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MODIFIED H-TRANSFORMS IN $\mathfrak{L}_{\nu,r}$-SPACES

Abstract. The paper is devoted to study the integral transforms

$$(\boldsymbol{H}^1 f)(x) = \int_0^\infty H^{m,n}_{p,q}\left[\frac{x}{t}\right] f(t)\,\frac{dt}{t}, \quad (\boldsymbol{H}^2 f)(x) = \int_0^\infty H^{m,n}_{p,q}\left[\frac{t}{x}\right] f(t)\,\frac{dt}{x},$$

containing the H-function in the kernel, on the space $\mathfrak{L}_{\nu,r}$ ($\nu \in (-\infty,\infty)$; $1 \leqq r \leqq \infty$) of Lebesgue measurable functions f on $(0,\infty)$ such that

$$\int_0^\infty |t^\nu f(t)|^r \,\frac{dt}{t} < \infty \text{ for } 1 \leqq r < \infty; \ \operatorname*{ess\,sup}_{x>0} |x^\nu f(x)| < \infty \text{ for } r = \infty.$$

We show the boundedness, the representation, the range and the inversion formulas for the transform.

1. Introduction

The paper deals with the integral transforms

$$(1.1) \qquad (\boldsymbol{H}^1 f)(x) = \int_0^\infty H^{m,n}_{p,q}\left[\frac{x}{t}\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right] f(t)\frac{dt}{t} \qquad (x>0)$$

and

$$(1.2) \qquad (\boldsymbol{H}^2 f)(x) = \int_0^\infty H^{m,n}_{p,q}\left[\frac{t}{x}\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right] f(t)\frac{dt}{x} \qquad (x>0)$$

containing the H-function in the kernel. For integers m,n,p,q ($0 \leqq m \leqq q$, $0 \leqq n \leqq p$), for complex a_i, b_j and positive α_i, β_j ($1 \leqq i \leqq p$; $1 \leqq j \leqq q$) such a function is defined by

$$(1.3) \qquad H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathfrak{L}} \mathcal{H}^{m,n}_{p,q}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\,\middle|\, s\right] z^{-s}\,ds,$$

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where

$$\mathcal{H}(s) \equiv \mathcal{H}^{m,n}_{p,q}\left[\begin{array}{c|c}(a_i,\alpha_i)_{1,p} & \\ (b_j,\beta_j)_{1,q} & s\end{array}\right] = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)}, \tag{1.4}$$

the contour $\mathfrak{L}$ is specially chosen in the complex plane $\mathbb{C}$ (if an empty product in (1.4) occurs, then it is taken to be one). The theory of this function may be found in [3, Section 1.19], [2], [16, Chapter 1], [25, Chapter 2] and [18, § 8.3].

In the paper we study the transforms (1.1) and (1.2) in the spaces $\mathfrak{L}_{\nu,r}$ of complex-valued Lebesgue measurable functions f on $\mathbb{R}_+ = (0,\infty)$ such that $\|f\|_{\nu,r} < \infty$, where $1 \leqq r \leqq \infty$, $\nu \in \mathbb{R} = (-\infty,\infty)$,

$$\|f\|_{\nu,r} = \left(\int_0^\infty |t^\nu f(t)|^r \frac{dt}{t}\right)^{1/r} \quad (1 \leqq r < \infty) \tag{1.5}$$

and

$$\|f\|_{\nu,\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}_+} |x^\nu f(x)| \quad (r = \infty). \tag{1.6}$$

We obtain the properties of the transforms $\boldsymbol{H}^1$ and $\boldsymbol{H}^2$ such as the boundedness, the representation and the range and give their inversion formulas.

We note that the boundedness and inversion of the modified $\boldsymbol{H}$-transform (1.1) were studied by McBride and Spratt [17] in a certain subspace of $\mathfrak{L}_{\nu,r}$. We also mention that several authors have investigated the mapping and composition properties for the transforms of the form (1.1) and (1.2), in which the integration over $\mathbb{R}_+$ is replaced by the integration over $(0,x)$ and (x,∞), respectively. Such operators with the $H^{m,0}_{m,m}$-function in the kernels were studied by Kiryakova [14] and by Kalla and Kiryakova [8] (see also the book by Kiryakova [15]) in the space $L_r(\mathbb{R}_+)$ $(r \geqq 1)$, and by Raina and Saigo [19] and by Saigo, Raina and Kilbas [22] in spaces of tested and generalized functions by McBride. The latter results were extended by the authors [10] and [21] to more general transforms with $H^{m,n}_{p,q}$-function as kernels.

The transforms $\boldsymbol{H}^1$ and $\boldsymbol{H}^2$ are the modifications of the following integral transform with the H-function kernel

$$(\boldsymbol{H}f)(x) = \int_0^\infty H^{m,n}_{p,q}\left[xt\left|\begin{array}{c}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{array}\right.\right] f(t)dt \quad (x > 0) \tag{1.7}$$

called $\boldsymbol{H}$-transform. When $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$, the function (1.3) turns to be the Meijer G-function [3, Chapter 5.3] and (1.7) is reduced to the so-called integral transform with G-function kernel or $\boldsymbol{G}$-transform [20]. The classical Laplace and Hankel transforms, the Riemann-Liouville fractional integrals, the even and odd Hilbert transforms, the integral transforms involving the Gauss hypergeometric function, etc. can be reduced to these $\boldsymbol{G}$-transforms, whose theory and historical notices can be found in [23, §36, §39]. There are other transforms which cannot be reduced to $\boldsymbol{G}$-transforms but can lead into the $\boldsymbol{H}$-transforms given in (1.5): the modified Laplace and Hankel transforms [23, §39.2, §36.4], the Erdélyi-Kober type fractional integration operators (2.15) and (2.16), below, [23, §18.1], the transforms involving the Gauss hypergeometric function as kernel [23, §23, §39], the Bessel-type integral transforms [5], and so on.

