DEMONSTRATIO MATHEMATICA
Vol. XXXIII No 3 2000

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ON MIKUSIŃSKI TYPE PRODUCTS OF DISTRIBUTIONS

Abstract. The well known result of Jan Mikisiński on distributional products: $x^{-1} . x^{-1} - \pi^2 \delta^2(x) = x^{-2}$, $x \in \mathbb{R}$, is generalized here for the distributions x^{-p} and $\delta^{(p-1)}(x)$ for arbitrary natural p. To this aim we follow the method of Mikusiński of employing a 'Fourier-product' formula, which is done in the setting of Colombeau algebra $\mathcal{G}_\tau$ of tempered generalized functions.

1. Introduction and basic definitions

Due to the large employment of Schwartz distributions in the natural sciences and other mathematical fields, where products of distributions with coinciding singularities often appear, the problem of multiplication of distributions has been for a long time an object of many research studies. Starting with the historically first construction of distributional multiplication proposed by König [13] and the sequential approach developed by Mikusiński and co-authors [2], there have been numerous attempts to define products for the distributions, or rather to enlarge the range of existing products (see [15] for a complete review and bibliography).

Several attempts have been also made to include the distributions into algebras of generalized functions with the differentiation always possible and subject to the Leibniz rule, or else — into differential algebras. According to the classical Schwartz counter-examples, however, in associative algebras of generalized functions, multiplication and differentiation can not simultaneously extend the corresponding classical operations unrestrainedly. One therefore has to reduce the requirements on the multiplication.

Most complete list of such properties so far possesses the associative differential algebra of generalized functions of J.-F. Colombeau [3]. The dis-

1991 *Mathematics Subject Classification*: 46F10, 46F12.
Key words and phrases: distribution, product, colombeau algebra, tempered generalized function.

This paper has been presented at the International Conference "Algebraic Analysis and Related Topics", held in Banach Center, Warsaw, September 21–25, 1999.

tributions are C-linearly embedded in that algebra and the multiplication is compatible with the operations of differentiation and products with C^∞-differentiable functions. The Colombeau algebra has a notion of 'association' that is a faithful generalization of the equality in $\mathscr{D}'(\mathbb{R})$. This is particularly useful for evaluation of distribution products, as they are embedded in $\mathscr{G}$, in terms of distributions again.

In 1966 Jan Mikusiński published in [14] his popular result

$$x^{-1}\,.\,x^{-1} - \pi^2\delta(x).\delta(x) = x^{-2}, \quad x \in \mathbb{R}. \tag{1}$$

Though, the products on the left-hand side here exists, their sum still has a correct meaning in the distribution space. Another formula of that type in dimension one—in a nonstandard approach to distribution theory — was given in [16]:

$$H\,.\,\delta'(x) + \delta(x)\,.\,\delta(x) \stackrel{*}{=} \delta'(x)/2. \tag{2}$$

H denotes here the Heaviside function, and '$\stackrel{*}{=}$' stands for the equality up to an infinitesimal quantity.

Formulas of that type can be found in the mathematical and physical literature. We proposed such equations to be named 'products of Mikusiński type' in a previous paper, where a generalization of (2) was obtained in Colombeau algebra (see equation (7) below). In this paper we generalize the basic Mikisiński formula (1) for the distributions x^{-p} and $\delta^{(p-1)}(x)$ for arbitrary $p \in \mathbb{N}$ and $x \in \mathbb{R}$. We follow the method of Mikusiński on applying a 'Fourier-product' formula, which is done in the setting of Colombeau algebra of tempered generalized functions.

We start by recalling the fundamentals of Colombeau theory, restricting ourselves to the algebra $\mathscr{G}_\tau$ of tempered generalized functions on the real line [4]. It contains the space S' of tempered distributions on $\mathbb{R}$ and is appropriate for the consideration we envisage in this paper.

NOTATION. Let $\mathbb{N}$, $\mathbb{N}_0$ stand for the natural numbers, respectively, the non-negative integers and $\delta_{mn} = \{1 \text{ for } m = n, \ = 0 \text{ otherwise}; \, m, n \in \mathbb{N}_0\}$. If $q \in \mathbb{N}_0$, we put $A_q = \{\varphi(x) \in \mathscr{D}(\mathbb{R}) : \int_{\mathbb{R}} x^n \varphi(x)\, dx = \delta_{0n}$ for each $n \in \mathbb{N}_0$, $n \le q\}$. We shall write $\varphi_\varepsilon = \varepsilon^{-1}\varphi(\varepsilon^{-1}x)$ for $\varphi \in A_0$ and $\varepsilon > 0$.

DEFINITION 1.1. Let $\mathscr{E}[\mathbb{R}]$ be the $\mathbb{C}$-algebra of functions $f(\varphi, x) : A_0 \times \mathbb{R} \to \mathbb{C}$ that are infinitely differentiable in respect to x by a fixed 'parameter' φ. Let the algebra $\mathscr{E}_{\mathrm{M},\tau}[\mathbb{R}]$ be the subset of $\mathscr{E}[\mathbb{R}]$ of 'moderate' functions $f(\varphi, x)$ in $\mathscr{E}[\mathbb{R}]$ such that for each $p \in \mathbb{N}_0$ there is a $q \in \mathbb{N}_0$ such that: for each $\varphi \in A_q$ there are $c > 0$, $\eta > 0$ satisfying $|\partial^p f(\varphi_\varepsilon, x)| \le c(1 + |x|^q)\varepsilon^{-q}$ for all $x \in \mathrm{R}$ and $0 < \varepsilon < \eta$. The symbol τ stands for 'tempered'. The ideal $\mathscr{N}_\tau[\mathbb{R}]$ of $\mathscr{E}_{\mathrm{M},\tau}[\mathbb{R}]$ is the set of functions $f(\varphi, x)$ such that for each $p \in \mathbb{N}_0$ there is $q \in \mathbb{N}$ such that: for every $r \ge q$ and each $\varphi \in A_r(\mathbb{R})$ there are

$c > 0$, $\eta > 0$ satisfying $|\partial^p f(\varphi_\varepsilon, x)| \leq c(1 + |x|^r)\varepsilon^{r-q}$, for all $x \in \mathbb{R}$ and $0 < \varepsilon < \eta$. Then the tempered generalized functions are defined as elements of the quotient algebra $\mathscr{G}_\tau = \mathscr{E}_{\mathrm{M},\tau}[\mathbb{R}] / \mathscr{N}_\tau[\mathbb{R}]$.

