 \quad \text{for } \lambda \in K', \\ E \quad \text{for } K' = \mathbb{C}, \\ S(D) = \bigcap_{g\in H(K')} S_g^\infty(D), \\ A_R(D) \ \text{if } \ R \in \mathfrak{R}_D \text{ is continuous,} \end{cases}$$

$$S_f^\infty(D) = \Big\{x \in D^\infty : \sum_{k=0}^{\infty} a_k h^k D^{k+n} x \text{ is convergent for all } h \in K',\ n \in \mathbb{N}_0\Big\}.$$

1. In this section, we shall assume that X is a complete linear metric space, $K = K_1$ or $K = K_\infty$. Let $D \in R(X)$, $\ker D \neq \{0\}$ and $T_{f,K} = \{T_{f,h}\}_{h\in K}$ be a family of functional R-shifts for D induced by a function $f \in H(K)$ and $R \in \mathfrak{R}_D$.

We consider the following equation

$$T_{f,h}x = y, \quad h \in K; \quad y \in X. \tag{1.1}$$

From Theorem 0.2 we can conclude

THEOREM 1.1. *Suppose that* $D \in R(X)$, $\ker D \neq \{0\}$ *and* $T_{f,K} = \{T_{f,h}\}_{h\in K}$, $T_{1/f,K} = \{T_{1/f,h}\}_{h\in K}$ *are families of functional shifts for* D *induced by functions* $f, 1/f \in H(K \cup \{0\})$ *respectively. If* $y \in S$ *then*

(i) *the unique solution of equation* (1.1) *which belongs to the set* S *has the form*

$$x = T_{1/f,h}y, \quad h \in K; \tag{1.2}$$

(ii) *every solution of equation (1.1) is of the form*

$$x = T_{1/f,h}y + w, \tag{1.3}$$

where $w \in \ker T_{f,h}$ *is arbitrary.*

EXAMPLE 1.1. Let $X = H(U)$, where U denotes the unit disk. The set $H(U)$ is a Fréchet space with the topology of uniform convergence on compact sets. We define the operators D, R as follows

$$(Dx)(t) = \frac{x(t) - x(0)}{t}; \quad (Rx)(t) = tx(t); \quad x \in X, \quad t \in U,$$

where

$$\left.\frac{x(t) - x(0)}{t}\right|_{t=0} := x'(0).$$

The operators D, R are uniquely determined on the whole space X, i.e. $D, R \in L_0(X)$, $\dim\ker D = 1$, $\operatorname{codim} RX = 1$ (cf. [21]). The operator D is called a Pommiez operator or a backward shift operator (cf. [22], [19], [20]). We can prove that $D \in R(X)$, $R \in \mathfrak{R}_D$ and

$$S = \operatorname{lin}\{R^k t : k = 0, 1, 2, \cdots\} = \operatorname{lin}\{1, t, t^2, \cdots\}.$$

Evidently, $\bar{S} = X$.

We take $f(h) = \frac{1}{1-h} \in X$ then $1/f(h) = 1 - h \in X$. Let $T_{f,U} = \{T_{f,h}\}_{h\in U}$, $T_{1/f,U} = \{T_{1/f,h}\}_{h\in U}$ be families of functional shifts for the operator D induced by $f, 1/f$, respectively. Then for $x \in X$, $h \in U$ (cf. [8]).

$$(T_{f,h}x)(t) = \sum_{n=0}^{\infty} h^n (D^n x)(t) = \begin{cases} \dfrac{tx(t) - hx(h)}{t - h} & \text{for } t \neq h, \\ x(h) + hx'(h) & \text{for } t = h, \end{cases}$$

$$(T_{1/f,h}x)(t) = [(I - hD)x](t) = \begin{cases} x(t) - h\dfrac{x(t) - x(0)}{t} & \text{for } t \neq 0, \\ x(0) - hx'(0) & \text{for } t = 0. \end{cases}$$

Theorem 1.1 implies that the equations

$$T_{f,h}x = y, \; y \in X; \; h \in U, \tag{1.4}$$

$$T_{1/f,h}x = u, \; u \in X; \; h \in U, \tag{1.5}$$

have the unique solutions for $y, u \in S$ which are determined by the formulae

$$x(t) = \begin{cases} y(t) - h\dfrac{y(t)-y(0)}{t} & \text{for } t \neq 0, \\ y(0) - hy'(0) & \text{for } t = 0, \end{cases} \tag{1.6}$$

$$x(t) = \begin{cases} \dfrac{tu(t) - hu(h)}{t-h} & \text{for } t \neq h, \\ u(h) + hu'(h) & \text{for } t = h, \end{cases} \tag{1.7}$$

respectively.