The integral transform (1.7) was first considered by Fox [4] while investigating G- and H-functions as symmetrical Fourier kernels. Many authors investigated the properties of the $\boldsymbol{H}$-transforms in $L_1(0,\infty)$, $L_2(0,\infty)$ and some special function spaces (see a short survey and bibliography in [11]–[13]). Mapping properties such as the boundedness, the representation and the range of the $\boldsymbol{H}$-transform (1.7) were proved independently by the authors together with Shlapakov in [11]–[13] and with Shlapakov and Glaeske in [6]–[7] and by Betancor and Jerez Diaz [1]. The invertibility of (1.7) in $\mathfrak{L}_{\nu,r}$ was given by the authors together with Shlapakov in [24]. We note that the investigation of the boundedness, representation and range of the $\boldsymbol{H}$-transform (1.7) is based on the technique of Mellin transform developed by Rooney [20] for the $\boldsymbol{G}$-transform, while the invertibility of such a transform is due to the asymptotic behavior of the function $H^{m,n}_{p,q}(z)$ in (1.4) considered by the authors in [9].

In this paper we apply the results of [11]–[13] and [6]–[7] to investigate such properties of the $\boldsymbol{H}^1$-and $\boldsymbol{H}^2$-transforms (1.1) and (1.2). Section 2 contains some auxiliary results, definitions and notations. The boundedness and the representations for the the modified $\boldsymbol{H}$-tranforms (1.1) and (1.2) in the space $\mathfrak{L}_{\nu,2}$ are presented in Section 3. Sections 4 and 5 deal with the same for the space $\mathfrak{L}_{\nu,r}$. Section 6 is devoted to the inversion of these tranforms in $\mathfrak{L}_{\nu,r}$.

2. Preliminaries

In this section we present some auxiliary results, definitions and notations which will be used later.

First we note that the modified $\boldsymbol{H}$-transforms (1.1) and (1.2) are connected with the $\boldsymbol{H}$-transform (1.7) by the relations

$$(\boldsymbol{H}^1 f)(x) = (\boldsymbol{H}Rf)(x) \tag{2.1}$$

and

$$(\boldsymbol{H}^2 f)(x) = (R\boldsymbol{H}f)(x), \tag{2.2}$$

in terms of the elementary operator R defined by

$$(Rf)(x) = \frac{1}{x} f\left(\frac{1}{x}\right). \tag{2.3}$$

Next we note that, for $f \in \mathfrak{L}_{\nu,r}$ with $1 \leqq r \leqq 2$, the Mellin transform $\mathfrak{M}f$ is defined by

$$(\mathfrak{M}f)(s) = \int_{-\infty}^{\infty} e^{(\sigma+it)\tau} f(e^{\tau}) d\tau \quad (s = \sigma + it,\ \sigma, t \in \mathbb{R}), \tag{2.4}$$

and, if $f \in \mathfrak{L}_{\nu,r} \bigcap \mathfrak{L}_{\nu,1}$ and $\mathrm{Re}(s) = \nu$, (2.4) coincides with the usual Mellin transform $\mathfrak{M}f$:

$$(\mathfrak{M}f)(s) = \int_0^{\infty} f(t) t^{s-1} dt \tag{2.5}$$

(see [20]). It can be directly checked that, for "sufficiently good functions" f, the Mellin transform of (1.7) is given by

$$(\mathfrak{M}\boldsymbol{H}f)(s) = \mathcal{H}_{p,q}^{m,n}(s)(\mathfrak{M}f)(1-s), \tag{2.6}$$

where $\mathcal{H}_{p,q}^{m,n}(s)$ is defined by (1.4).

Following [11]–[13], [6]–[7] and [24] we use the notation

$$\alpha = \begin{cases} \max\left[-\dfrac{\mathrm{Re}(b_1)}{\beta_1}, \cdots, -\dfrac{\mathrm{Re}(b_m)}{\beta_m}\right] & \text{if } m > 0, \\ -\infty & \text{if } m = 0; \end{cases} \tag{2.7}$$

$$\beta = \begin{cases} \min\left[\dfrac{1-\mathrm{Re}(a_1)}{\alpha_1}, \cdots, \dfrac{1-\mathrm{Re}(a_n)}{\alpha_n}\right] & \text{if } n > 0, \\ \infty & \text{if } n = 0; \end{cases} \tag{2.8}$$

$$a^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j; \tag{2.9}$$

$$a_1^* = \sum_{j=1}^{m} \beta_j - \sum_{i=n+1}^{p} \alpha_i; \quad a_2^* = \sum_{i=1}^{n} \alpha_i - \sum_{j=m+1}^{q} \beta_j; \tag{2.10}$$

$$\Delta = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i; \quad \delta = \prod_{i=1}^{p} \alpha_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^{\beta_j}; \tag{2.11}$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}; \tag{2.12}$$

$$(2.13)\qquad \alpha_0 = \begin{cases} \max\left[\dfrac{\mathrm{Re}(b_{m+1})-1}{\beta_{m+1}}+1,\cdots,\dfrac{\mathrm{Re}(b_q)-1}{\beta_q}+1\right] & \text{if } q>m,\\ -\infty & \text{if } q=m;\end{cases}$$

$$(2.14)\qquad \beta_0 = \begin{cases} \min\left[\dfrac{\mathrm{Re}(a_{n+1})}{\alpha_{n+1}}+1,\cdots,\dfrac{\mathrm{Re}(a_p)}{\alpha_p}+1\right] & \text{if } p>n,\\ \infty & \text{if } p=n.\end{cases}$$

Note that $a^* = a_1^* + a_2^*$ and $\Delta = a_1^* - a_2^*$.