The algebra $\mathscr{G}_\tau$ contains the tempered distributions [4], canonically embedded by the map $i : S' \to \mathscr{G}_\tau : u \mapsto \{\widetilde{u}(\varphi, x) = (u * \check{\varphi})(x)\}$, where $\check{\varphi}(x) = \varphi(-x)$, and φ is running the set A_0. Basic examples are the embeddings $\widetilde{x_+^p}$, $\widetilde{x^{-p}}$, and $\widetilde{\delta^{(p)}}(x)$ of the distributions $x_+^p = \{x^p$ for $x \geq 0$, $= 0$ for $x < 0\}$, $x^{-p} = (-1)^{p-1}/(p-1)!\partial^p(\ln|x|)$, and $\delta^{(p)}(x)$, $p \in \mathbb{N}$. We note that similar, but different schemes of 'new generalized functions' were introduced by Antonevich and Radyno [1] and by Egorov [7].

We recall some properties of the parameter functions $\varphi \in A_q$, that will be in use later. From the equation for their Fourier transform

$$\widehat{\varphi_\varepsilon}(x) = \int_{\mathbb{R}} e^{-ixy}\varphi_\varepsilon(y)\,dy = \frac{1}{\varepsilon}\int_{\mathbb{R}} e^{-ixy}\varphi\left(\frac{y}{\varepsilon}\right)dy = \int_{\mathbb{R}} e^{-ix\varepsilon t}\varphi(t)\,dt = \widehat{\varphi}(\varepsilon x)$$

and the definition of A_q, it follows that, for any $q \in \mathbb{N}$ and $\varphi \in A_q$,

$$\widehat{\varphi}(0) = 1, \qquad \widehat{\varphi}^{(j)}(0) = 0 \quad \text{for all } j \in \mathbb{N}_0,\ j < q. \tag{3}$$

Then, from the Taylor Expansion formula up to order $q+1$, we easily obtain the estimation

$$|\widehat{\varphi}(\varepsilon x) - 1| \leq \frac{c(\varphi)\varepsilon^{q+1}}{1 + |x|^r}, \quad \text{for each } r \in \mathbb{N},\ x \in \mathbb{R}. \tag{4}$$

2. Association and results on distribution products in $\mathscr{G}_\tau$

The equality in Colombeau algebra is very strict, so the next weaker concepts for 'association' are introduced.

DEFINITION 2.1. Two generalized functions f, g of $\mathscr{G}_\tau$ are said to be strongly associated, written as $f \overset{s}{\approx} g$, if for any $\psi \in S$, denoting

$$d_\psi(\varphi_\varepsilon) = \int_{\mathbb{R}} [f(\varphi_\varepsilon, x) - g(\varphi_\varepsilon, x)]\psi(x)\widehat{\varphi}(\varepsilon x)\,dx,$$

there exists $N \in \mathbb{N}_0$, such that for any $q \geq N$, $\varphi \in A_q$ there exist $c, \eta > 0$ such that $|d_\psi(\varphi_\varepsilon)| \leq c\varepsilon^{q-N}$ for all $\varepsilon \in (0, \eta)$.

DEFINITION 2.2. Two generalized functions $f, g \in \mathscr{G}_\tau$ are said to be associated (in weak sense), denoted as $f \approx g$, if for some representatives of theirs $f(\varphi_\varepsilon, x)$, $g(\varphi_\varepsilon, x)$ and for each test function $\psi(x) \in S$ there is $q \in \mathbb{N}_0$ such that, for all $\varphi(x) \in A_q$,

$$\lim_{\varepsilon \to 0_+} \int_{\mathbb{R}} [f(\varphi_\varepsilon, x) - g(\varphi_\varepsilon, x)]\psi(x)\widehat{\varphi}(\varepsilon x)dx = 0.$$

DEFINITION 2.3. A generalized function $f \in \mathcal{G}_\tau$ is said to admit some $u \in S'$ as an 'associated distribution', denoted by $f \approx u$, if for some representative $f(\varphi_\varepsilon, x) \in \mathcal{E}_{M,\tau}[\mathbb{R}]$ of the function f and for each test function $\psi(x) \in S$ there is a $q \in \mathbb{N}_0$ such that, for all $\varphi(x) \in A_q$, $\lim_{\varepsilon \to 0_+} \int_{\mathbb{R}} f(\varphi_\varepsilon, x)\psi(x)\widehat{\varphi}(\varepsilon x)dx = \langle u, \psi \rangle$.

REMARK. These definitions are independent of the representative chosen. The distribution associated by Definition 2.3, if it exists, is unique. The embedding of every distribution in Colombeau algebra is associated with the latter [3], the association thus being a generalization of the equality of distributions in classical distribution theory. This fact also implies that two definitions are equivalent on the embedding of distributions:

$$f \approx \widetilde{u} \Longleftrightarrow f \approx u \qquad \text{for any } f \in \mathcal{G}_\tau,\ u \in S'. \tag{5}$$

Now, by a distribution product in Colombeau algebra, sometimes called 'Colombeau product', is meant the product of some distributions as they are embedded in Colombeau algebra, whenever the result admits an associated distribution (see [11] for comparison with other distribution products, and [5] for particular results).

The following result on products of distributions of several variables in Colombeau algebra proved in [6] will be needed in the sequel. For any multiindex $p \in N^d$, it is

$$\widetilde{x^{-p}} \,.\, \widetilde{\delta^{(p-1)}}(x) \;\approx\; \frac{(-1)^p (p-1)!}{2^d (2p-1)!} \delta^{(2p-1)}(x), \quad x \in \mathbb{R}^d. \tag{6}$$

Note that equation (6) was derived in the one-dimensional case by Fisher [8] and Itano [10] as a regularized model product (in the termonology proposed by Kamiński [12]), but only under (different) additional requirements on the regularizing δ-nets.

