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ON VARIOUS INTEGRAL TRANSFORMATIONS OF TEMPERED ULTRADISTRIBUTIONS

Abstract. We introduce and study Hermite expansions and various integral transformations on the spaces $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$ of tempered ultradistributions of Beurling and Roumieu type. In particular, we investigate the Wigner distribution and the Fourier, Bargmann and Laplace transforms.

1. Introduction

Beurling and Roumieu ultradistribution spaces $\mathcal{D}'^{(M_p)}$and $\mathcal{D}'^{\{M_p\}}$, defined in [3] and [31] for an arbitrary sequence (M_p) of positive numbers satisfying certain growth conditions, have been studied by many authors and various approaches have been used in the studies (see [3], [31], [4], [23], [10], [26]). The importance of the spaces of ultradistributions in the theory of partial differential equations was acknowledged by Björk in [4]. The approach of Komatsu (see [23]) was chosen in [25] and [30] for introducing and investigating the spaces $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$ of Beurling and Roumieu tempered ultradistributions. From the results of [25], [24] as well as of the present paper it follows that the spaces of tempered ultradistributions and the integral transformations on them are natural generalizations of the space of Schwartz's tempered distributions and the corresponding integral transforms. In the special case, if the sequence (M_p) is of the form $M_p = p^{\alpha p}$ ($p \in \mathbf{N}_0$) with $\alpha > 1/2$, the space $\mathcal{S}'^{(M_p)}$ coincides with the space $\sum'_\alpha$, considered in [28], while the space $\mathcal{S}^{\{M_p\}}$ of test functions for the space $\mathcal{S}'^{\{M_p\}}$is the well known Gel'fand-Shilov space S^α_α (see [14], [11], [12], [5], [22], [8]). Important subspaces of tempered ultradistributions were investigated and various representation theorems for ultradistributions were obtained in [12], [27], [7], [9] and [6].

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We shall introduce in this paper various integral transformations on the spaces $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$of tempered ultradistributions of Beurling and Roumieu type. The Hermite expansions of generalized functions (see [1], [2], [33]) can be regarded as a generalized integral transform in the sense of [33], Chapt. IX. We give the Hermite expansions of elements of the basic spaces for the spaces of tempered ultradistributions as well as of their duals. This enables us to obtain, in a similar way as it was done in [29], results about the Wigner distribution, the Fourier, Bargmann and Laplace transforms and the boundary value representation of elements of $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$(cf. [5], [17], [18], [19]).

For the sake of simplicity we give all definitions, theorems and proofs in the one-dimensional case, though the results can be generalized to the multi-dimensional case.

Björk studied in [4] the spaces $\mathcal{S}'_\omega$ of so-called ω-tempered distributions and Grudzinski in [16] investigated the spaces of tempered Beurling distributions, being generalizations of the space of tempered distributions (see [3]). We use in this paper a different approach to Beurling's theory of generalized distributions and consider, in general, different problems than ones considered in [4] and [16] with the exception of the problem of characterizing the spaces by the Fourier transform.

Janssen and van Eijndhoven ([19]) studied the Gel'fand-Shilov inductive limit type spaces $W_M^{M^\times}$ (see [15]), where $M^\times$ is the Young conjugate of a suitable function M. They characterized them by the Fourier transform, the Wigner distribution, the Bargmann transform and by expansions into the Hermite series. In the special case where $M(x) = \alpha x^{1/\alpha}$, $x > 0$ for $1/2 \leq \alpha < 1$ and $M_p = p^{\alpha p}$, $p \in \mathbf{N}_0$, both the spaces $W_M^{M^\times}$ and $\mathcal{S}^{\{M_p\}}$ coincide with the Gel'fand-Shilov space $\mathcal{S}_\alpha^\alpha$. In the general case, however, the spaces $W_M^{M^\times}$ and $\mathcal{S}^*$ are different. In the case of the space $W_M^{M^\times}$, the function M tends to infinity faster than x and slower than x^2, and for $\mathcal{S}^*$ the role of M plays the function associated to the sequence (M_p), which is increasing and tends to infinity slower than x. For example, if $M_p = p!^\alpha$ with $\alpha > 1$ for $p \in \mathbf{N}_0$, then $M(x) \sim Cx^{1/\alpha}$ and Young's conjugate for such a function does not exist at all. Using a different method than Janssen and van Eijndhoven's, we prove that the theorems analogous to their results are also true for the spaces $\mathcal{S}^*$ ($*$ stands for (M_p) or $\{M_p\}$) which are investigated in the paper.

The paper is organized as follows. In section 2 we recall the basic notions of L^2 theory and the definitions of test function spaces. In section 3 we state some structural theorems (Theorems 3.1 - 3.4). Theorem 3.2 is the main assertion of the paper and a tool for proving other results of this and the next section. The proof of Theorem 3.2 is given in [21] and, since it is long and complicated, will be published in a separate paper. In section 4, we characterize spaces of test functions by the Fourier and Laplace transforms.

In [20], we shall study the Hilbert transform and, more generally, singular integral operators on the spaces of tempered ultradistributions of Beurling and Roumieu type.

2. Notation

The sets of non-negative integers, positive integers, real, complex and complex numbers with positive imaginary parts are denoted by $\mathbf{N}_0$, $\mathbf{N}$, $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{C}_+$, respectively.

The letter $\mathcal{C}$ (without super- or subscript) denotes a positive constant, not necessarily the same at every occurrence. We denote by $\mathcal{R}$ a family of positive sequences which increase to infinity. This set is partially ordered and directed by the relation $r_p \preceq s_p$, which means that there exists p_0 such that $r_p \leq s_p$ for every $p > p_0$.

Since we shall often use products of functions and power factors, it will be convenient to have the following notation for these factors:

$$\chi^\alpha(x) := x^\alpha; \quad <\chi>^\alpha(x) :=<x>^\alpha= (1+|x|^2)^{\alpha/2}$$

for $x \in \mathbf{R}$ and $\alpha \in \mathbf{N}_0$. Moreover, we shall often write $(\partial/\partial x)^\alpha \varphi(x)$ instead of $\varphi^{(\alpha)}(x)$.