We denote by $\mathcal{E}_{\mathcal{H}}$ the exceptional set of the function $\mathcal{H}$ defined in (1.4) which is the set of real numbers ν such that $\alpha < 1-\nu < \beta$ and $\mathcal{H}(s)$ has a zero on the line $\mathrm{Re}(s) = 1-\nu$. The symbol $[X, Y]$ represents the collection of bounded linear operators from a Banach space X into a Banach space Y. Specially, $[X, X]$ is denoted by $[X]$.

We also need the Erdélyi-Kober type fractional integral operators [23, §18.1] defined for α, $\eta \in \mathbb{C}$ ($\mathrm{Re}(\alpha) > 0$) and $\sigma > 0$ by

$$(2.15)\qquad (I^{\alpha}_{0+;\sigma,\eta}f)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_0^x (x^\sigma - t^\sigma)^{\alpha-1}t^{\sigma\eta+\sigma-1}f(t)dt \quad (x>0),$$

$$(2.16)\qquad (I^{\alpha}_{-;\sigma,\eta}f)(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)}\int_x^\infty (t^\sigma - x^\sigma)^{\alpha-1}t^{\sigma(1-\alpha-\eta)-1}f(t)dt \quad (x>0),$$

the modified Hankel and Laplace transforms [23, §39.2, §36.4] defined, for real $k \neq 0$ and for the Bessel function of first kind $J_\eta(z)$ of order $\eta \in \mathbb{C}$, by

$$(2.17)\qquad (\mathbb{H}_{k,\eta}f)(x) = \int_0^\infty (xt)^{1/k-1/2}J_\eta(|k|(xt)^{1/k})f(t)dt \quad (\mathrm{Re}(\eta) > -1, x>0),$$

$$(2.18)\qquad (\mathbb{L}_{k,\alpha}f)(x) = \int_0^\infty (xt)^{-\alpha}e^{-|k|(xt)^{1/k}}f(t)dt \quad (\alpha\in\mathbb{C}, x>0),$$

and elementary operators M_ζ and W_δ defined for $\zeta \in \mathbb{C}$ and $\delta > 0$ by

$$(2.19)\qquad (M_\zeta f)(x) = x^\zeta f(x),$$

$$(2.20)\qquad (W_\delta f)(x) = f\left(\frac{x}{\delta}\right).$$

The following assertions for the operators R in (2.3) and above M_ζ, W_δ are verified directly (see also [20, Section 2]).

LEMMA 2.1. *Let $\nu \in \mathbb{R}$ and $1 \leqq r \leqq \infty$.*

(a) *R is an isometric isomorphism of $\mathfrak{L}_{\nu,r}$ onto $\mathfrak{L}_{1-\nu,r}$ and*

$$(2.21)\qquad \|Rf\|_{1-\nu,r} = \|f\|_{\nu,r}.$$

(b) *M_ζ with $\zeta \in \mathbb{C}$ is an isometric isomorphism of $\mathfrak{L}_{\nu,r}$ onto $\mathfrak{L}_{\nu-\mathrm{Re}(\zeta),r}$ and*

$$\|M_\zeta f\|_{\nu-\mathrm{Re}(\zeta),r} = \|f\|_{\nu,r}. \tag{2.22}$$

(c) *W_δ with $\delta > 0$ is an isomorphism of $\mathfrak{L}_{\nu,r}$ onto $\mathfrak{L}_{\nu,r}$ and*

$$\|W_\delta f\|_{\nu,r} = \delta^{-\nu}\|f\|_{\nu,r}. \tag{2.23}$$

LEMMA 2.2. *Let $f \in \mathfrak{L}_{\nu,r}$, where $\nu \in \mathbb{R}$ and $1 \leqq r \leqq 2$, and let $\mathfrak{M}$ be the Mellin transform* (2.4). *Then there the following relations are satisfied*

$$(\mathfrak{M}Rf)(s) = (\mathfrak{M}f)(1-s) \quad (\mathrm{Re}(s) = 1-\nu); \tag{2.24}$$

$$(\mathfrak{M}M_\zeta f)(s) = (\mathfrak{M}f)(s+\zeta) \quad (\mathrm{Re}(s) = \nu - \mathrm{Re}(\zeta)); \tag{2.25}$$

$$(\mathfrak{M}W_\delta f)(s) = \delta^{-s}(\mathfrak{M}f)(s) \quad (\mathrm{Re}(s) = \nu). \tag{2.26}$$

3. H^1- and H^2-transforms in the space $\mathfrak{L}_{\nu,2}$

In this section we present some properties, including the boundedness and the representation, of the modified $\boldsymbol{H}$-transforms (1.1) and (1.2) in the space $\mathfrak{L}_{\nu,2}$. The first result for the $\boldsymbol{H}^1$-transform is given by the following statement in which α, β, a^*, Δ and μ are defined in (2.7), (2.8), (2.9), (2.11) and (2.12), respectively.