Passing further to Mikusiński type distribution products, we recall the results of this type in Colombeau algebra, given by this.

THEOREM 2.1 ([6]). *For arbitrary $p \in \mathbb{N}_0$, the embeddings in $\mathcal{G}_\tau$ of the distributions x_+^p, x_-^p and $\delta^{(p+1)}(x)$ satisfy:*

$$\frac{(-1)^p}{p!} \widetilde{x_+^p} \,.\, \widetilde{\delta^{(p+1)}}(x) + \widetilde{\delta}(x) \,.\, \widetilde{\delta}(x) \approx \frac{p+1}{2} \delta'(x), \tag{7}$$

$$\frac{1}{p!} \widetilde{x_-^p} \,.\, \widetilde{\delta^{(p+1)}}(x) - \widetilde{\delta}(x) \,.\, \widetilde{\delta}(x) \approx \frac{p+1}{2}\, \delta'(x). \tag{8}$$

REMARK. Clearly, the particular case $p = 0$ of (7) gives equation (2) obtained in Colombeau algebra. Note also that (7) and (8) are easily shown to be consistent with following known equation in the distribution space: $x^p \delta^{(p+1)}(x) = (-1)^p (p+1) \delta'(x)$, $p \in \mathbb{N}_0$.

Consider next the 'even' and 'odd' sums of the distributions $x_+^p, x_-^p, x \in \mathbb{R}$ and $p \in \mathbb{N}_0$, as defined in [9]: $|x|^p = x_+^p + x_-^p$, $|x|^p \operatorname{sgn} x = x_+^p - x_-^p$. Combining now equations (7) and (8), we obtain the following.

COROLLARY 2.1. *Let for arbitrary* $p \in \mathbb{N}$, $\widetilde{|x|^p}$ and $\widetilde{|x|^p \operatorname{sgn} x}$ be the embeddings in $\mathcal{G}_\tau$ of the distributions $|x|^p, |x|^p\operatorname{sgn}x$. Then it holds:

$$\widetilde{|x|^{2p-1}\operatorname{sgn}x} \,.\, \widetilde{\delta^{(2p)}}(x) \approx -(2p)!\delta'(x);$$
$$\widetilde{|x|^{2p}} \,.\, \widetilde{\delta^{(2p+1)}}(x) \approx (2p+1)!\delta'(x);$$
$$\widetilde{|x|^{2p-1}} \,.\, \widetilde{\delta^{(2p)}}(x) - 2(2p-1)!\widetilde{\delta}(x) \,.\, \widetilde{\delta}(x) \approx 0;$$
$$\widetilde{|x|^{2p}\operatorname{sgn} x} \,.\, \widetilde{\delta^{(2p+1)}}(x) + 2(2p)!\widetilde{\delta}(x) \,.\, \widetilde{\delta}(x) \approx 0.$$

Observe that the first two equations here are "ordinary" products of distributions in Colombeau algebra, while the other two represent Mikusiński type products.

3. Fourier transform and Fourier-product formula in $\mathcal{G}_\tau$

In compliance with the general definition of integral in $\mathcal{G}_\tau$, the Fourier transform $\mathcal{F}f \equiv \widehat{f}$, the inverse Fourier transform $\mathcal{F}^{-1}f \equiv \overline{f}$, and the convolution of $f, g \in \mathcal{G}_\tau$ are introduced by the following equations for the representatives:

$$\widehat{f}(\varphi_\varepsilon, x) = \int_{\mathbb{R}} e^{-ixy} f(\varphi_\varepsilon, y)\widehat{\varphi}(\varepsilon y)\, dy, \tag{9}$$
$$\overline{f}(\varphi_\varepsilon, x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixy} f(\varphi_\varepsilon, y)\widehat{\varphi}(\varepsilon y)\, dy,$$
$$(f * g)(\varphi_\varepsilon, x) = \int_{\mathbb{R}} f(\varphi_\varepsilon, x-y) g(\varphi_\varepsilon, y)\widehat{\varphi}(\varepsilon y)\, dy. \tag{10}$$

Note that these equations make sense since $\widehat{\varphi} \in S$, and according to (3) and (4), the factor $\widehat{\varphi}(\varepsilon x)$ can be omitted whenever $f(\varphi_\varepsilon, x)$ is supported in a compact subset of $\mathbb{R}$. It can be shown [17] that these definitions preserve both the representative classes of generalized functions in $\mathcal{G}_\tau$ and the association relations. Moreover, the following basic properties were demonstrated by Colombeau [4] for any function $f \in \mathcal{G}_\tau$:

$$(a)\ \mathcal{F}^{-1}\mathcal{F}f \overset{s}{\approx} \mathcal{F}\mathcal{F}^{-1}f \overset{s}{\approx} f \qquad (b)\ f * \widetilde{\delta} \overset{s}{\approx} f. \tag{11}$$

Note that the corresponding strict equalities are not valid in the algebra $\mathcal{G}_\tau$.

We shall need an 'exchange formula' between the operations of multiplication and convolution, via the Fourier transform. Such a formula was proved by Colombeau in [4] in a strong-association sense whenever one of the multipliers is decreasing fast at infinity. Again, it is not valid as a strict equality in $\mathcal{G}_\tau$. We will prove an exchange formula in weak-association sense, enlarging its validity for arbitrary generalized functions in $\mathcal{G}_\tau$. We formulate it for the inverse Fourier transform in order to get later a 'Fourier-product' equation appropriate for our purposes.