The multi-dimensional notation corresponds to this one.

The sequence of Hermite functions h_α is given by

$$h_\alpha(x) = (-1)^\alpha(\sqrt[4]{\pi}\sqrt{2^\alpha \alpha!})^{-1}\, e^{x^2/2}\,(e^{-x^2})^{(\alpha)}, \quad \alpha \in \mathbf{N},\ x \in \mathbf{R}. \tag{1}$$

We will use the fact that the set of Hermite functions makes an othonormal base of the space $L^2(\mathbf{R})$.

The norm in $L^s = L^s(\mathbf{R})$, $s \in [1,\infty]$, is denoted by $\|\cdot\|_s$.

The Fourier transform, Wigner distribution and Bargmann transform are defined respectively by

$$(\mathcal{F}f)(\xi) = \int_{\mathbf{R}} e^{-ix\xi} f(x)dx,\ \xi \in \mathbf{R},\ f \in L^1,$$

$$\mathbf{W}(x,y;f) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \exp(-iyt) f(x+t/2)\overline{f(y-t/2)}dt, \quad x,y \in \mathbf{R},\ f \in L^2,$$

$$(\mathbf{A}f)(\zeta) = \pi^{-1/4} \int_{\mathbf{R}} \exp(-1/2(\zeta^2+x^2) + \sqrt{2}\zeta x) f(x)\,dx, \quad \zeta \in \mathbf{C},\ f \in L^2.$$

For the properties of the Wigner distribution and the Bargmann transform we refer to [5], [17] and [18].

By (M_p), we denote a given sequence of positive numbers and put $m_p := M_p/M_{p-1}$ for $p \in \mathbf{N}$.

The following conditions are frequently imposed on this sequence (cf. [3] and [23]):

$(M.1)$ $\quad M_p^2 \leq M_{p-1}M_{p+1}, \quad p \in \mathbf{N};$

$(M.2)$ $\quad M_p \leq AH^p M_q M_{p-q}, \quad p, q \in \mathbf{N}, 0 \leq q \leq p;$

$(M.2')$ $\quad M_p \leq AH^p M_{p-1}, \quad p \in \mathbf{N};$

$(M.3)$ $\quad \sum\limits_{q=p+1}^{\infty} M_{q-1}/M_q < pM_p/M_{p+1}, \quad p \in \mathbf{N};$

$(M.3')$ $\quad \sum_{q=1}^{\infty} M_{q-1}M_q^{-1} < \infty$

for some constants $A > 0$, $H > 0$.

Throughout the paper we assume conditions $(M.1)$, $(M.3')$ and, for convenience, $M_0 = 1$. The letter H will always denote the constant mentioned in $(M.2')$ or $(M.2)$. The so-called associated functions for the sequence (M_p) are

$$M(\rho) = \sup_{p \in \mathbf{N}_0} \log(\rho^p/M_p), \quad \tilde{M}(\rho) = \sup_{p \in \mathbf{N}_0} \log(\rho^p p!/M_p), \qquad \rho > 0.$$

For a given sequence (M_p) and a given $(a_p) \in \mathcal{R}$, we consider the corresponding sequence (N_p), defined by $N_p = M_p(\prod_{k=1}^{p} a_k)$ for $p \in \mathbf{N}$. Then the associated functions for the sequence (N_p) are denoted by $N_{(a_p)}$ and $\tilde{N}_{(a_p)}$, respectively.

We say that a formal series $P(\xi) = \sum_{\alpha \in \mathbf{N}_0} a_\alpha \xi^\alpha$, $\xi \in \mathbf{R}$, is an ultrapolynomial of the class (M_p) (resp. $\{M_p\}$) whenever the coefficients a_α satisfy the estimation $|a_\alpha| \leq CL^\alpha M_\alpha$, $\alpha \in \mathbf{N}_0$, for some $L > 0$ and C (resp. for every $L > 0$ and some C). The corresponding operator $P(D) = \sum_\alpha a_\alpha D^\alpha$ defines an ultradifferential operator of the class (M_p) (resp. $\{M_p\}$). Let us stress once again that the symbol $*$ is the common notation for the symbols (M_p) and $\{M_p\}$. We say that function f is is of ultrapolynomial growth of the class $*$ if there is an ultrapolynomial P of the class $*$ such that

$$|f(x)| \leq P(|x|), \qquad x \in \mathbf{R}.$$

Let us recall the definition of Beurling and Roumieu spaces of ultradifferentiable functions (see [23]). If K is a compact subset of $\mathbf{R}$, $h > 0$ and φ is C^∞-function we let

$$\|\varphi\|_{K,h} = \sup_{\substack{\alpha \in \mathbf{N}_0 \\ x \in K}} \frac{|\, \partial^\alpha \varphi(x)\, |}{h^\alpha M_\alpha}.$$

Denote by $\mathcal{D}_{K,h}^{(M_p)}$ the space of C^∞-functions φ on $\mathbf{R}$ with $\operatorname{supp} \varphi \subset K$ and $\|\varphi\|_{K,h} < \infty$. The basic spaces of functions of the class (M_p) and of the class

$\{M_p\}$ are defined by

$$\mathcal{D}^{(M_p)} = \text{ind}\lim_{K\subset\subset\mathbf{R}}\text{proj}\lim_{h\to 0}\mathcal{D}^{(M_p)}_{K,h},$$
$$\mathcal{D}^{\{(M_p)\}} = \text{ind}\lim_{K\subset\subset\mathbf{R}}\text{ind}\lim_{h\to\infty}\mathcal{D}^{(M_p)}_{K,h}.$$

The notation $K \subset\subset \mathbf{R}$ means that K is compact and grows up to $\mathbf{R}$.

For properties of the space $\mathcal{D}'^* = \mathcal{D}'^*(\mathbf{R})$, the definition and properties of $\mathcal{E}'^* = \mathcal{E}'^*(\mathbf{R})$ we refer to [23].