Observe that for $h \in U$ we have $\ker T_{f,h} = \ker T_{1/f,h} = \{0\}$. This implies that equations (1.4), (1.5) with $y, u \in X$ have the unique solutions which are determined by formulae (1.6), (1.7), respectively.

EXAMPLE 1.2 (cf. [9], [30]). Let X, U and f be defined as in Example 1.1. Let $D = \frac{d}{dt}$. Then $R = \int_0^t \in \mathfrak{R}_D$, $S = \operatorname{lin}\{R^k 1 : k \in \mathbb{N}_0\} = \operatorname{lin}\left\{1, t, \frac{t^2}{2!}, \frac{t^3}{3!}, \dots\right\}$ and $\bar{S} = X$. Let $T_{f,U} = \{T_{f,h}\}_{h \in U}$, $T_{1/f,U} = \{T_{1/f,h}\}_{h\in U}$ be families of functional shifts for the operator D induced by the functions $f, 1/f$ respectively. Then for all $x \in S$, $h \in U$ the following formulae holds:

$$(T_{f,h}x)(t) = \sum_{n=0}^{\infty} h^n (D^n x)(t) = \sum_{n=0}^{\infty} h^n x^{(n)}(t),$$

and

$$(T_{1/f,h}x)(t) = \left[(I - hD)x\right](t) = x(t) - hx'(t),$$

respectively.

By Theorem 1.1 we obtain that the equations of the form (1.4) (1.5) have the unique solutions for $y, u \in S$, which are determined by the formulae

$$x(t) = y(t) - hy'(t), \quad h \in U,$$

and

$$x(t) = \sum_{n=0}^{\infty} h^n u^{(n)}(t), \quad h \in U,$$

respectively.

Observe that

$$\ker T_{1/f,h} = \begin{cases} \{0\} & \text{for } h = 0, \\ \{Ce^{t/h}\} & \text{for } 0 \neq h \in U, \end{cases}$$

where C denote arbitrary scalars. Clearly, for $0 \neq h \in U$

$$\ker T_{1/f,h} \subset S_{1/f} \setminus S \quad \text{and} \quad \ker T_{1/f,h} \cap S_f(D) = \{0\}.$$

Theorem 1.1 implies that every solution of equation (1.5) with $u \in S$, $0 \neq h \in U$ is of the form

$$x(t) = Ce^{t/h} + \sum_{n=0}^{\infty} h^n u^{(n)}(t),$$

where C is an arbitrary scalar.

Obviously, we have also that the formula

$$x(t) = Ce^{t/h} - \frac{1}{h} e^{t/h} \int_0^t e^{-s/h} u(s)\, ds,$$

where, as above, C is an arbitrary scalar, determines all solution of equation (1.5) with $u \in X$ and $0 \neq h \in U$.

Moreover, this shows that the family $T_{f,U} = \{T_{f,h}\}_{h \in U}$, where $T_{f,h}$ is defined as follows $T_{f,0} = I$,

$$(T_{f,h}x)(t) = \Big(\sum_{n=0}^{\infty} h^n x^{(n)}(0)\Big) e^{t/h} - \frac{1}{h} e^{t/h} \int_0^t e^{-s/h} x(s) ds \quad 0 \neq h \in U;\ x \in S$$

is a family of functional shifts for $D = d/dt$ induced by the function $f = (1-h)^{-1}$.

2. In this section we assume that X is a complete linear metric space. Let a function $f \in H(K)$, where K denotes either the ring K_1 or the ring $K_\infty = \mathbb{C} \setminus \{0\}$, has the expansion

$$f(h) = \sum_{k=-n}^{\infty} a_k h^k; \quad h \in K, \tag{2.1}$$

where $a_{-n} \neq 0$, i.e. f has a pole of order $n \in \mathbb{N}$ at $h = 0$.