THEOREM 3.1. *For each two generalized functions* $f_{1,2} \in \mathcal{G}_\tau$, it holds

$$\overline{f_1} \cdot \overline{f_2} \approx \frac{1}{2\pi} \overline{f_1 * f_2}. \tag{12}$$

Proof. Taking into account Definition 2.3 and equations (9), (10), we have to evaluate the difference between wo sides of (12) for some representatives of $f_{1,2}$ and an arbitrary $\psi \in S$:

$$\begin{aligned}
\Delta_\psi(\varphi_\varepsilon) &:= \int_{\mathbb{R}} \left[\frac{1}{2\pi} \overline{f_1 * f_2}(\varphi_\varepsilon, x) - \overline{f_1}(\varphi_\varepsilon, x)\overline{f_2}(\varphi_\varepsilon, x) \right] \psi(x)\widehat{\varphi}(\varepsilon x)dx \\
&= \frac{1}{4\pi^2} \int_{\mathbb{R}} \Big[\int_{\mathbb{R}\times\mathbb{R}} e^{ixy} f_1(\varphi_\varepsilon, y - z) f_2(\varphi_\varepsilon, z)\widehat{\varphi}(\varepsilon z)\widehat{\varphi}(\varepsilon y)dzdy \\
&\quad - \int_{\mathbb{R}\times\mathbb{R}} e^{ix(y_1+y_2)} f_1(\varphi_\varepsilon, y_1) f_2(\varphi_\varepsilon, y_2)\widehat{\varphi}(\varepsilon y_1)\widehat{\varphi}(\varepsilon y_2)dy_1dy_2 \Big] \\
&\quad \times \psi(x)\widehat{\varphi}(\varepsilon x)dx \\
&= \frac{1}{4\pi^2} \int_{(\mathbb{R})^3} e^{ixy} f_1(\varphi_\varepsilon, y - z) f_2(\varphi_\varepsilon, z)\widehat{\varphi}(\varepsilon z)[\widehat{\varphi}(\varepsilon y - \varepsilon z) \\
&\quad - \widehat{\varphi}(\varepsilon y)]\psi(x)\widehat{\varphi}(\varepsilon x)dzdydx,
\end{aligned} \tag{13}$$

where the change $y_1 + y_2 = y,\ y_2 = z$ is made.

Evaluate first the integral in z. Suppose that, in accordance with Definition 1, we have the estimate for the representatives

$$|f_1(\varphi_\varepsilon, z)| \le \varepsilon^{-l}(1 + |z|^l), \quad |f_2(\varphi_\varepsilon, z)| \le \varepsilon^{-m}(1 + |z|^m), \tag{14}$$

for some $l, m \in \mathbb{N}_0$ and all $z \in \mathbb{R}$. Supposing further that $\varphi \in A_q$, we fix some $p \in \mathbb{N}$, $p < q$. Replace then $\widehat{\varphi}(\varepsilon y - \varepsilon z)$ with its Taylor expansion, first around the point εy and then — around 0, up to order p. Taking into account equation (3), we get

$$\widehat{\varphi}(\varepsilon y - \varepsilon z) - \widehat{\varphi}(\varepsilon y) = (-\varepsilon)^p \sum_{j=1}^{p} \frac{(-1)^j}{j!(p-j)!} z^j y^{p-j} \widehat{\varphi}_j^{(p)}(\varepsilon\theta_j y) \tag{15}$$

Here each of the parameters θ_j ($j = 1, \dots, p$) is a fixed number in the interval $(0, 1)$. Further, the next estimate of any $\varphi \in A_0$ easily follows from (4):

$$|\widehat{\varphi}(\varepsilon z)| \le \frac{1 + o(1)}{1 + |z|^r} \quad \text{for all } r \in \mathbb{N},\ z \in \mathbb{R}, \text{ and } \varepsilon \to 0_+. \tag{16}$$

Now, in view of (14) and (16), we obtain the following estimate, for any $j = 1, \dots, p$:

$$
\begin{aligned}
(17) \qquad |I_j(y)| &:= \left| \int_{\mathbb{R}} f_1(\varphi_\varepsilon, y - z) f_2(\varphi_\varepsilon, z) \widehat{\varphi}(\varepsilon z) z^j \, dz \right| \\
&\leq \varepsilon^{-l-m} \int_{\mathbb{R}} (1 + |y - z|^l)(1 + |z|^m)|z|^p (1 + |z|^r)^{-1} dz \\
&\leq c \,.\, \varepsilon^{-l-m-n}[1 + o(1)](1 + |y|^n), \qquad \text{for some } c \in \mathbb{R}_+ \text{ and } n \in \mathbb{N}.
\end{aligned}
$$

Then, taking into account equations (15) and (17), we evaluate further

$$
\begin{aligned}
(18) \qquad 4\pi^2 \, |T(\varphi_\varepsilon)| &:= \Big| \int_{\mathbb{R}\times\mathbb{R}} e^{ixy} f_1(\varphi_\varepsilon, y - z) f_2(\varphi_\varepsilon, z) \widehat{\varphi}(\varepsilon z) \\
&\qquad \times [\widehat{\varphi}(\varepsilon y - \varepsilon z) - \widehat{\varphi}(\varepsilon y)] \, dz \, dy \Big| \\
&= \left| (-\varepsilon)^p \int_{\mathbb{R}} e^{ixy} \sum_{j=1}^{p} \frac{(-1)^j y^{p-j} \widehat{\varphi}^{(p)}(\varepsilon \theta_j y)}{j!(p-j)!} I_j(y) \, dy \right| \\
(19) \qquad &\leq c \,.\, \varepsilon^{p-l-m-n}[1 + o(1)] \int_{\mathbb{R}} (1 + |y|^n) |\widehat{\varphi}_p(\varepsilon y)| \, dy,
\end{aligned}
$$

for some $n \in \mathbb{N}_0$. Here we have put

$$
\widehat{\varphi}_p(\varepsilon y) := \sum_{j=1}^{p} \max_{0 \leq \theta_j \leq 1} \left| \frac{y^{p-j} \widehat{\varphi}^{(p)}(\varepsilon \theta_j y)}{j!(p-j)!} \right| .
$$

Now we can choose the paremeter $q \in \mathbb{N}$ to be such that $q - l - m - n =: t > 0$. Then, $\widehat{\varphi}_p(\varepsilon y)$ is a function in S such that $\widehat{\varphi}_p^{(k)}(0) = 0$, for $k = 0, \dots, q-p-1$, and therefore $|\widehat{\varphi}_p(\varepsilon y)| \leq c(\varphi_\varepsilon) \,.\, \varepsilon^{q-p}(1 + |y|^s)^{-1}$, for any $s \in \mathbb{N}$. Taking then into account this latter estimate and equation (19), we obtain

$$
\begin{aligned}
4\pi^2 |T(\varphi_\varepsilon)| &\leq c \,.\, \varepsilon^t [1 + o(1)] \int_{\mathbb{R}} (1 + |y|^n)(1 + |y|^s)^{-1} dy \\
(20) \qquad &\leq c_1 \,.\, \varepsilon^t [1 + o(1)].
\end{aligned}
$$

Applying successively equations (13), (18), (20), and (16), we finally get the estimate

$$
|\Delta_\psi(\varphi_\varepsilon)| \leq c_1 \,.\, \varepsilon^t \int_{\mathbb{R}} |\psi(x).(1 + |x|^r)^{-1}| \; [1 + o(1)] dx \;\leq\; c_2 \,.\, \varepsilon^t [1 + o(1)].
$$

Now, since $t > 0$, we have that $\lim_{\varepsilon \to 0_+} \Delta_\psi(\varphi_\varepsilon) = 0$. This completes the proof of the theorem.