As in [27] and [29], we define, for $s \in [1,\infty]$,

$$\mathcal{D}^{(M_p)}_{L^s} = \text{proj}\lim_{h\to 0}\mathcal{D}^{(M_p)}_{L^s,h}, \qquad \mathcal{D}^{\{M_p\}}_{L^s} = \text{ind}\lim_{h\to\infty}\mathcal{D}^{(M_p)}_{L^s,h},$$

where $\mathcal{D}^{(M_p)}_{L^s,h}$ is the space of smooth functions φ such that

$$\|\varphi\|_{L^s,h} = \sup_{\alpha\in\mathbf{N}_0}\frac{\|\partial^\alpha\varphi\|_s}{h^\alpha M_\alpha} < \infty, \tag{2}$$

equipped with the norm $\|\cdot\|_{L^s,h}$. For the spaces $\mathcal{D}^*_{L^s}$, we denote the corresponding strong duals by $\mathcal{D}'^*_{L^t}$, where $t = s/(s-1)$; they are subspaces of Beurling and Roumieau spaces of ultradistributions. We denote by $\dot{\mathcal{B}}^*$ the completion of $\mathcal{D}^*$ in $\mathcal{D}^*_{L^\infty}$. The strong dual of $\dot{\mathcal{B}}^*$ is denoted by $\mathcal{D}'^*_{L^1}$.

Recall that a locally convex topological vector space is an (F-S)-space (resp. an (LS)-space) if it is a projective limit (resp. inductive limit) of a countable, compact specter of spaces. If the mentioned specter is also nuclear the space is an (FN)-space (resp. an (LN)-space) - for more details see [13].

Let us remark that the basic spaces and ultrapolynomials are defined in d-dimensional case via the multi-indexed sequence (M_α), where $M_\alpha = M_{\alpha_1}\cdot\ldots\cdot M_{\alpha_d}$ for $\alpha = (\alpha_1,\ldots,\alpha_d) \in \mathbf{N}^d$. For example, the seminorm $\|\cdot\|_{K,h}$ is defined on the space $\mathcal{D}^{(M_p)}_{K,h}(\mathbf{R^d})$ by

$$\|\varphi\|_{K,h} = \sup_{\substack{\alpha\in\mathbf{N}_0^d\\ x\in K}}\frac{|\,\partial^\alpha\varphi(x)\,|}{h^\alpha M_\alpha},$$

where $h^\alpha = h^{\alpha_1+\ldots+\alpha_d}$. Notice that under the condition $(M.2)$ the above defined multi-indexed sequence (M_α) and the sequence $(M_{\alpha_1+\ldots+\alpha_d})$ define the same spaces of ultradistributions and ultrapolynomials (see [26]).

The Wigner distribution and the Bargman transform are investigated in [5], [17] and [18] only in the one-dimensional case. However, their d-dimensional analogues may be simply examined.

3. Structural theorems

Let us start from the definitions of the spaces $\mathcal{S}^{(M_p)}$ and $\mathcal{S}^{\{M_p\}}$ and various families of norms in these spaces.

DEFINITION 3.1. Fix $m > 0$ and $r \in [1, \infty)$. The spaces $\mathcal{S}_r^{M_p,m} = \mathcal{S}_r^{M_p,m}(\mathbf{R})$, $\mathcal{S}_\infty^{(M_p),m} = \mathcal{S}_\infty^{(M_p),m}(\mathbf{R})$ are defined to be the sets of all smooth functions φ on $\mathbf{R}$ such that

$$\sigma_{m,r}(\varphi) := \Big(\sum_{\alpha,\beta\in\mathbf{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} (\| < \chi >^\beta \varphi^{(\alpha)} \|_r)^r \Big)^{1/r} < \infty;$$

$$\sigma_{m,\infty}(\varphi) := \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \| < \chi >^\beta \varphi^{(\alpha)} \|_\infty < \infty,$$

respectively, equipped with the topologies induced by the families $\Sigma_r = \{\sigma_{m,r} :\ m > 0\}$, $\Sigma_\infty = \{\sigma_{m,\infty} :\ m > 0\}$ of norms, respectively.

DEFINITION 3.2. The spaces $\mathcal{S}^{(M_p)} = \mathcal{S}^{(M_p)}(\mathbf{R})$ and $\mathcal{S}^{\{M_p\}} = \mathcal{S}^{\{M_p\}}(\mathbf{R})$ are defined to be the projective (as $m \to \infty$) and inductive (as $m \to 0$) limits of the spaces $\mathcal{S}_2^{(M_p),m}$, respectively.

The dual spaces of $\mathcal{S}^{(M_p)}$ and $\mathcal{S}^{\{M_p\}}$ are denoted by $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$, respectively.

It follows from the results given in this paper that if condition $(M.2')$ is satisfied, then $\mathcal{S}^{(M_p)} = \mathcal{S}^{(M_p)}(\mathbf{R})$ and $\mathcal{S}^{\{M_p\}} = \mathcal{S}^{\{M_p\}}(\mathbf{R})$ are the projective (as $m \to \infty$) and inductive (as $m \to 0$) limits, respectively, of the spaces $\mathcal{S}_r^{(M_p),m}$ for an arbitrary $r \in [1, \infty]$.

We will consider various norms in the spaces $\mathcal{S}^{(M_p)}$ and $\mathcal{S}^{\{M_p\}}$. For given $m > 0$, $(a_p) \in \mathcal{R}$ and associated functions M and $N_{(a_p)}$, denote $e_{M,m}(t) := \exp[M(m|t|)]$ and $e_{N,(a_p)}(t) := \exp[N_{(a_p)}(|t|)]$.