PROPOSITION 2.1. *Suppose that* $D \in R(X)$, $\dim \ker D > 0$ *and* $T_{f,K} = \{T_{f,h}\}_{h \in K}$, $W_{f,K} = \{W_{f,h}\}_{h \in K}$ *are families of functional R-shifts for* D *and* f, *induced by* $R_1, R_2 \in \mathfrak{R}_D$, *respectively. Then for* $h \in K$

$$(T_{f,h} - W_{f,h})(S_f(D)) \subset \ker D^n,$$

where the set $S_f(D)$ *is defined by formula* (0.9). *In other words: A difference of two families of functional R-shifts for a given element* $x \in S_f(D)$ *is a subset of the set of constants for* D^n.

Proof. Let $x \in S_f(D)$; $h \in K$ be arbitrarily fixed. Then by the definition we have

$$T_{f,h}x - W_{f,h}x = \sum_{k=1}^{n} a_{-k} h^{-k} (R_1^k x - R_2^k x).$$

Hence,

$$\begin{aligned} D^n(T_{f,h} - W_{f,h})x &= \sum_{k=1}^{n} a_{-k}h^{-k}(D^nR_1^k - D^nR_2^k)x \\ &= \sum_{k=1}^{n} a_{-k}h^{-k}(D^{n-k} - D^{n-k})x = 0. \end{aligned}$$

Let $T_{f,K} = \{T_{f,h}\}_{h\in K}$, $T_{g,K} = \{T_{g,h}\}_{h\in K}$ be families of functional R-shifts for D, R induced by functions f, g, respectively, where $g \in H(K)$ has a pole of order $m \in \mathbb{N}$ at $h = 0$. We can show (cf. [15]) that in general, on the set S, we have

$$T_{f,h}T_{g,h} \neq T_{g,h}T_{f,h}, \quad T_{f,h}T_{g,h} \neq T_{fg,h} \quad \text{for} \quad h \in K.$$

In connection with the above, we suppose that the function $1/f \in H(K)$, where $f \in H(K)$ is determined by formula (2.1). In this case $1/f$ has the following expansion

$$1/f(h) = \sum_{k=n}^{\infty} b_k h^k; \quad h \in K, \tag{2.2}$$

where the coefficients b_k $(n \leq k \in \mathbb{N})$ satisfies the equations

$$\begin{aligned} &a_{-n}b_n = 1, \\ &\sum_{k=n}^{l+n} a_{l-k}b_k = 0 \quad \text{for} \quad l \in \mathbb{N}. \end{aligned} \tag{2.3}$$

In order to show the right invertibility of the operators $T_{1/f,h}$ $(h \in K)$ on the set S, we will prove the next two propositions.

PROPOSITION 2.2. *Suppose that $D \in R(X)$ and an $R \in \mathfrak{R}_D$ is arbitrarily fixed. Let $T_{f,K} = \{T_{f,h}\}_{h\in K}$ be a family of functional R-shifts for the operator D induced by the function $f \in H(K)$ and R. Let $T_{1/f,K} = \{T_{1/f,h}\}_{h\in K}$ be a family of functional shifts for D induced by the function $1/f \in H(K)$. Then on the set S*

$$T_{1/f,h}T_{f,h} = I \quad \text{for all} \quad h \in K. \tag{2.4}$$

Proof. Let $h \in K$, $x \in S$ be arbitrarily fixed. This implies that there exists a number $n \leq q \in \mathbb{N}$ such that $D^{q+1}x = 0$. We have

$$
\begin{aligned}
(T_{1/f,h}T_{f,h})x &= T_{1/f,h}(T_{f,h}x) \\
&= \sum_{k=n}^{\infty} b_k h^k D^k \Big(\sum_{j=1}^{n} a_{-j} h^{-j} R^j x + \sum_{j=0}^{\infty} a_j h^j D^j x \Big) \\
&= \sum_{k=n}^{\infty} \sum_{j=1}^{n} a_{-j} b_k h^{k-j} D^{k-j} x + \sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_j b_k h^{k+j} D^{k+j} x.
\end{aligned}
$$