THEOREM 3.1. *We suppose that* **(i)** *$\alpha < \nu < \beta$ and that either of conditions* **(ii)** *$a^* > 0$ or* **(iii)** *$a^* = 0$, $\Delta\nu + \mathrm{Re}(\mu) \leqq 0$ hold. Then we have the following results:*

(a) *There is a one-to-one transform $\boldsymbol{H}^1 \in [\mathfrak{L}_{\nu,2}]$ so that the relation*

$$(\mathfrak{M}\boldsymbol{H}^1 f)(s) = \mathcal{H}^{m,n}_{p,q}(s)(\mathfrak{M}f)(s) \tag{3.1}$$

holds for $f \in \mathfrak{L}_{\nu,2}$ and $\mathrm{Re}(s) = \nu$. If $a^* = 0$, $\Delta\nu + \mathrm{Re}(\mu) = 0$ and $1-\nu \notin \mathcal{E}_{\mathcal{H}}$, then the transform $\boldsymbol{H}^1$ maps $\mathfrak{L}_{\nu,2}$ onto $\mathfrak{L}_{\nu,2}$.

(b) *If $f \in \mathfrak{L}_{\nu,2}$ and $g \in \mathfrak{L}_{1-\nu,2}$, then the relation of fractional integration by parts*

$$\int_0^\infty f(x)(\boldsymbol{H}^1 g)(x)\,dx = \int_0^\infty g(x)(\boldsymbol{H}^2 f)(x)\,dx \tag{3.2}$$

holds, where $\boldsymbol{H}^2$ is the modified $\boldsymbol{H}$-transform (1.2).

(c) *Let $\lambda \in \mathbb{C}, h \in \mathbb{R}_+$ and $f \in \mathfrak{L}_{\nu,2}$. If $\mathrm{Re}(\lambda) > \nu h - 1$, then $\boldsymbol{H}^1 f$ is given by*

$$(3.3)\quad (\boldsymbol{H}^1 f)(x) = hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h} \times \int_0^\infty H^{m,n+1}_{p+1,q+1}\left[\frac{x}{t}\left| \begin{matrix}(-\lambda,h),(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\ (b_1,\beta_1),\cdots,(b_q,\beta_q),(-\lambda-1,h)\end{matrix}\right.\right]\frac{f(t)}{t}dt \quad (x>0).$$

If $\mathrm{Re}(\lambda) < \nu h - 1$, *then* $\boldsymbol{H}^1 f$ *is givn by*

$$(3.4)\quad (\boldsymbol{H}^1 f)(x) = -hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h} \times \int_0^\infty H^{m+1,n}_{p+1,q+1}\left[\frac{x}{t}\left| \begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p),(-\lambda,h)\\ (-\lambda-1,h),(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right.\right]\frac{f(t)}{t}dt \quad (x>0).$$

(d) *The transform* $\boldsymbol{H}^1$ *is independent on* ν *in the sense that, if* ν *and* $\widetilde{\nu}$ *satisfy* (i), *and* (ii) *or* (iii), *and if the transforms* $\boldsymbol{H}^1$ *and* $\widetilde{\boldsymbol{H}^1}$ *are defined in* $\mathfrak{L}_{\nu,2}$ *and* $\mathfrak{L}_{\widetilde{\nu},2}$, *respectively, by* (3.1), *then* $\boldsymbol{H}^1 f = \widetilde{\boldsymbol{H}^1} f$ *for* $f \in \mathfrak{L}_{\nu,2} \bigcap \mathfrak{L}_{\widetilde{\nu},2}$.

(e) *If* $a^* > 0$ *or if* $a^* = 0$, $\Delta\nu + \mathrm{Re}(\mu) < -1$, *then for* $f \in \mathfrak{L}_{\nu,2}$, $\boldsymbol{H}^1 f$ *is given by* (1.1).

P r o o f. Due to the equality $R(\mathfrak{L}_{\nu,2}) = \mathfrak{L}_{1-\nu,2}$ from Lemma 2.1(a), the results in (a), (d) and (e) follow by virtue of the corresponding statements for the $\boldsymbol{H}$-transform (1.7) given in [11, Theorems 3 and 4]. In fact, taking into account (2.6) and (2.24), we have

$$(3.5)\quad (\mathfrak{M}\boldsymbol{H}^1 f)(s) = (\mathfrak{M}\boldsymbol{H}Rf)(s) = \mathcal{H}^{m,n}_{p,q}(s)(\mathfrak{M}Rf)(1-s) = \mathcal{H}^{m,n}_{p,q}(s)(\mathfrak{M}f)(s),$$

which yields (3.1).

For "sufficiently good" functions f and g, the formula (3.2) is proved directly. For $f \in \mathfrak{L}_{\nu,2}$ and $g \in \mathfrak{L}_{1-\nu,2}$ it is sufficient to show that the both sides of (3.2) represent bounded bilinear functionals on $\mathfrak{L}_{\nu,2} \times \mathfrak{L}_{1-\nu,2}$. In view of (i) and the Schwarz inequality we have

$$(3.6)\quad \left|\int_0^\infty f(x)(\boldsymbol{H}^1 g)(x)dx\right| = \left|\int_0^\infty [x^{\nu-1/2}f(x)][x^{-\nu+1/2}(\boldsymbol{H}^1 g)(x)]dx\right| \leqq \|f\|_{\nu,2}\|\boldsymbol{H}^1 g\|_{1-\nu,2} \leqq K\|f\|_{\nu,2}\|g\|_{1-\nu,2},$$

where $K > 0$ is a bound for $\boldsymbol{H}^1 \in [\mathfrak{L}_{1-\nu,2}]$. Hence, the left-hand side of (3.2) represents a bounded bilinear functional on $\mathfrak{L}_{\nu,2} \times \mathfrak{L}_{1-\nu,2}$, and the same is for the right-hand side of (3.2), which proves (b).