DEFINITION 3.3. For an arbitrary function $\varphi \in \mathcal{S}^{(M_p)}$ with $\varphi \overset{L^2}{=} \sum_{n\in\mathbf{N}_0} d_n h_n$, and arbitrary numbers $r \in [1, \infty)$ and $m \in (0, \infty)$, we define the following norms:

$$\sigma'_{m,r}(\varphi) := \sum_{\alpha,\beta\in\mathbf{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|\chi^\beta \varphi^{(\alpha)}\|_r; \quad \sigma'_{m,\infty}(\varphi) := \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|\chi^\beta \varphi^{(\alpha)}\|_\infty;$$

$$\sigma''_{m,r}(\varphi) := \sum_{\alpha,\beta\in\mathbf{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|(\chi^\beta \varphi)^{(\alpha)}\|_r; \quad \sigma''_{m,\infty}(\varphi) := \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|(\chi^\beta \varphi)^{(\alpha)}\|_\infty$$

and, moreover,

$$\upsilon_m(\varphi) := \sup_{\alpha\in\mathbf{N}_0} \frac{m^\alpha}{M_\alpha} \|\varphi^{(\alpha)} e_{M,m}\|_\infty; \quad \theta_m(\varphi) := \sum_{n\in\mathbf{N}_0} |d_n|^2 e_{2M,m}(\sqrt{2n+1})].$$

Before giving the next definition let us introduce, for given sequences $(a_p), (b_p) \in \mathcal{R}$, the following notation:

$$A_\alpha := \prod_{p=1}^{\alpha} a_p, \quad B_\beta := \prod_{p=1}^{\beta} b_p$$

if $\alpha, \beta \in \mathbf{N}$ and, moreover, $A_0 := 1 =: B_0$.

DEFINITION 3.4. For an arbitrary function $\varphi \in \mathcal{S}^{(M_p)}$ with $\varphi \overset{L^2}{=} \sum_{n\in\mathbf{N}_0} d_n h_n$, an arbitrary number $r \in [1, \infty)$ and arbitrary sequences $(a_p), (b_p) \in \mathcal{R}$, we define the following norms:

$$\sigma_{(a_p),(b_p),r}(\varphi) := \sum_{\alpha,\beta\in\mathbf{N}_0} \frac{\| < \chi >^\beta \varphi^{(\alpha)} \|_r}{M_\alpha A_\alpha M_\beta B_\beta};$$

$$\sigma_{(a_p),(b_p),\infty}(\varphi) := \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{\| < \chi >^\beta \varphi^{(\alpha)} \|_\infty}{M_\alpha A_\alpha M_\beta B_\beta};$$

$$\sigma'_{(a_p),(b_p),r}(\varphi) := \sum_{\alpha,\beta\in\mathbf{N}_0} \frac{\|\chi^\beta \varphi^{(\alpha)}\|_r}{M_\alpha A_\alpha M_\beta B_\beta};$$

$$\sigma'_{(a_p),(b_p),\infty}(\varphi) := \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{\|\chi^\beta \varphi^{(\alpha)}\|_\infty}{M_\alpha A_\alpha M_\beta B_\beta};$$

$$\sigma''_{(a_p),(b_p),r}(\varphi) := \sum_{\alpha,\beta\in\mathbf{N}} \frac{\|(\chi^\beta \varphi)^{(\alpha)}\|_r}{M_\alpha A_\alpha M_\beta B_\beta};$$

$$\sigma''_{(a_p),(b_p),\infty}(\varphi) := \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{\|(\chi^\beta \varphi)^{(\alpha)}\|_\infty}{M_\alpha A_\alpha M_\beta B_\beta}$$

and, moreover,

$$v_{(a_p),(b_p)}(\varphi) := \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{\|\varphi^{(\alpha)} e_{N,(b_p)}\|_\infty}{M_\alpha A_\alpha};$$

$$\theta_{(a_p)}(\varphi) := \sum_{n\in\mathbf{N}_0} |d_n|^2 e_{2N,(a_p)}(\sqrt{2n+1}).$$

DEFINITION 3.5. Now let us introduce the following notation for various families of norms:

$$\Sigma_r := \{\sigma_{m,r} :\ m > 0\}, \quad \tilde{\Sigma}_r := \{\sigma_{(a_p),(b_p),r} :\ (a_p), (b_p) \in \mathcal{R}\},$$

$$\Sigma'_r := \{\sigma'_{m,r} :\ m > 0\}, \quad \tilde{\Sigma}'_r := \{\sigma'_{(a_p),(b_p),r} :\ (a_p), (b_p) \in \mathcal{R}\},$$

$$\Sigma''_r := \{\sigma''_{m,r} :\ m > 0\}, \quad \tilde{\Sigma}''_r := \{\sigma''_{(a_p),(b_p),r} :\ (a_p), (b_p) \in \mathcal{R}\}$$

for $r \in [1, \infty]$ and, moreover,

$$\Upsilon := \{\upsilon_m : \ m > 0\}, \qquad \tilde{\Upsilon} := \{\upsilon_{(a_p),(b_p)} : \ (a_p), (b_p) \in \mathcal{R}\},$$
$$\Theta := \{\theta_m : \ m > 0\}, \qquad \tilde{\Theta} := \{\theta_{(a_p)} : \ (a_p) \in \mathcal{R}\}.$$

In the sequel we will need the following two theorems. The proof of the first one is given in [30]. The proof of the second one is given in [21] and will be published in a forthcoming paper.

THEOREM 3.1 (see [30]). *Let (a_p), $(b_p) \in \mathcal{R}$ and let $\mathcal{S}^{(M_p)}_{(a_p),(b_p)}$ be the space of smooth functions φ on $\mathbf{R}$ such that $\sigma_{(a_p),(b_p),\infty}(\varphi) < \infty$, equipped with the topology induced by the norm $\sigma_{(a_p),(b_p),\infty}$. Then*

$$\mathcal{S}^{\{M_p\}} = \operatorname{proj}\lim_{(a_p),(b_p)\in\mathcal{R}} \mathcal{S}^{(M_p)}_{(a_p),(b_p)}.$$

THEOREM 3.2. *The following equivalence relations between the families of norms defined above hold true:*

1. The families Σ_∞ and Σ'_∞ (resp. $\tilde{\Sigma}_\infty$ and $\tilde{\Sigma}'_\infty$) of norms in the space $\mathcal{S}^{(M_p)}$ (resp. $\mathcal{S}^{\{M_p\}}$) are equivalent;

2. If condition $(M.2')$ holds, then for every $r \in [1, \infty]$ the families Σ_r, Σ'_r and Υ (resp. $\tilde{\Sigma}_r$, $\tilde{\Sigma}'_r$ and $\tilde{\Upsilon}$) of norms in the space $\mathcal{S}^{(M_p)}$ (resp. $\mathcal{S}^{\{M_p\}}$) are equivalent;