Our assumptions and equations (2.3) together imply

$$
\begin{aligned}
&\sum_{k=n}^{\infty} \sum_{j=1}^{n} a_{-j} b_k h^{k-j} D^{k-j} x \\
&\quad = \sum_{j=1}^{n} \sum_{k=n}^{q+n} a_{-j} b_k h^{k-j} D^{k-j} x = \sum_{j=1}^{n} \sum_{l=n-j}^{q+n-j} a_{-j} b_{l+j} h^l D^l x \\
&\quad = \sum_{j=1}^{n} \sum_{l=n-j}^{q} a_{-j} b_{l+j} h^l D^l x = \sum_{r=0}^{n-1} \sum_{l=r}^{q} a_{r-n} b_{l+n-r} h^l D^l x \\
&\quad = \sum_{r=0}^{n-1} \Big(\sum_{l=r}^{n-1} a_{r-n} b_{l+n-r} h^l D^l x + \sum_{l=n}^{q} a_{r-n} b_{l+n-r} h^l D^l x \Big) \\
&\quad = \sum_{l=0}^{n-1} \Big(\sum_{r=0}^{l} a_{r-n} b_{l+n-r} \Big) h^l D^l x + \sum_{l=n}^{q} \Big(\sum_{r=0}^{n-1} a_{r-n} b_{l+n-r} \Big) h^l D^l x \\
&\quad = a_{-n} b_n x + \sum_{l=n}^{q} \Big(\sum_{r=0}^{n-1} a_{r-n} b_{l+n-r} \Big) h^l D^l x \\
&\quad = x + \sum_{l=n}^{q} \Big(\sum_{r=0}^{n-1} a_{r-n} b_{l+n-r} \Big) h^l D^l x,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_j b_k h^{k+j} D^{k+j} x \\
&\qquad = \sum_{k=n}^{\infty} \sum_{j=0}^{q} a_j b_k h^{k+j} D^{k+j} x = \sum_{k=n}^{q} \sum_{j=0}^{q} a_j b_k h^{k+j} D^{k+j} x \\
&\qquad = \sum_{k=n}^{q} \sum_{l=k}^{k+q} a_{l-k} b_k h^l D^l x = \sum_{k=n}^{q} \sum_{l=k}^{q} a_{l-k} b_k h^l D^l x \\
&\qquad = \sum_{l=n}^{q} \Big(\sum_{k=n}^{l} a_{l-k} b_k \Big) h^l D^l x.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(T_{1/f,h}T_{f,h})x &= x + \sum_{l=n}^{q}\Big(\sum_{r=0}^{n-1} a_{r-n}b_{l+n-r}\Big)h^l D^l x + \sum_{l=n}^{q}\Big(\sum_{k=n}^{l} a_{l-k}b_k\Big)h^l D^l x \\
&= x + \sum_{l=n}^{q}\Big(\sum_{r=0}^{n-1} a_{r-n}b_{l+n-r} + \sum_{k=n}^{l} a_{l-k}b_k\Big)h^l D^l x \\
&= x + \sum_{l=n}^{q}\Big(\sum_{k=l+1}^{l+n} a_{l-k}b_k + \sum_{k=n}^{l} a_{l-k}b_k\Big)h^l D^l x \\
&= x + \sum_{l=n}^{q}\Big(\sum_{k=n}^{l+n} a_{l-k}b_k\Big)h^l D^l x = x.
\end{aligned}
$$

PROPOSITION 2.3. *Suppose that all assumptions of Proposition* 2.2 *are satisfied. Then on the set* S

$$
T_{f,h}T_{1/f,h} = I - \sum_{k=n}^{\infty}\sum_{j=1}^{n}\sum_{p=0}^{j-1} a_{-j}b_k h^{k-j} R^p F D^{p+k-j}, \quad h \in K, \tag{2.5}
$$

where $F \in \mathfrak{F}_D$ *is an initial operator for* D *corresponding to the operator* $R \in \mathfrak{R}_D$.

Proof. Let $h \in K$; $x \in S$ be arbitrarily fixed and let $D^{q+1}x = 0$ for a $n < q \in \mathbb{N}$. Write

$$
\begin{aligned}
T_{f,h}T_{1/f,h}x &= T_{f,h}(T_{1/f,h}x) \\
&= \Big(\sum_{j=1}^{n} a_{-j}h^{-j}R^j + \sum_{j=0}^{\infty} a_j h^j D^j\Big)\Big(\sum_{k=n}^{\infty} b_k h^k D^k x\Big) \\
&= \sum_{j=1}^{n} a_{-j}h^{-j}R^j\Big(\sum_{k=n}^{q} b_k h^k D^k x\Big) + \sum_{j=0}^{\infty} a_j h^j D^j\Big(\sum_{k=n}^{q} b_k h^k D^k x\Big) \\
&= \sum_{j=1}^{n}\sum_{k=n}^{q} a_{-j}b_k h^{k-j} R^j D^k x + \sum_{j=0}^{\infty}\sum_{k=n}^{q} a_j b_k h^{k+j} D^{k+j} x \\
&= \sum_{j=1}^{n}\sum_{k=n}^{q+n} a_{-j}b_k h^{k-j} R^j D^k x + \sum_{j=0}^{q}\sum_{k=n}^{q} a_j b_k h^{k+j} D^{k+j} x \\
&= \sum_{j=1}^{n}\sum_{k=n}^{q+n} a_{-j}b_k h^{k-j} R^j D^j D^{k-j} x + \sum_{k=n}^{q}\sum_{j=0}^{q} a_j b_k h^{k+j} D^{k+j} x.
\end{aligned}
$$