Clearly the above proof remains valid for the Fourier transform as well. Now, if we set $\overline{f_i} = g_i, i = 1, 2$, equations $(11(a))$ and (12) imply the Fourier-product formula given by this.

COROLLARY 3.1. *For each two generalized functions $g_1, g_2 \in \mathscr{G}_\tau$, the following holds*

$$g_1 \cdot g_2 \approx \frac{1}{2\pi} \overline{\widehat{g_1} * \widehat{g_2}}. \tag{21}$$

4. Preliminary results on Fourier transform in $\mathscr{G}_\tau$

We proceed further to particular tempered distributions embedded in Colombeau algebra $\mathscr{G}_\tau$. Before this, we will prove a basic property of their elements. It is known that the inverse Fourier transform of a distribution with support in $\mathbb{R}_+$ can be obtained as a weak limit of its (one-sided) Laplace transform, as the imaginary part of the variable tends to 0. The same holds for the elements of $\mathscr{G}_\tau$ with support in $\mathbb{R}_+$.

PROPOSITION 4.1. *For each generalized function $f \in \mathscr{G}_\tau$ supported in $\mathbb{R}_+$, is*

$$\lim_{\chi\to 0_+} \int_{\mathbb{R}_+} e^{izt} f(t)\, dt = 2\pi \overline{f}(x), \quad \textit{where } z = x + i\chi \in \mathbb{C}. \tag{22}$$

Proof. According to definition (9), we have to evaluate the difference between two sides of (22) for some representative $f(\varphi_\varepsilon, t)$ of f:

$$\begin{aligned}\Delta(\varphi_\varepsilon) &:= \lim_{\chi\to 0_+} \int_{\mathbb{R}_+} e^{i(x+i\chi)t} f(\varphi_\varepsilon, t)\widehat{\varphi}(\varepsilon t)\, dt - \int_{\mathbb{R}_+} e^{ixt} f(\varphi_\varepsilon, t)\widehat{\varphi}(\varepsilon t)\, dt \\ &= \lim_{\chi\to 0_+} \int_{\mathbb{R}_+} [e^{-\chi t} - 1] e^{ixt} f(\varphi_\varepsilon, t)\widehat{\varphi}(\varepsilon t) dt.\end{aligned}$$

Observe that $e^{-\chi t} \to 1$, as $\chi \to 0_+$, uniformly regarding x. Since $|e^{-\chi t} - 1| \le 0$ on the half-line, the last integral above is uniformly convergent at infinity, and we can therefore pass to the limit as $\chi \to 0_+$ under the integral sign. This gives $\Delta(\varphi_\varepsilon) = 0$, and the proof is complete.

Recall next the definition of the distribution

$$(x \pm i0)^{-p} : \psi \mapsto \lim_{\chi\to 0_+} \langle (x \pm i\chi)^{-p}, \psi\rangle, \quad p \in \mathbb{N}, \tag{23}$$

for an arbitrary $\psi \in S(\mathbb{R})$. Then we prove the following:

THEOREM 4.1. *For each $p \in \mathbb{N}$, the embeddings in $\mathscr{G}_\tau$ of the distributions x_+^{p-1} and $(x + i0)^{-p}$ satisfy:*

$$\mathscr{F}^{-1}(\widetilde{x_+^{p-1}}) \approx \frac{i^p}{2\pi} \widetilde{(x + i0)^{-p}}. \tag{24}$$

Proof. One can easily check that

$$\partial_x \widetilde{x_+^p}(\varphi_\varepsilon, x) = p\widetilde{x_+^{p-1}}(\varphi_\varepsilon, x), \quad p \in \mathbb{N}; \quad \partial_x \widetilde{H}(\varphi_\varepsilon, x) = \widetilde{\delta}(\varphi_\varepsilon, x). \tag{25}$$

Applying Proposition 4.1 and defining equation (9), we get that for each $p \in \mathbb{N}$:

$$\begin{aligned}\mathscr{F}^{-1}(\widetilde{x_+^{p-1}})(\varphi_\varepsilon, x) &= \lim_{\chi\to 0_+} \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x+i\chi)y} \widetilde{x_+^{p-1}}(\varphi_\varepsilon, y)\widehat{\varphi}(\varepsilon y)\, dy \\ &= \lim_{\chi\to 0_+} \frac{i}{2\pi}(x+i\chi)^{-1}\Big[\int_{\mathbb{R}} e^{ixy-\chi y}\widetilde{x_+^{p-1}}(\varphi_\varepsilon, y)\varepsilon\widehat{\varphi}'(\varepsilon y)\, dy \\ &\quad + \int_{\mathbb{R}} e^{i(x+i\chi)y}\partial_y[\widetilde{x_+^{p-1}}(\varphi_\varepsilon, y)]\widehat{\varphi}(\varepsilon y)\, dy\Big] \\ &=: \lim_{\chi\to 0_+} [I_{p-1}(\varphi_\varepsilon, x+i\chi) + J_{p-1}(\varphi_\varepsilon, x+i\chi)]\,.\end{aligned}$$

Integrating by parts, we have used here that $|t|^n\widehat{\varphi}(t)\to 0$ as $|t|\to\infty$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$, the integrated term thus being 0. For any $\psi \in S(\mathbb{R})$ and $f \in \mathscr{G}_\tau$ with a representative $f(\varphi_\varepsilon, x)$ denote by

$$L_\psi(f) := \lim_{\varepsilon\to 0_+} \langle \lim_{\chi\to 0_+} f(\varphi_\varepsilon, x+i\chi), \psi\rangle.$$