3. If condition $(M.2)$ holds, then the families Σ_2, Σ'_2 and Θ (resp. $\tilde{\Sigma}_2$, $\tilde{\Sigma}'_2$ and $\tilde{\Theta}$) of norms in the space $\mathcal{S}^{(M_p)}$ (resp. $\mathcal{S}^{\{M_p\}}$) are equivalent;

4. If condition $(M.2)$ holds and, given a smooth function φ on $\mathbf{R}$, $p_{\lambda,\beta}(\varphi) < \infty$ and $q_{\lambda,\alpha}(\varphi) < \infty$ for every (resp. some) $\lambda > 0$ and all $\alpha, \beta \in \mathbf{N}_0$, then $r_\lambda(\varphi) < \infty$ for every (resp. some) $\lambda > 0$, where

$$p_{\lambda,\beta}(\varphi) := \sup_{\alpha\in\mathbf{N}_0} \frac{\|\chi^\beta \varphi^{(\alpha)}\|_2}{\lambda^\alpha M_\alpha}, \quad q_{\lambda,\alpha}(\varphi) := \sup_{\beta\in\mathbf{N}_0} \frac{\|\chi^\beta \varphi^{(\alpha)}\|_2}{\lambda^\beta M_\beta},$$

$$r_\lambda(\varphi) := \sum_{\alpha,\beta\in\mathbf{N}_0} \frac{\|\chi^\beta \varphi^{(\alpha)}\|_2}{\lambda^{\alpha+\beta} M_\alpha M_\beta}.$$

REMARK 3.1. Notice that:

1° If condition $(M.2')$ is fulfilled, then the space $\mathcal{S}_2^{(M_p),m}$ in the definition of $\mathcal{S}^*$ can be replaced by $\mathcal{S}_r^{(M_p),m}$, $r \in [1, \infty]$;

2° Part 3 of Theorem 3.2 gives a characterization of Hermite expansions of elements of the space of test functions for the space of tempered ultradistributions;

3° The last part of Theorem 3.2 is an analogue of the following result of Kashpirovski: $\mathcal{S}^\alpha_\alpha = \mathcal{S}^\alpha \cap \mathcal{S}_\alpha$ (see [22]; see also [11]).

It was observed in [25] that $\mathcal{S}^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$ are (F-S)-spaces, while $\mathcal{S}^{\{M_p\}}$ and $\mathcal{S}'^{(M_p)}$ are (LS)-spaces. Moreover, if $(M.2')$ is satisfied, we have the following embeddings: a) $\mathcal{D}^* \hookrightarrow \mathcal{S}^* \hookrightarrow \mathcal{E}^*$, b) $\mathcal{S}^* \hookrightarrow \mathcal{S}$, c) $\mathcal{E}'^* \hookrightarrow \mathcal{S}'^* \hookrightarrow \mathcal{D}'^*$ and d) $\mathcal{S}' \hookrightarrow \mathcal{S}'^*$, where the symbol $A \hookrightarrow B$ means that the inclusion mapping of the space A into the space B is continuous and that A is dense in B (see [25]). The following theorem, proved in [25] (Part 1) and in [30] (Part 2) gives a characterization of Hermite expansions of tempered ultradistributions:

THEOREM 3.3. *Let $f \in \mathcal{D}'^{(M_p)}$ (resp. $f \in \mathcal{D}'^{\{M_p\}}$).*

1. If condition $(M.2)$ is satisfied, then $f \in \mathcal{S}'^{(M_p)}$(resp. $f \in \mathcal{S}'^{\{M_p\}}$) if and only if

$$f = \sum_{n \in \mathbf{N}_0} d_n h_n, \quad in\, \mathcal{S}'^*$$

and, for some (resp. every) $\delta > 0$, we have

$$\sum_{n \in \mathbf{N}_0} |d_n|^2 \exp(-2M(\delta\sqrt{2n+1})) < \infty.$$

2. If conditions $(M.2)$ and $(M.3)$ are satisfied, then $f \in \mathcal{S}'^*$ if and only if f is of the form $f = P(D)F$, where P is an ultradifferentiable operator of the class $*$ and F is a continuous function on $\mathbf{R}$ of ultrapolynomial growth of the class $*$.

THEOREM 3.4. *If condition $(M.2)$ is satisfied, then $\mathcal{S}^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$ are (FN)-spaces $\mathcal{S}^{\{M_p\}}$ and $\mathcal{S}'^{(M_p)}$ are (LN)-spaces, respectively.*

Proof. If $(M.2)$ is fulfilled, then the spaces $\mathcal{S}^{(M_p)}$ and $\mathcal{S}^{\{M_p\}}$ are isomorphic to the projective and inductive limits of the Köthe spaces $\ell^2(b_k)$ and $\ell^2(c_k)$ (see [13]), respectively, where (b_k) and (c_k) are sequences of the form:

$$b_k = (b_{1,k}, b_{2,k}, \ldots), \quad b_{n,k} = \exp(M(k\sqrt{2n+1})), \quad k, n \in \mathbf{N},$$

and

$$c_k = (c_{1,k}, c_{2,k}, \ldots), \quad c_{n,k} = \exp(M(\sqrt{2n+1}/k)), \quad k, n \in \mathbf{N}.$$

The isomorphism is established by the mapping $\varphi \mapsto (d_n)$, where

$$\varphi = \sum_{n=0}^{\infty} d_n h_n$$

(see Part 3 of Theorem 3.2). In order to prove the assertion it is enough to show that

$$\sum_{n \in \mathbf{N}_0} b_{n,k}/b_{n,l} < \infty \quad \text{and} \quad \sum_{n \in \mathbf{N}_0} c_{n,k}/c_{n,l} < \infty \tag{3}$$

for some $l > k$ (see e.g. [13], p. 112). The inequalities

$$M(k\rho) + M(\rho) \leq 2M((k+1)\rho), \quad \rho > 0$$

and

$$2M(\rho) \leq M(H\rho) + \log A, \quad \rho > 0$$

(see Proposition 3.6 in [23]) imply that

$$\sum_{n\in\mathbf{N}_0} \frac{b_{n,k}}{b_{n,l}} \leq \sum_{n\in\mathbf{N}_0} \exp(-M(\sqrt{2n+1})) < \infty.$$

for $l > H(k+1)$, which proves the first inequality in (3). The second one follows in a similar way.