By the Taylor Expansion Formula for right invertible operators (cf. [24]) and equations (2.3) we obtain

$$T_{f,h}T_{1/f,h}x$$

$$= \sum_{j=1}^{n}\sum_{k=n}^{q+n} a_{-j}b_k h^{k-j}\Big(I - \sum_{p=0}^{j-1} R^p F D^p\Big)D^{k-j}x + \sum_{l=n}^{q}\Big(\sum_{k=n}^{l} a_{l-k}b_k\Big)h^l D^l x$$

$$= x + \sum_{l=n}^{q}\Big(\sum_{r=0}^{n-1} a_{r-n}b_{l+n-r}\Big)h^l D^l x - \sum_{k=n}^{q+n}\sum_{j=1}^{n}\sum_{p=0}^{j-1} a_{-j}b_k h^{k-j}R^p F D^{p+k-j}x$$

$$+ \sum_{l=n}^{q}\Big(\sum_{k=n}^{l} a_{l-k}b_k\Big)h^l D^l x$$

$$= x + \sum_{l=n}^{q}\Big(\sum_{k=n}^{l+n} a_{l-k}b_k\Big)h^l D^l x - \sum_{k=n}^{q+n}\sum_{j=1}^{n}\sum_{p=0}^{j-1} a_{-j}b_k h^{k-j}R^p F D^{p+k-j}x$$

$$= x - \sum_{k=n}^{q+n}\sum_{j=1}^{n}\sum_{p=0}^{j-1} a_{-j}b_k h^{k-j}R^p F D^{p+k-j}x,$$

where F is an initial operator for D corresponding to R.

Proposition 2.2 and Proposition 2.3 together imply

THEOREM 2.1. *Suppose that all assumptions of Proposition 2.2 are satisfied and $F \in \mathfrak{F}_D$ is an initial operator for D corresponding to the operator $R \in \mathfrak{R}_D$. Then*

(i) *the operator $T_{1/f,h}$ $(h \in K)$ is right invertible on the set S and the operator $T_{f,h}$ is a right inverse of $T_{1/f,h}$,*

(ii) *the operator*

$$F_{1/f,h} := \sum_{k=n}^{\infty}\sum_{j=1}^{n}\sum_{p=0}^{j-1} a_{-j}b_k h^{k-j}R^p F D^{p+k-j}, \quad h \in K \tag{2.6}$$

is an initial operator $T_{1/f,h}$ corresponding to the operator $T_{f,h}$.

REMARK 2.1. Suppose that all assumptions of Theorem 2.1 are satisfied. Then the operator $F_{1/f,h}$ $(h \in K)$ defined by formula (2.6) preserve constants of the operator D, i.e.

$$F_{1/f,h}z = z \quad \text{for all} \quad z \in \ker D, \quad h \in K.$$

Theorem 2.1. implies (cf. [24]).

THEOREM 2.2. *Suppose that all assumptions of Proposition 2.2 are satisfied. Then*

(i) *the equation*

$$T_{f,h}x = y, \quad y \in S; \quad h \in K$$

has the unique solution of the form

$$x = T_{1/f,h}y;$$

(ii) *every solution of the equation*

$$T_{1/f,h}x = y, \quad y \in S; \quad h \in K$$

is of the form

$$x = T_{f,h}y + v, \tag{2.7}$$

where $v \in \ker T_{1/f,h}$ *is arbitrary and* $T_{f,K} = \{T_{f,h}\}_{h\in K}$ *is induced by arbitrarily fixed* $R \in \mathfrak{R}_D$.

REMARK 2.2. The inclusion $\ker D^n \subset \ker T_{1/f,h}$, and Proposition 2.1 imply that a change of the right inverse $R \in \mathfrak{R}_D$ implies only a change of the constant v (which is arbitrarily fixed) in formula (2.7).