The formulas (3.3) and (3.4) are proved on the basis of the corresponding representations for the $\boldsymbol{H}$-transform (1.7). If $f \in \mathfrak{L}_{\nu,2}$ and $\mathrm{Re}(\lambda) > (1-\nu)h-1$

then it is known [11, Theorem 3] that

$$(3.7)\qquad (\boldsymbol{H}f)(x) = hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}\int_0^\infty H^{m,n+1}_{p+1,q+1}[xt]f(t)dt,$$

where

$$(3.8)\qquad H^{m,n+1}_{p+1,q+1}(z) = H^{m,n+1}_{p+1,q+1}\left[z\middle|\begin{matrix}(-\lambda,h),(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q),(-\lambda-1,h)\end{matrix}\right].$$

For $f \in \mathfrak{L}_{\nu,2}$ and $\mathrm{Re}(\lambda) > \nu h - 1$, in view of Lemma 2.1(a), (3.7) with f being replaced by Rf, and (2.1) we have

$$(3.9)\quad (\boldsymbol{H}^1f)(x) = (\boldsymbol{H}Rf)(x) = hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}\int_0^\infty H^{m,n+1}_{p+1,q+1}[xt]\frac{1}{t}f\left(\frac{1}{t}\right)dt,$$

which yields (3.3) (after the change t by $1/t$ in the integrand). (3.4) is proved similarly by using the fact [11, Theorem 3] that, for $f \in \mathfrak{L}_{\nu,2}$ and $\mathrm{Re}(\lambda) < \nu h - 1$, the following the relation is satisfied

$$(3.10)\qquad (\boldsymbol{H}f)(x) = -hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}\int_0^\infty H^{m+1,n}_{p+1,q+1}[xt]f(t)dt,$$

where

$$(3.11)\qquad H^{m+1,n}_{p+1,q+1}(z) = H^{m+1,n}_{p+1,q+1}\left[z\middle|\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p),(-\lambda,h)\\(-\lambda-1,h),(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right].$$

Thus (c) is established, and the proof of the theorem is completed.

Similar examination yields $\mathfrak{L}_{\nu,2}$-theory of the $\boldsymbol{H}^2$-transform.

THEOREM 3.2. *We suppose that* **(i)** $\alpha < 1-\nu < \beta$ *and that either of conditions* **(ii)** $a^* > 0$ *or* **(iii)** $a^* = 0$, $\Delta(1-\nu) + \mathrm{Re}(\mu) \leqq 0$ *holds. Then we have the following results:*

(a) *There is a one-to-one transform* $\boldsymbol{H}^2 \in [\mathfrak{L}_{\nu,2},]$ *so that the relation*

$$(3.12)\qquad (\mathfrak{M}\boldsymbol{H}^2f)(s) = \mathcal{H}^{m,n}_{p,q}(1-s)(\mathfrak{M}f)(s)$$

holds for $f \in \mathfrak{L}_{\nu,2}$ *and* $\mathrm{Re}(s) = \nu$. *If* $a^* = 0$, $\Delta(1-\nu) + \mathrm{Re}(\mu) = 0$ *and* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *then the transform* $\boldsymbol{H}^2$ *maps* $\mathfrak{L}_{\nu,2}$ *onto* $\mathfrak{L}_{\nu,2}$.

(b) *If* $f \in \mathfrak{L}_{\nu,2}$ *and* $g \in \mathfrak{L}_{1-\nu,2}$, *then the relation of fractional integration by parts*

$$(3.13)\qquad \int_0^\infty f(x)(\boldsymbol{H}^2g)(x)\,dx = \int_0^\infty g(x)(\boldsymbol{H}^1f)(x)\,dx$$

holds, where $\boldsymbol{H}^1$ *is the modified* $\boldsymbol{H}$*-transform* (1.1).

(c) *Let $\lambda \in \mathbb{C}, h \in \mathbb{R}_+$ and $f \in \mathfrak{L}_{\nu,2}$. If* $\mathrm{Re}(\lambda) > (1-\nu)h - 1$, *then* $\boldsymbol{H}^2 f$ *is given by*

$$(3.14)\quad (\boldsymbol{H}^2 f)(x) = -hx^{(\lambda+1)/h}\frac{d}{dx}x^{-(\lambda+1)/h} \times \int_0^\infty H^{m,n+1}_{p+1,q+1}\left[\frac{t}{x}\middle| \begin{matrix}(-\lambda,h),(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q),(-\lambda-1,h)\end{matrix}\right] f(t)dt \quad (x>0).$$

If $\mathrm{Re}(\lambda) < (1-\nu)h - 1$, *then*

$$(3.15)\quad (\boldsymbol{H}^2 f)(x) = hx^{(\lambda+1)/h}\frac{d}{dx}x^{-(\lambda+1)/h} \times \int_0^\infty H^{m+1,n}_{p+1,q+1}\left[\frac{t}{x}\middle| \begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p),(-\lambda,h)\\(-\lambda-1,h),(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right] f(t)dt \quad (x>0).$$

(d) $\boldsymbol{H}^2$ *is independent* ν *in the sense that, if* ν *and* $\widetilde{\nu}$ *satisfy* (**i**), *and* (**ii**) *or* (**iii**), *and if the transforms* $\boldsymbol{H}^2$ *and* $\widetilde{\boldsymbol{H}^2}$ *are defined in* $\mathfrak{L}_{\nu,2}$ *and* $\mathfrak{L}_{\widetilde{\nu},2}$ *respectively by* (3.12), *then* $\boldsymbol{H}^2 f = \widetilde{\boldsymbol{H}^2} f$ *for* $f \in \mathfrak{L}_{\nu,2} \bigcap \mathfrak{L}_{\widetilde{\nu},2}$.

(e) *If* $a^* > 0$ *or if* $a^* = 0$, $\Delta(1-\nu) + \mathrm{Re}(\mu) < -1$, *then for* $f \in \mathfrak{L}_{\nu,2}$, *the transform* $\boldsymbol{H}^2 f$ *is given by* (1.2).