REMARK 3.2. All the definitions given in this section can be easily generalized to the d-dimensional case. For example $\mathcal{S}_r^{M_p,m}(\mathbf{R}^d)$ with $r \geq 0$ and $m > 0$ is the space of all smooth functions φ on $\mathbf{R}^d$ satisfying

$$\sigma_{m,r}(\varphi) = \Big(\sum_{\alpha,\beta\in\mathbf{N}_0^d} \int_{\mathbf{R}^d} \Big| \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} < x >^\beta \varphi^{(\alpha)}(x) \Big|^r dx \Big)^{1/r} < \infty,$$

equipped with the topology induced by the norm $\sigma_{m,r}$. We used above the usual convention $a^\beta = a^{\beta_1+\dots+\beta_d}$ for $a \in \mathbf{R}$ and $\beta = (\beta_1, \dots, \beta_d) \in \mathbf{N}_0^d$.

Moreover, in the d-dimensional case, we have the multi-indexed sequence of Hermite functions defined by

$$h_\alpha(x) = h_{\alpha_1}(x_1)h_{\alpha_2}(x_2) \cdot \ldots \cdot h_{\alpha_d}(x_d),$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{N}^d$, $x = (x_1, \dots, x_d) \in \mathbf{R}^d$ and $h_{\alpha_i}(x_i)$ are given by (1).

It is easy to verify that the proofs of the theorems of this section are valid in the d-dimensional case. They can be written down in the same way if we use the fact that the multi-indexed sequences (M_α) and $(M_{\alpha_1+\dots+\alpha_d})$ with $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{N}^d$ determine the same spaces of ultradistributions, under the assumption that condition $(M.2)$ is satisfied.

4. Integral transforms

Suppose that conditions $(M.1)$, $(M.2)$ and $(M.3')$ are fulfilled. The Fourier transform is an isomorphism of $\mathcal{S}^*$ onto itself. In the next theorem we give the characterizations which are similar to the ones given in [19].

THEOREM 4.1 (characterization via the Fourier transform). *A function φ belongs to $\mathcal{S}^{(M_p)}$ (resp. $\mathcal{S}^{\{M_p\}}$) if and only if it is square integrable and for every (resp. some) $h > 0$,*

$$\varphi(x) = \mathcal{O}\left(\exp(-M(h \mid x \mid))\right) \text{ and } (\mathcal{F}\varphi)(x) = \mathcal{O}\left(\exp(-M(h \mid x \mid))\right).$$

Proof. The assertion follows from Parts 1 and 3 of Theorem 3.2.

THEOREM 4.2 (characterization via the Wigner distribution). *A function* $\varphi \in \mathcal{S}^{(M_p)}$ (*resp.* $\varphi \in \mathcal{S}^{\{M_p\}}$) *if and only if for every* (*resp. some*) $\lambda > 0$

$$\mathbf{W}(x, y; \varphi) = \mathcal{O}(\exp(-M(\lambda(x^2 + y^2)^{1/2}))).$$

Proof. The assertion follows from Parts 2 and 4 of Theorem 3.2, properties of the function M and the preceding theorem.

THEOREM 4.3 (characterization via the Bargmann transform). *A function* $\varphi \in \mathcal{S}^{(M_p)}$ (*resp.* $\varphi \in \mathcal{S}^{\{M_p\}}$) *if and only if for every* (*resp. some*) $\lambda > 0$ *there exists* C *such that*

$$|(\mathbf{A}\varphi)(\zeta)| \leq C \exp\left(\frac{1}{2}|\zeta|^2 - M(\lambda|\zeta|)\right), \qquad \zeta \in \mathbf{C}.$$

Proof. The assertion follows from Parts 2 and 4 of Theorem 3.2, properties of the function M and Theorem 4.1.

As usual, we define the Fourier transform of $f \in \mathcal{S}'^*$ by

$$\langle \mathcal{F}f, \varphi\rangle = \langle f, \mathcal{F}\varphi\rangle, \qquad \varphi \in \mathcal{S}^*$$

(see [25]).

Assume that $(M.1)$, $(M.2)$ and $(M.3)$ are satisfied.

Denote by $\mathcal{S}'^*_+$ the subspace of $\mathcal{S}'^*$ consisting of elements supported by $[0, \infty)$. Let $g \in \mathcal{S}'^*_+$. For fixed $y > 0$ we define $g\exp(-y\,\cdot)$ as an element of $\mathcal{S}'^*$ by

$$\langle g\exp(-y\,\cdot),\, \varphi\rangle = \langle g,\, \varrho\exp(-y\cdot)\varphi\rangle, \quad \varphi \in \mathcal{S}^*,$$

where ϱ is an element of $\mathcal{E}^*$ such that $\varrho(x) = 1$, if $x \in (-\varepsilon, \infty)$ and $\varrho(x) = 0$, if $x \in (-\infty, -2\varepsilon)$, for some $\varepsilon > 0$.

An example of a function satisfying the above conditions is $\varrho = f * \omega$, where ω is a function with the properties: $\omega \in \mathcal{D}^*$, $\int\omega = 1$, $\operatorname{supp}\omega \subset [-\epsilon/2, \epsilon/2]$ (for the existence of such a function see [23], Theorem 4.2). The function ϱ defined above belongs to $\mathcal{E}^*$(see [23], Theorem 6.10) and, moreover, $f(x) = 1$ for $x \geq -3\epsilon/2$ and $f(x) = 0$ for $x < -3\epsilon/2$. It is easy to see that the definition does not depend on the choice of ϱ.

As in the case of $\mathcal{S}'_+$ (see e.g. [32]) we define the Laplace transform of $g \in \mathcal{S}'^*_+$ by

$$(\mathcal{L}g)(\zeta) = \mathcal{F}(g\exp(-y\,\cdot))(x), \quad \zeta = x + iy \in \mathbf{C}_+.$$

Clearly, if $y > 0$ is fixed, the right hand side is an element of $\mathcal{S}'^*$.