EXAMPLE 2.1. Suppose that X, K, D, R are defined as in Example 1.1. Take $f(h) = (1-h)/h$. Then $1/f(h) = h/(1-h)$ and $f \in H(K)$, $1/f \in H(K \cup \{0\})$. We have

$$f(h) = \frac{1}{h} - 1, \quad 1/f(h) = h + h^2 + \cdots \quad \text{for} \quad h \in K.$$

The equality (cf. [8])

$$\sum_{n=0}^{\infty} h^n (D^n x)(t) = \begin{cases} (t-h)^{-1}(tx(t) - hx(h)) & \text{for } t \neq h, \\ x(h) + hx'(h) & \text{for } t = h, \quad h \in K; \ x \in X \end{cases}$$

implies for $h \in K$; $x \in X$

$$\sum_{n=1}^{\infty} h^n (D^n x)(t) = \begin{cases} h\frac{x(t)-x(h)}{t-h} & \text{for } t \neq h, \\ hx'(h) & \text{for } t = h. \end{cases}$$

This proves that $S_{1/f}(D) = X$. Obviously, also $S_f(D) = X$.

Moreover, this shows that the families $T_{f,K} = \{T_{f,h}\}_{h\in K}$, $T_{1/f,K} = \{T_{1/f,h}\}_{h\in K}$, where $T_{f,h}, T_{1/f,h}$ are defined as follows

$$(T_{f,h}x)(t) = \frac{t-h}{h}x(t), \quad h \in K; \quad x \in X,$$

and

$$(T_{1/f,h}x)(t) = \begin{cases} h\frac{x(t)-x(h)}{t-h} & \text{for } t \neq h, \\ hx'(h) & \text{for } t = h, \end{cases} \qquad h \in K; \ x \in X$$

are the unique family of functional R-shifts for D induced by the function f and the operator R and the unique family of functional shifts for D induced by the function $1/f$, respectively.

For $h \in K,\ x \in X$ we have

$$(T_{f,h}T_{1/f,h}x)(t) = x(t) - x(h),$$

and

$$(T_{1/f,h}T_{f,h}x)(t) = x(t).$$

This implies that for $h \in K$ the operator $T_{1/f,h}$ is right invertible on X and $T_{f,h}$ is a right inverse of $T_{1/f,h}$. The operator

$$(F_{1/f,h}x)(t) = x(t) - (T_{f,h}T_{1/f}x)(t) = x(h), \quad h \in K; \quad x \in X$$

is the initial operator for $T_{1/f,h}$ corresponding to the operator $T_{f,h}$.

Evidently, the every solution of the equation

$$T_{1/f,h}x = y, \quad y \in X; \quad h \in K,$$

is of the form

$$x(t) = (T_{f,h}y)(t) + C = \frac{t-h}{h}y(t) + C,$$

where $C \in \mathbb{C}$ is arbitrary.

References

[1] Z. Binderman, *Complex R-shifts for right invertible operators*, Demonstratio Math. 23 (1990), 1043–1053.

[2] Z. Binderman, *On some properties of complex R-shifts*, Demonstratio Math. 25 (1992), 207–217.

[3] Z. Binderman, *Cauchy integral formula induced by right invertible operators*, Demonstratio Math. 25 (1992), 671–690.

[4] Z. Binderman, *Functional shifts induced by right invertible operators*, Math. Nachr. 157 (1992), 211–224.

[5] Z. Binderman, *Periodic solutions of equations of higher order with a right invertible operator induced by functional shifts*, Ann. Univ. M. Curie-Sklodowska 46 (1993), 9–22.

[6] Z. Binderman, *Some properties of operators of complex differentiation and shifts*, Sci. Bull. Lodz. Technical Univ. Math. 24 (1993), 5–18.

[7] Z. Binderman, *On periodic solutions of equations with right operators induced by functional shifts*, Demonstratio Math. 26 (1993), 535–543.

[8] Z. Binderman, *A unified approach to shifts induced by right invertible operators*, Math. Nachr. 161 (1993), 239–252.

[9] Z. Binderman, *On some functional shifts induced by operators of complex differentiation*, Opuscula Math. 14 (1993), 45–57.

[10] Z. Binderman, *Some remarks on sequential shifts induced by right invertible operators*, Funct. Approx. 22 (1993), 71–84.