Proof. Statements (a), (b), (d) and (e) are proved similarly to the corresponding ones in Theorem 3.1 on the basis of (2.2) and by applying the results in [11, Theorems 3 and 4] for the $\boldsymbol{H}$-transform. In particular (3.12) is proved similarly to (3.5) by using (2.24) and (2.6), while (3.13) by similar arguments to (3.6).

The representations (3.14) and (3.15) are deduced from the corresponding ones for the $\boldsymbol{H}$-transform [11, Theorem 3], where the operator relation

$$(3.16)\qquad RM_{-\zeta}DM_{\zeta+1} = -M_{\zeta+1}DM_{-\zeta}R$$

is invoked for the elementary operators (2.3), (2.19) and the differentiation operator $D = d/dx$. In fact, for $f \in \mathfrak{L}_{\nu,2}$ and $\mathrm{Re}(\lambda) > (1-\nu)h - 1$, the relations (2.2), (3.7), (3.16) and (2.3) imply that

$$(\boldsymbol{H}^2 f)(x) = (R\boldsymbol{H}f)(x) = RM_{1-(\lambda+1)/h}DM_{(\lambda+1)/h}\int_0^\infty H^{m,n+1}_{p+1,q+1}[xt]f(t)dt$$
$$= -M_{(\lambda+1)/h}DM_{1-(\lambda+1)/h}\int_0^\infty H^{m,n+1}_{p+1,q+1}\left[\frac{t}{x}\right]f(t)\frac{dt}{x}.$$

This gives (3.14) in accordance with (3.8) and (2.19), and (3.15) may be proved similarly. Thus the theorem is established.

4. Boundedness, representation and range of H^1-transform in $\mathfrak{L}_{\nu,r}$

It is known ([12]–[13], [6]–[7]) that from the existence of the $\boldsymbol{H}$-transform (1.7) on the space $\mathfrak{L}_{\nu,2}$ the transform can be extended to $\mathfrak{L}_{\nu,r}$ for $1 < r < \infty$ such that $\boldsymbol{H} \in [\mathfrak{L}_{\nu,r}, \mathfrak{L}_{1-\nu,s}]$ for a certain range of the value s. Moreover, the range of $\boldsymbol{H}$ on $\mathfrak{L}_{\nu,r}$ is characterized in terms of the Erdélyi-Kober type fractional integral operators $I^{\alpha}_{0+;\sigma,\eta}$ in (2.15) and $I^{\alpha}_{-;\sigma,\eta}$ in (2.16) and the modified Hankel transform $\mathbb{H}_{k,\eta}$ in (2.17) and Laplace transform $\mathbb{L}_{k,\alpha}$ in (2.18). The results are different in eight cases: (1) $a^* = \Delta = \mathrm{Re}(\mu) = 0$; (2) $a^* = \Delta = 0, \mathrm{Re}(\mu) < 0$; (3) $a^* = 0, \Delta \neq 0$; (4) $a_1^* > 0, a_2^* > 0$; (5) $a_1^* > 0, a_2^* = 0$; (6) $a_1^* = 0, a_2^* > 0$; (7) $a^* > 0, a_1^* > 0, a_2^* < 0$ and (8) $a^* > 0, a_1^* < 0, a_2^* > 0$. Here the constants a^*, a_1^*, a_2^*, Δ and μ are defined in (2.9) to (2.12).

Such results can also be proved for the $\boldsymbol{H}^1$- and $\boldsymbol{H}^2$-transforms on the basis of their existence on the space $\mathfrak{L}_{\nu,2}$ guaranteed in Theorems 3.1 and 3.2. We shall obtain these results in view of the relations (2.1), (2.2) and corresponding results for the $\boldsymbol{H}$-transform (1.7). In this section we exhibit the statements for the transform $\boldsymbol{H}^1$.

In the case $a^* = 0$ we present the following three theorems.

THEOREM 4.1. *Let $a^* = \Delta = 0$, $\mathrm{Re}(\mu) = 0$, $\alpha < \nu < \beta$ and let $1 < r < \infty$.*

(a) *The transform $\boldsymbol{H}^1$ defined on $\mathfrak{L}_{\nu,2}$ can be extended to $\mathfrak{L}_{\nu,r}$ as an element of $[\mathfrak{L}_{\nu,r}]$.*

(b) *If $1 < r \leqq 2$, the transform $\boldsymbol{H}^1$ is one-to-one on $\mathfrak{L}_{\nu,r}$ and there holds the equality* (3.1) *for $f \in \mathfrak{L}_{\nu,r}$ and* $\mathrm{Re}(s) = \nu$.

(c) *If $f \in \mathfrak{L}_{\nu,r}$ and $g \in \mathfrak{L}_{1-\nu,r'}$ with $r' = r/(r-1)$, then the relation* (3.2) *holds.*

(d) *If $1 - \nu \notin \mathcal{E}_{\mathcal{H}}$, then the transform $\boldsymbol{H}^1$ is one-to-one on $\mathfrak{L}_{\nu,r}$ and*

$$\boldsymbol{H}^1(\mathfrak{L}_{\nu,r}) = \mathfrak{L}_{\nu,r}. \tag{4.1}$$

(e) *If $f \in \mathfrak{L}_{\nu,r}, \lambda \in \mathbb{C}$ and $h > 0$, then $\boldsymbol{H}^1 f$ is given by* (3.3) *for* $\mathrm{Re}(\lambda) > \nu h - 1$, *while by* (3.4) *for* $\mathrm{Re}(\lambda) < \nu h - 1$.