Let

$$G(\zeta) = \langle g, \omega\exp(i\zeta\cdot)\rangle, \quad \zeta = x + iy \in \mathbf{C}_+, \tag{4}$$

where ω is as above. The function G is holomorphic on $\mathbf{C}_+$ and does not depend on ω.

The next two theorems, that we give here for the sake of completness, were proved in [29] in the one-dimensional case for the spaces of Beurling-Gevrey tempered ultradistributions $\Sigma'_\alpha = \mathcal{S}^{(p^{\alpha p})}$ with $\alpha > 1/2$, supported by $[0,\infty)$. The proofs in the general case are analogous.

THEOREM 4.4. *Let* $g \in \mathcal{S}'^{(M_\alpha)}_+$ *(resp.* $g \in \mathcal{S}'^{\{M_p\}}$*) and let* G *be defined by* (4). *Then*

1. for every $\varepsilon > 0$ *there are a* $k > 0$ *and a* $\mathcal{C} > 0$ *(resp. for every* $\varepsilon > 0$ *and every* $k > 0$ *there exists a* $\mathcal{C} > 0$*) such that*

$$|G(\zeta)| \leq \mathcal{C} \exp\left[\varepsilon y + \left(M(k|x|) + \tilde{M}\left(k|y|^{-1}\right)\right)\right], \quad \zeta = x + iy \in \mathbf{C}_+;$$

2. for every fixed $y > 0$*, we have* $(\mathcal{L}g)(\cdot + iy) = G(\cdot + iy)$*;*

3. there exists a tempered ultradistribution $h =: G(\cdot + i0) \in \mathcal{S}'^{(M_p)}$ *(resp.* $h \in \mathcal{S}'^{\{M_p\}}$*) such that*

$$G(\cdot + iy) \to h = G(\cdot + i0) \quad as\, y \to 0^+$$

in the sense of convergence in $\mathcal{S}'^{(M_p)}$ *(resp. in* $\mathcal{S}'^{\{M_p\}}$*) and*

$$h = G(\cdot + i0) = \mathcal{F}g;$$

4. if $G_k(\zeta) = (\mathcal{L}g_k)(\zeta)$ *for* $\zeta \in \mathbf{C}_+$*,* $k = 1, 2$ *and* $G_1(\cdot + i0) = G_2(\cdot + i0)$*, then* $g_1 = g_2$*.*

Applying the above theorem one can represent tempered ultradistributions, similarly as it was done in [29] in a particular case (i.e. for elements of $\mathcal{S}'^{(M_p)}$with the sequence (M_p) given by $M_p = p^{\alpha p}$ for $p \in \mathbf{N}$) as boundary values of appropriate harmonic functions in the upper half-plane, expanded into series of Hermite functions of the second type.

Let $b > 0$ (resp. $(b_p) \in \mathcal{R}$) be given and let P_b (resp. $P_{(b_p)}$) be an entire function such that, for some constants $L > 0$ and $\mathcal{C}$, we have

$$(5) \qquad |P_b(\zeta)| \leq \mathcal{C} \exp M(L|\zeta|) \quad \left(\text{resp. } |P_{(b_p)}(\zeta)| \leq \mathcal{C} \exp N_{(b_p)}(L|\zeta|)\right)$$

for all $\zeta \in \mathbf{C}$, and

$$(6) \qquad \exp M(b|\zeta|) \leq P_b(\zeta) \quad \left(\text{resp. } \exp N_{(b_p)}(|\zeta|) \leq P_{(b_p)}(\zeta)\right)$$

for all $\zeta = x + iy \in \mathbf{C}$ such that $|x| \geq |y|$. In case that conditions $(M.1)$, $(M.2)$ and $(M.3)$ are satisfied, an example of such an entire function is

$$P_b(\zeta) = \prod_{\alpha=1}^{\infty}(1 + \frac{\zeta^2}{b^2 m_\alpha^2}) \quad \left(\text{resp. } P_{(b_p)}(\zeta) = \prod_{\alpha=1}^{\infty}(1 + \frac{\zeta^2}{{b_\alpha}^2 m_\alpha^2})\right), \quad \zeta \in \mathbf{C}.$$

From [23, p. 91], it follows that, in the Roumieu case, the entire function $P_{(b_p)}$ satisfies (5). Moreover, we have

$$\left|\prod_{\alpha=1}^{\infty}(1+\frac{\zeta^2}{b_\alpha^2 m_\alpha^2})\right| \geq \sup_{\beta\in\mathbf{N}}\prod_{\alpha=1}^{\beta}\left|1+\frac{\zeta^2}{b_\alpha^2 m_\alpha^2}\right|$$
$$\geq \sup_{\beta\in\mathbf{N}}\prod_{\alpha=1}^{\beta}\left|\frac{\zeta^2}{b_\alpha^2 m_\alpha^2}\right| = \exp[2N_{(b_p)}(|\zeta|)]$$

for $\zeta = x+iy \in \mathbf{C}$ such that $|x| \geq |y|$. This means that also (6) is fulfilled by the function $P_{(b_p)}$. A similar reasoning can be made in the Beurling case for the function P_b.

THEOREM 4.5 (characterization via the Laplace transform). *Let G be a holomorphic function on $\mathbf{C}_+$. Then G is the Laplace transform of some $g \in \mathcal{S}_+^{\prime(M_\alpha)}$ (resp. $\mathcal{S}'^*$) if and only if for every $\epsilon > 0$ there are a $k > 0$, an ultradifferential operator P_b and a constant $C > 0$ (resp. for every $\epsilon > 0$ there exist an ultradifferential operator $P_{(b_p)}$ that for every $k > 0$ there is a constant $C > 0$) such that*

$$\left\|\frac{G(\cdot+iy)}{P_b(\cdot+iy)}\right\|_2 \leq C\exp(\epsilon y+\tilde{M}(k|y|^{-1})), \quad y>0$$
$$\left(resp.\ \left\|\frac{G(\cdot+iy)}{P_{(b_p)}(\cdot+iy)}\right\|_2 \leq C\exp(\epsilon y+\tilde{M}(k|y|^{-1})), \quad y>0\right).$$

Moreover, $(\mathcal{F}g) = \lim_{y\to 0^+} G(\cdot+iy)$ in $\mathcal{S}_+^{\prime(M_\alpha)}$ (resp. in $\mathcal{S}'^*$).