[11] Z. Binderman, *Applications of sequential shifts to an interpolation problem*, Collect. Math. 44 (1993), 47–57.

[12] Z. Binderman, *Some fundamental properties of functional R-shifts for right invertible operators*, Comm. Math. 33 (1993), 9-22.

[13] Z. Binderman, *A note of functional R-sfifts for right invertible operators*, Discuss. Math. 14 (1994), 63–76.

[14] Z. Binderman, *On summation formulas induced by functional shifts of right invertible operators*, Demonstratio Math. 28 (1995), 301–314.
[15] Z. Binderman, *On isomorphisms of spaces of functional R-shifts for right invertible operators*, in: Different Aspects of Differentiability, Proc. Intern. Conf. Warszawa, September 1993. Ed. D. Przeworska-Rolewicz. Dissertations Math. 340, Warszawa, 1995, 23–36.
[16] Z. Binderman, *Remarks on functional R-shifts for right invertible operators*, in: Proc. Intern. Conf., Different Aspects of Differentiability II. Warszawa, September 1995. Integral Transforms and Special Functions, 1–2, vol 4, 1996, 39–48.
[17] Z. Binderman and D. Przeworska-Rolewicz, *The limit property and perturbations of functional shifts*, Demonstratio Math. 28 (1995), 835–851.
[18] Z. Binderman and D. Przeworska-Rolewicz, *Almost quasinilpotent right inverses*, In: Proc. Intern. Conf. Different Aspects of Differentiability II. Warszawa, September 1995. Integral Transforms and Special Functions, 1–2, vol 4, 1996, 23–38.
[19] I. H. Dimovski *Convolutional Calculus*, Kluwer Acad. Publishers, Dordrecht, 1990, (1st. ed. Publish. House of the Bulg. Acad. Sci., Sofia), 1982.
[20] R.G. Douglas and H.S. Shapiro and A.L.Shields, *Cyclic vectors and invariant subspaces for the bacward shift operator* , Ann. 1' Inst. Fourier 20, 1 (1970), 37-76.
[21] M.K. Fage and N.I. Nagnibida, *An Equivalence Problem of Ordinary Linear Differential Operators*, Nauka, Novosibirsk, 1987 (In Russian).
[22] N. Linchuk , *Representation of commutants of the Pommiez operator and their applications*, Mat. Zametki 44 (1988), 794–802 (in Russian).
[23] D. Przeworska-Rolewicz, *Shifts and Periodicity for Right Invertible Operators Research Notes in Mathematics*, vol 43, Pitman Avanced Publishing Program, Boston–London–Melbourne, 1980.
[24] D. Przeworska-Rolewicz, *Algebraic Analysis* , PWN-Polish Scientific Publishers and D. Reidel. Warszawa-Dordrecht, 1988.
[25] D. Przeworska-Rolewicz, *Spaces of D-paraanalytic Elements*, Dissertationes Math. 302, Warszawa, 1990.
[26] D. Przeworska-Rolewicz, *True shifts*, J. Math. Anal. Appl. 170 (1992), 27–48.
[27] D. Przeworska-Rolewicz, *The operator* $\exp(hD)$ *and its inverse formula*, Demonstratio Math. 26 (1993), 545–552.
[28] D. Przeworska-Rolewicz, *Functional stability of linear systems*, Demonstratio Math. 28 (1995), 449–464.
[29] D. Przeworska-Rolewicz, *Conditions for interpolation by polynomials in right invertible operators to be admissible*, Math. Balcanica 4, 9 (1995), 353–363.
[30] D. Przeworska-Rolewicz, *Logarithms and Antilogarithms*, An Algebraic Approach with Appendix by Z. Binderman, Kluwer Acad. Publishers, Dordrecht, 1998.
[31] D. Przeworska-Rolewicz, *Isomorphisms preserving Leibniz condition, Fractional Calculus and Applied Analysis*, vol 2, No 2, (1999), 149–161.
[32] S. Rolewicz, *Metric Linear Spaces*, 2nd extended ed. PWN-Polish Scientific Publishers and D. Reidel, Warszawa-Dordrecht, 1985.

WARSAW UNIVERSITY OF AGRICULTURE
Nowoursynowska 166
02-787 WARSZAWA, POLAND
E-mail: binder@bri.kei.sggw.waw.pl

Received March 22nd, 2000.