Then we have

$$L_\psi(I_{p-1}) = \frac{i}{2\pi}(x+i0)^{-1}\langle \lim_{\varepsilon\to 0_+} \int_{\mathbb{R}} 1.e^{ixy}\widetilde{x_+^{p-1}}(\varphi_\varepsilon, y)\varepsilon\widehat{\varphi}'(\varepsilon y)\, dy,\ \psi(x)\rangle.$$

We have passed to the limit as $\chi\to 0_+$ under the integral sign on the same argument as in the proof of Proposition 4.1. Now, it can be checked that (as required by Definition 1.1) the following estimate holds: $|\widetilde{x_+^{p-1}}(\varphi_\varepsilon, y)| \le c\,.\,\varepsilon^{-p}(1+|y|^p)$. Choosing some $q \in \mathbb{N}$, $q > p$, we have for any $\varphi \in A_q$, by the Taylor theorem and (3): $\varepsilon\widehat{\varphi}'(\varepsilon y) = \sum_{j=1}^{q-1}(j!)^{-1}\varepsilon^j y^{j-1}\widehat{\varphi}^{(j)}(0) + \varepsilon^q\widehat{\varphi}^{(q)}(\varepsilon\theta y) = \varepsilon^q\widehat{\varphi}^{(q)}(\varepsilon\theta y)$, where $\theta \in (0,1)$. Denote further $\widehat{\varphi}_q(y) := \sup_{\varepsilon\in\mathbb{R}_+} \widehat{\varphi}^{(q)}(\varepsilon\theta y)$, which is well defined since $\widehat{\varphi} \in S(\mathbb{R})$. Due to the estimate

$$\begin{aligned}|\widetilde{x_+^{p-1}}(\varphi_\varepsilon, y)\varepsilon^q\widehat{\varphi}^{(q)}(\varepsilon\theta y)| &\le |\widetilde{x_+^{p-1}}(\varphi_\varepsilon, y)\varepsilon^q\widehat{\varphi}_q(y)| \\ &\le c\,.\,\varepsilon^{q-p}(1+|y|^p)\,.\,c_1(1+|y|^n)^{-1}\end{aligned}$$

for some $c \in \mathbb{R}_+$ and all $n \in \mathbb{N}$, $y \in \mathbb{R}$, and by the theorem for dominated convergence, we can pass to the limit as $\varepsilon\to 0_+$ under the integral sign. Since for any $p \in \mathbb{N}$ we can choose $q > p$, then

$$\begin{aligned}(26)\quad & L_\psi(I_{p-1}) \\ &= \frac{i}{2\pi}(x+i0)^{-1}\Big\langle \lim_{\varepsilon\to 0_+} \int_{\mathbb{R}} e^{ixy}\widetilde{x_+^{p-1}}(\varphi_\varepsilon, y)\varepsilon^q\widehat{\varphi}^{(q)}(\varepsilon\theta y)dy, \psi(x)\Big\rangle = 0.\end{aligned}$$

Consider next the term J_{p-1} above, by $p = 1$. Taking into account relation (5), equations (25), and the fact that $\delta(\varphi_\varepsilon, y)$ is supported in a compact subset of R, we get

$$(27)\quad L_\psi(J_0) = \lim_{\varepsilon\to 0_+}\left\langle \lim_{\chi\to 0_+}\frac{i}{2\pi}(x+i\chi)^{-1}\int_{\mathbb{R}} e^{-\chi y}e^{ixy}\widetilde{\delta}(\varphi_\varepsilon, y)\,dy, \psi(x)\right\rangle = \left\langle \frac{i}{2\pi}(x+i0)^{-1}, \psi(x)\right\rangle.$$

Employing further equation (25) and integrating by parts (the integrated part being again 0) we obtain

$$J_{p-1} = \frac{-i^2}{2\pi}(x+i\chi)^{-2}\int_{\mathbb{R}}\widetilde{x_+^{p-2}}(\varphi_\varepsilon, y)\widehat{\varphi}(\varepsilon y)d\left[e^{i(x+\chi)y}\right] = i(x+i\chi)^{-1}\,[I_{p-2}+J_{p-2}].$$

By (26), which holds for each $p\in\mathbb{N}$, we get: $L_\psi(J_{p-1}) = L_\psi(i(x+i\chi)^{-1}J_{p-2})$.

Iterating the above manipulation $(p-2)$ times and taking into account (27), we obtain

$$\lim_{\varepsilon\to 0_+}\langle\, \mathscr{F}^{-1}(\widetilde{x_+^{p-1}})(\varphi_\varepsilon, x), \psi\rangle = \lim_{\varepsilon\to 0_+}\langle \lim_{\chi\to 0_+} i^{p-1}(p-1)!(x+i\chi)^{-p+1}J_0, \psi\rangle = \left\langle \frac{i^p(p-1)!}{2\pi}(x+i0)^{-p}, \psi(x)\right\rangle.$$

This, in view of (5), proves the relation (24) for each $p\in\mathbb{N}$. The proof of the theorem is complete.

Combining now equations (11a) and (24), we get the following

COROLLARY 4.1. *For each* $p\in\mathbb{N}$, *it holds*

$$(28)\qquad \mathscr{F}\widetilde{(x+i0)}^{-p} \approx 2\pi i^{-p}\,\widetilde{x_+^{p-1}}.$$

We need also the following result.