THEOREM 4.2. *Let $a^* = \Delta = 0$, $\mathrm{Re}(\mu) < 0$, $\alpha < \nu < \beta$, and let either $m > 0$ or $n > 0$. Let $1 < r < \infty$.*

(a) *The transform $\boldsymbol{H}^1$ defined on $\mathfrak{L}_{\nu,2}$ can be extended to $\mathfrak{L}_{\nu,r}$ as an element of $[\mathfrak{L}_{\nu,r}, \mathfrak{L}_{\nu,s}]$ for all $s \geqq r$ such that $1/s > 1/r + \mathrm{Re}(\mu)$.*

(b) *If $1 < r \leqq 2$, then the transform $\boldsymbol{H}^1$ is one-to-one on $\mathfrak{L}_{\nu,r}$ and the equality* (3.1) *holds for $f \in \mathfrak{L}_{\nu,r}$ and* $\mathrm{Re}(s) = \nu$.

(c) *If $f \in \mathfrak{L}_{\nu,r}$ and $g \in \mathfrak{L}_{1-\nu,s}$ with $1 < s < \infty$ and $1 \leqq 1/r + 1/s < 1 - \mathrm{Re}(\mu)$, then the relation* (3.2) *holds.*

(d) *If* $1-\nu \notin \mathcal{E}_{\mathcal{H}}$, *then the transform* $\boldsymbol{H}^1$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$ *and*

$$\boldsymbol{H}^1(\mathfrak{L}_{\nu,r}) = I^{-\mu}_{-;k,-\alpha/k}(\mathfrak{L}_{\nu,r}) \tag{4.2}$$

for $k \geqq 1$ *and* $m > 0$, *and*

$$\boldsymbol{H}^1(\mathfrak{L}_{\nu,r}) = I^{-\mu}_{0+;k,\beta/k-1}(\mathfrak{L}_{\nu,r}) \tag{4.3}$$

for $0 < k \leqq 1$ *and* $n > 0$. *If* $1-\nu \in \mathcal{E}_{\mathcal{H}}$, *then* $\boldsymbol{H}^1(\mathfrak{L}_{\nu,r})$ *is a subset of right hand sides of* (4.2) *and* (4.3) *in respective cases.*

(e) *If* $f \in \mathfrak{L}_{\nu,r}, \lambda \in \mathbb{C}$ *and* $h > 0$, *then* $\boldsymbol{H}^1 f$ *is given by* (3.3) *for* $\mathrm{Re}(\lambda) > \nu h - 1$, *while by* (3.4) *for* $\mathrm{Re}(\lambda) < \nu h - 1$. *If furthemore* $\mathrm{Re}(\mu) < -1$, *then* $\boldsymbol{H}^1 f$ *is given by* (1.1).

THEOREM 4.3. *Let* $a^* = 0$, $\Delta \neq 0$, $\alpha < \nu < \beta$, $1 < r < \infty$ *and* $\Delta\nu + \mathrm{Re}(\mu) \leqq 1/2 - \gamma(r)$, *where*

$$\gamma(r) = \max\left[\frac{1}{r}, 1 - \frac{1}{r}\right]. \tag{4.4}$$

Assume that $m > 0$ *if* $\Delta > 0$ *and* $n > 0$ *if* $\Delta < 0$.

(a) *The transform* $\boldsymbol{H}^1$ *defined on* $\mathfrak{L}_{\nu,2}$ *can be extended to* $\mathfrak{L}_{\nu,r}$ *as an element of* $[\mathfrak{L}_{\nu,r}, \mathfrak{L}_{\nu,s}]$ *for all* s *with* $r \leqq s < \infty$ *such that* $s' \geqq [1/2 - \Delta\nu - \mathrm{Re}(\mu)]^{-1}$ *with* $s' = s/(s-1)$.

(b) *If* $1 < r \leqq 2$, *then the transform* $\boldsymbol{H}^1$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$ *and the equality* (3.1) *holds for* $f \in \mathfrak{L}_{\nu,r}$ *and* $\mathrm{Re}(s) = \nu$.

(c) *If* $f \in \mathfrak{L}_{\nu,r}$ *and* $g \in \mathfrak{L}_{1-\nu,s}$ *with* $1 < s < \infty$, $1/r + 1/s \geqq 1$ *and* $\Delta\nu + \mathrm{Re}(\mu) \leqq 1/2 - \max[\gamma(r), \gamma(s)]$, *then the relation* (3.2) *holds.*

(d) *If* $1-\nu \notin \mathcal{E}_{\mathcal{H}}$, *then the transform* $\boldsymbol{H}^1$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$. *If we set* $\eta = -\Delta\alpha - \mu - 1$ *for* $\Delta > 0$ *and* $\eta = -\Delta\beta - \mu - 1$ *for* $\Delta < 0$, *then* $\mathrm{Re}(\eta) > -1$ *and*

$$\boldsymbol{H}^1(\mathfrak{L}_{\nu,r}) = (M_{\mu/\Delta+1/2}\mathbb{H}_{\Delta,\eta})(\mathfrak{L}_{1/2-\nu-\mathrm{Re}(\mu)/\Delta,r}). \tag{4.5}$$

When $1-\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^1(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (4.5).

(e) *If* $f \in \mathfrak{L}_{\nu,r}, \lambda \in \mathbb{C}, h > 0$ *and* $\Delta\nu + \mathrm{Re}(\mu) \leqq 1/2 - \gamma(r)$, *then* $\boldsymbol{H}^1 f$ *is given by* (3.3) *for* $\mathrm{Re}(\lambda) > \nu h - 1$, *while by* (3.4) *for* $\mathrm{Re}(\lambda) < \nu h - 1$. *If furthemore* $\Delta\nu + \mathrm{Re}(\mu) < -1$, *then* $\boldsymbol{H}^1 f$ *is given by* (1.1).