REMARK 4.1. The proofs of the theorems of this section also hold in the d-dimensional case. Moreover, they can be written down in the same way if we use the conventions mentioned at the end of the previous section.

References

[1] P. Antosik, J. Mikusiński, R. Sikorski, *The theory of Distributions. The Sequential Approach*, Elsevier, Amsterdam, 1973.

[2] A. Avantaggiati, *$\mathcal{S}$-spaces by means of the behaviour of Hermite-Fourier coefficients*, Boll. Un. Mat. Ital. (6) 4-A (1985), 487–495.

[3] A. Beurling, *Quasi-analyticity and general distributions*, Lectures 4. and 5., A.M.S. Summer Institute, Stanford, 1961.

[4] G. Björk, *Linear partial differential operators and generalized distributions*, Ark. Mat. 6 (1966), 351–407.

[5] N. G. de Bruin, *A theory of generalized functions, with applications to Wigner distribution and Weyl correspondence*, Nieuw Arch. Wisk. (3) 21 (1973), 205–280.

[6] M. Budincević, Z. Lozanov-Crvenković, D. Perišić, *Representation theorems for tempered ultradistributions*, Publ. Inst. Math. (Beograd) 65 (1999), 142–160.

[7] J. Chung, S.-Y. Chung, D. Kim, *Characterization of the Gelfand-Shilov spaces via Fourier transforms*, Proc. Amer. Math. Soc. 124 (1996), 2101–2108.

[8] J. Chung, S.-Y. Chung, D. Kim, *Equivalence of the Gelfand-Shilov spaces*, J. Math. Anal. Appl. 203 (1996), 828–839.

[9] S.-Y. Chung, D. Kim, *Representation of quasianalytic ultradistributions*, Ark. Math. 31 (1993), 51–60.

[10] I. Ciorănescu, L. Zsidó, *ω-ultradistributions and their applications to operator theory*, in: Spectral Theory, Banach Center Publications 8, PWN, Warsaw 1982, 77-220.

[11] S. J. L. van Eijndhoven, *Functional analytic characterization of the Gelfand-Shilov spaces* S_α^β, Proc. Nederl. Akad. Wetensch. Proc. Ser. A 90 (1987), 133-144.

[12] S. J. L. van Eijndhoven, J. de Graaf, *Trajectory Spaces, Generalized Functions and Unbounded Operators*, Lecture Notes in Math. 1162, Springer, Berlin, 1985.

[13] K. Floret, J. Wloka, *Einführung in die Theorie der lokalkonvexen Räume*, Lecture Notes in Math. 56, Springer, Berlin, 1968.

[14] I. M. Gel'fand, G. E. Shilov, *Generalized Functions*, Vol. 2, Academic Press, New York, 1968.

[15] I. M. Gel'fand, G. E. Shilov, *Generalized Functions*, Vol. 3, Academic Press, New York, 1967.

[16] V. O. Grudzinski, *Temperierte Beurling-distributions*, Math. Nachr. 91 (1979), 197–220.

[17] A. J. E. M. Jansen, *Positivity of weighted Wigner distributions*, SIAM J. Math. Anal. 12 (1981), 752–758.

[18] A. J. E. M. Jansen, *Bargmann transform, Zak transform and coherent states*, J. Math. Phys. 23 (1982), 720–731.

[19] A. J. E. M. Jansen, S. J. L. van Eindhoven, *Spaces of type W, growth of Hermite coefficients, Wigner distribution and Bargmann transform*, J. Math. Anal. Appl. 152 (1990), 368–390.

[20] A. Kamiński, D. Perišić, S. Pilipović, *Hilbert transform and singular integrals on the spaces of tempered ultradistributions*, in: Algebraic Analysis and Related Topics, Banach Center Publications 53, PWN, Warsaw 2000.

[21] A. Kamiński, D. Perišić, S. Pilipović, *Integral transformations on the spaces of tempered ultradistributions*, Preprint.

[22] O. I. Kashpirovskij, *Equality of the spaces* S_α^β *and* $S_\alpha \cap S^\beta$, Functional Anal. Appl. 14 (1978), 60.

[23] H. Komatsu, *Ultradistributions I*, J. Fac. Sci. Tokyo Univ. Sect. IA Math. 20 (1973), 25–105.

[24] D. Kovačević, *Some operations on the space* $\mathcal{S}'^{(M_p)}$ *of tempered ultradistributions*, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 23 (1993), 87–107.

[25] D. Kovačević, S. Pilipović, *Structural properties of the spaces of tempered ultradistributions*, in: Complex Analysis and Generalized Functions, Varna 1991, Publ. House of the Bugarian Academy of Sciences, Sofia 1993, 169–184.

[26] H.-J. Petzsche, *Die nuklearität der Ultradistributionsräume und der Satz vom Kern I*, Manuscripta Math. 24 (1978), 133–171.

[27] S. Pilipović, *Hilbert transformation of Beurling ultradistributions*, Rend. Sem. Mat. Univ. Padova 77 (1987), 1–13.

[28] S. Pilipović, *Tempered ultradistributions*, Boll. Un. Mat. Ital. (7) 2-B (1988), 235–251.
[29] S. Pilipović, *Beurling-Gevrey tempered ultradistributions as boundary values*, Portug. Math. 48 (1991), 483–504.
[30] S. Pilipović, *Characterization of bounded sets in spaces of ultradistributions*, Proc. Amer. Math. Soc. 120 (1994), 1191–1206.
[31] C. Roumieu, *Sur quelques extensions de la notion de distribution*, Ann. Sci. École Norm. Sup. (3) 77 (1960), 41–121.
[32] V. S. Vladimirov, *Generalized Functions in Mathematical Physics*, Mir, Moscow, 1979.
[33] A. H. Zemanian, *Generalized Integral Transformations*, Dover Publications, New York, 1987.

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