THEOREM 4.2. *For each* $p\in\mathbb{N}$, *the embedding in* $\mathscr{G}_\tau$ *of the distribution* x_+^{p-1} *satisfies*:

$$(29)\qquad \widetilde{x_+^{p-1}} * \widetilde{x_+^{p-1}} \approx \widetilde{x_+^{2p-1}}.$$

Proof. Applying equation (10), we obtain for each $p\in\mathbb{N}$:

$$(30)\ C_{p-1}(x) := (\widetilde{x_+^{p-1}} * \widetilde{x_+^{p-1}})(\varphi_\varepsilon, x) = \int_{\mathbb{R}}\widetilde{x_+^{p-1}}(\varphi_\varepsilon, y)\widetilde{x_+^{p-1}}(\varphi_\varepsilon, x-y)\widehat{\varphi}(\varepsilon y)\,dy$$
$$= \int_{\mathbb{R}}\widetilde{x_+^{p}}(\varphi_\varepsilon, x-y)\widetilde{x_+^{p-1}}(\varphi_\varepsilon, y)\varepsilon\widehat{\varphi}'(\varepsilon y)\,dy$$
$$+\frac{p-1}{p}\int_{\mathbb{R}}\widetilde{x_+^{p}}(\varphi_\varepsilon, x-y)\widetilde{x_+^{p-2}}(\varphi_\varepsilon, y)\widehat{\varphi}(\varepsilon y)\,dy$$
$$=: I_p + J_{p,p-2}.$$

Integrating by parts, we have again used that $|t|^n\widehat{\varphi}(t)\to 0$ as $|t|\to\infty$ for all $n\in\mathbb{N}$ and $t\in\mathbb{R}$, the integrated term being 0. For an arbitrary $\psi\in S(\mathbb{R})$, we can write

$$\lim_{\varepsilon\to 0_+}\langle I_p,\psi\rangle=\Big\langle\lim_{\varepsilon\to 0_+}\int_{\mathbb{R}}\widetilde{x_+^p}(\varphi_\varepsilon,x-y)\widetilde{x_+^{p-1}}(\varphi_\varepsilon,y)\varepsilon\widehat{\varphi}'(\varepsilon y)\,dy,\ \psi(x)\Big\rangle.$$

The following estimate holds:

$$|\widetilde{x_+^p}(\varphi_\varepsilon,x-y)\widetilde{x_+^{p-1}}(\varphi_\varepsilon,y)|\le c\,.\,\varepsilon^{-2p+1}(1+|x-y|^p)(1+|y|^{p-1}).$$

Choosing some $q\in\mathbb{N}$, $q>2p-1$, we have for any $\varphi\in A_q$, by the Taylor theorem and (3): $\varepsilon\widehat{\varphi}'(\varepsilon y)=\varepsilon^q\widehat{\varphi}^{(q)}(\varepsilon\theta y)$, $\theta\in(0,1)$. Denote $\widehat{\varphi}_q(y):=\sup_{\varepsilon\in\mathbb{R}_+}\widehat{\varphi}^{(q)}(\varepsilon\theta y)$. Now, on the same argument as in the proof of (24), we get the estimate:

$$\begin{aligned}&|\widetilde{x_+^p}(\varphi_\varepsilon,x-y)\widetilde{x_+^{p-1}}(\varphi_\varepsilon,y)\varepsilon^q\widehat{\varphi}^{(q)}(\varepsilon\theta y)|\\&\qquad\le c\,.\,\varepsilon^{q-2p+1}(1+|x-y|^p)(1+|y|^{p-1})(1+|y|^n)^{-1},\end{aligned}$$

for some $c\in\mathbb{R}_+$ and all $n\in\mathbb{N}$, $x,y\in\mathbb{R}$. Since $\psi(x)\in S$, by the theorem for dominated convergence, we can pass to the limit as $\varepsilon\to 0_+$ under the integral. For we can choose $q>2p-1$, it therefore holds for each $p\in\mathbb{N}$:

$$\lim_{\varepsilon\to 0_+}\langle\, I_p,\psi\rangle=0. \tag{31}$$

Employing further relation(25) and integrating by parts — the integrated part being again 0 — we obtain:

$$J_{p,p-2}=-\int_{\mathbb{R}}\widetilde{x_+^{p-2}}(\varphi_\varepsilon,y)\widehat{\varphi}(\varepsilon y)d[\widetilde{{}_+^{p+1}}(\varphi_\varepsilon,x-y)]=I_{p+1}+J_{p+1,p-3}.$$

Thus, by (31), we get $\lim_{\varepsilon\to 0_+}\langle J_{p,p-2},\psi\rangle=\lim_{\varepsilon\to 0_+}\langle J_{p+1,p-3},\psi\rangle$. Iterating the above manipulation $(p-2)$ times, taking into account successively equations (30), (31), (25), and (11b), we eventually get

$$\begin{aligned}&\lim_{\varepsilon\to 0_+}\langle C_{p-1}(x),\psi(x)\rangle\\&\qquad=\lim_{\varepsilon\to 0_+}\langle J_{2p-1,-1},\psi(x)\rangle\\&\qquad=\lim_{\varepsilon\to 0_+}\frac{[(p-1)!]^2}{(2p-1)!}\Big\langle\int_{\mathbb{R}}\widetilde{x_+^{2p-1}}(\varphi_\varepsilon,x-y)\widetilde{\delta}(\varphi_\varepsilon,y)\widehat{\varphi}(\varepsilon y)\,dy,\psi(x)\Big\rangle\\&\qquad=\lim_{\varepsilon\to 0_+}\frac{[(p-1)!]^2}{(2p-1)!}\langle(\widetilde{x_+^{2p-1}}*\widetilde{\delta})(\varphi_\varepsilon,x),\psi(x)\rangle\\&\qquad=\frac{[(p-1)!]^2}{(2p-1)!}\langle\widetilde{x_+^{2p-1}},\psi(x)\rangle.\end{aligned}$$

This proves relation (29), for arbitrary $p\in\mathbb{N}$.

5. Generalization of the basic Mikusiński equation

We are now ready for the final step towards the extension of the basic Mikusiński equation.

THEOREM 5.1. *For arbitrary* $p \in \mathbb{N}$, *the embedding in* $\mathscr{G}_\tau$ *of the distributions* x^{-p} *and* $\delta^{(p-1)}(x)$ *satisfy:*

$$\widetilde{x^{-p}} \,.\, \widetilde{x^{-p}} - \frac{\pi^2}{[(p-1)]^2} \widetilde{\delta^{(p-1)}}(x) \,.\, \widetilde{\delta^{(2-1)}}(x) \approx x^{-2p}. \tag{32}$$

P r o o f. (i) Applying consecutively the Fourier-product formula (21) and the results given by relations (28), (29), (24), we have, for arbitrary $p \in \mathbb{N}$:

$$\begin{aligned} \widetilde{(x+i0)^{-p}} \,.\, \widetilde{(x+i0)^{-p}} &\approx \frac{1}{2\pi} \mathscr{F}^{-1}[(\mathscr{F}\widetilde{(x+i0)^{-p}}) * (\mathscr{F}\widetilde{(x+i0)^{-p}})] \\ &\approx \frac{1}{2\pi} \frac{4\pi^2}{i^{2p}(p-1)!(p-1)!} \mathscr{F}^{-1}(\widetilde{x_+^{p-1}} * \widetilde{x_+^{p-1}}) \\ &\approx \frac{2\pi}{i^{2p}(2p-1)!} \mathscr{F}^{-1}(\widetilde{x_+^{2p-1}}) \approx \widetilde{(x+i0)^{-2p}}. \end{aligned} \tag{33}$$

(ii) Next we prove the following formula that translates exactly from distribution theory [9] into Colombeau algebra:

$$\widetilde{(x+i0)^{-p}} = \widetilde{x^{-p}} - i\pi \frac{(-1)^{p-1}}{(p-1)!} \widetilde{\delta^{(p-1)}}(x), \quad x \in \mathbb{R}. \tag{34}$$

By definition, $x^{-p} = (-1)^{p-1}/(p-1)! d^p/dx^p(\ln x)$, $p \in \mathbb{N}$. Thus, for $x \in \mathbb{R}$, we have the representation

$$\widetilde{x^{-p}}(\varphi_\varepsilon, x) = \frac{(-1)^{2p-1}}{(p-1)!\varepsilon^{p+1}} \int_{-\varepsilon l+x}^{\varepsilon l+x} \ln|y| \varphi^{(p)}\left(\frac{y-x}{\varepsilon}\right) dy. \tag{35}$$

Here, it is taken into account that, if $\operatorname{supp} \varphi(x) \subseteq [-l, l]$ for some $l \in \mathbb{R}$, then $\operatorname{supp} \varphi((y-x)/\varepsilon) \subseteq [-\varepsilon l + x, \varepsilon l + x]$. Also,

$$\begin{aligned} \widetilde{\delta^{(p-1)}}(\varphi_\varepsilon, x) &= (-1)^{p-1}\varepsilon^{-p}\langle \delta_y, \varphi^{(p-1)}((y-x)/\varepsilon)\rangle \\ &= (-1)^{p-1}\varepsilon^{-p}\varphi^{(p-1)}(-x/\varepsilon). \end{aligned} \tag{36}$$

Now, replacing $(y-x)/\varepsilon = v$, and taking into account (35) and (36), we have:

$$\begin{aligned} &\widetilde{(x+i0)^{-p}}(\varphi_\varepsilon, x) \\ &\quad = \lim_{\chi \to 0_+} \frac{(-1)^{2p-1}}{(p-1)!\varepsilon^{p+1}} \int_{-\varepsilon l+x}^{\varepsilon l+x} \ln(y+i\chi)\varphi^{(p)}\left(\frac{y-x}{\varepsilon}\right) dy \\ &\quad = \lim_{\chi \to 0_+} \frac{(-1)^{2p-1}}{(p-1)!\varepsilon^{p}} \int_{-l}^{l} (\ln|v+\varepsilon x+i\chi| + i\arg(v+\varepsilon x+i\chi))\varphi^{(p)}(v) dv \end{aligned}$$

$$= \frac{(-1)^{2p-1}}{(p-1)!\varepsilon^p} \int_{-l}^{l} \Big(\ln|v+\varepsilon x| + i\pi[1 - H(v+\varepsilon x)]\Big) \varphi^{(p)}(v)dv$$

$$= \frac{(-1)^{2p-1}}{(p-1)!\varepsilon^p} \Big(\int_{-l}^{l} \ln|v+\varepsilon x| \varphi^{(p)}(v)dv - i\pi \int_{-x/\varepsilon}^{l} \varphi^{(p)}(v)dv \Big)$$

$$= \widetilde{x^{-p}}(\varphi_\varepsilon, x) - i\pi \frac{(-1)^{p-1}}{(p-1)!} \widetilde{\delta^{(p-1)}}(\varphi_\varepsilon, x).$$

This proves equality (34) for a fixed parameter function φ. When φ is running the set A_q, we get the representative class of the embeddings in $\mathcal{G}_\tau$ of the distributions on the left-hand side, respectively, right-hand side of (34). This establishes a one-to-one correspondence between these classes, which amounts to an equality in $\mathcal{G}_\tau$ of the corresponding generalized functions.

(iii) Consider for any $p \in \mathbb{N}$, the difference of the two sides of (33), taking into account (5):

$$\widetilde{(x+i0)^{-p}} \,.\, \widetilde{(x+i0)^{-p}} - \widetilde{(x+i0)^{-2p}} \approx 0.$$

Applying equality (34) and taking into account (6) (by $d = 1$), we have for the latter equation

$$\widetilde{x^{-p}} \,.\, \widetilde{x^{-p}} - \frac{\pi^2}{[(p-1)!]^2} \widetilde{\delta^{(p-1)}} \,.\, \widetilde{\delta^{(p-1)}} - 2i\pi \frac{(-1)^{p-1}}{(p-1)!} \widetilde{x^{-p}} \,.\, \widetilde{\delta^{(p-1)}}$$

$$-\widetilde{x^{-2p}} + i\pi \frac{(-1)^{2p-1}}{(2p-1)!} \widetilde{\delta^{(2p-1)}} = \widetilde{x^{-p}} \,.\, \widetilde{x^{-p}} - \frac{\pi^2}{[(p-1)!]^2} \widetilde{\delta^{(p-1)}} \,.\, \widetilde{\delta^{(p-1)}} - \widetilde{x^{-2p}} \approx 0.$$

This gives equation (32) for arbitrary $p \in \mathbb{N}$, and the theorem is proved.

REMARK. As is specific for the Mikusiński type products, the individual summands in (32) do not admit associated distribution, but their sum considered as a single entity is associated with the distribution x^{-2p}.

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Received February 22nd, 2000.