The results in Theorems 4.1–4.3 follow from (2.1) and the corresponding assertions for the $\boldsymbol{H}$-transform [7, Theorems 4.1, 4.2, 5.1 and 5.2], if we take into account the isometric property

$$R(\mathfrak{L}_{\nu,r}) = \mathfrak{L}_{1-\nu,r} \tag{4.6}$$

given in Lemma 2.1(a).

From (2.1), Lemma 2.1(a) and [7, Theorem 6.1] we obtain the extension of the $\boldsymbol{H}^1$-transform from $\mathfrak{L}_{\nu,2}$ to $\mathfrak{L}_{\nu,r}$ ($\nu \in \mathbb{R}, 1 \leqq r \leqq \infty$) in the case $a^* > 0$.

THEOREM 4.4. *Let* $a^* > 0$, $\alpha < \nu < \beta$ *and* $1 \leqq r \leqq s \leqq \infty$.

(a) *The transform* $\boldsymbol{H}^1$ *defined on* $\mathfrak{L}_{\nu,2}$ *can be extended to* $\mathfrak{L}_{\nu,r}$ *as an element of* $[\mathfrak{L}_{\nu,r}, \mathfrak{L}_{\nu,s}]$. *If* $1 \leqq r \leqq 2$, *then the transform* $\boldsymbol{H}^1$ *is one-to-one from* $\mathfrak{L}_{\nu,r}$ *onto* $\mathfrak{L}_{\nu,s}$.

(b) *If* $f \in \mathfrak{L}_{\nu,r}$ *and* $g \in \mathfrak{L}_{1-\nu,s'}$ *with* $s' = s/(s-1)$, *then the relation* (3.2) *holds.*

Similarly, by virtue of (2.1), (4.6) and the results for the $\boldsymbol{H}$-transform (1.7), we obtain the one-to-one property and the range for the $\boldsymbol{H}^1$-transform (1.1) in the space $\mathfrak{L}_{\nu,r}$ when $a^* > 0$.

THEOREM 4.5. *Let* $a_1^* > 0$, $a_2^* > 0$, $m > 0$, $n > 0$, $\alpha < \nu < \beta$ *and* $\omega = \mu + a_1^*\alpha - a_2^*\beta + 1$ *and let* $1 < r < \infty$.

(a) *If* $1-\nu \notin \mathcal{E}_{\mathcal{H}}$, *or if* $1 < r \leqq 2$, *then the transform* $\boldsymbol{H}^1$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$.

(b) *If* $\mathrm{Re}(\omega) \geqq 0$ *and* $1-\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\boldsymbol{H}^1(\mathfrak{L}_{\nu,r}) = (\mathbb{L}_{a_1^*,\alpha}\mathbb{L}_{a_2^*,1-\beta-\omega/a_2^*})(\mathfrak{L}_{\nu,r}). \tag{4.7}$$

When $1-\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^1(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (4.7).

(c) *If* $\mathrm{Re}(\omega) < 0$ *and* $1-\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\boldsymbol{H}^1(\mathfrak{L}_{\nu,r}) = (I^{-\omega}_{-;1/a_1^*,-a_1^*\alpha}\mathbb{L}_{a_1^*,\alpha}\mathbb{L}_{a_2^*,1-\beta})(\mathfrak{L}_{\nu,r}). \tag{4.8}$$

When $1-\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^1(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (4.8).

THEOREM 4.6. *Let* $a_1^* > 0$, $a_2^* = 0$, $m > 0$, $\alpha < \nu < \beta$, $\omega = \mu + a_1^*\alpha + 1/2$ *and let* $1 < r < \infty$.

(a) *If* $1-\nu \notin \mathcal{E}_{\mathcal{H}}$, *or if* $1 < r \leqq 2$, *then the transform* $\boldsymbol{H}^1$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$.

(b) *If* $\mathrm{Re}(\omega) \geqq 0$ *and* $1-\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\boldsymbol{H}^1(\mathfrak{L}_{\nu,r}) = \mathbb{L}_{a_1^*,\alpha-\omega/a_1^*}(\mathfrak{L}_{1-\nu,r}). \tag{4.9}$$

When $1-\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^1(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (4.9).

(c) *If* $\mathrm{Re}(\omega) < 0$ *and* $1-\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\boldsymbol{H}^1(\mathfrak{L}_{\nu,r}) = (I^{-\omega}_{-;1/a_1^*,-a_1^*\alpha}\mathbb{L}_{a_1^*,\alpha})(\mathfrak{L}_{1-\nu,r}). \tag{4.10}$$

When $1-\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^1(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (4.10).

THEOREM 4.7. *Let* $a_1^* = 0$, $a_2^* > 0$, $n > 0$, $\alpha < \nu < \beta$, $\omega = \mu - a_2^*\beta + 1/2$ *and let* $1 < r < \infty$.

(a) *If* $1-\nu \notin \mathcal{E}_{\mathcal{H}}$, *or if* $1 < r \leqq 2$, *then the transform* $\boldsymbol{H}^1$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$.

(b) *If* $\mathrm{Re}(\omega) \geqq 0$ *and* $1-\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\boldsymbol{H}^1(\mathfrak{L}_{\nu,r}) = \mathbb{L}_{-a_2^*,\beta+\omega/a_2^*}(\mathfrak{L}_{1-\nu,r}). \tag{4.11}$$

When $1-\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^1(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (4.11).

(c) *If* $\mathrm{Re}(\omega) < 0$ *and* $1-\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\boldsymbol{H}^1(\mathfrak{L}_{\nu,r}) = (I^{-\omega}_{0+;1/a_2^*,a_2^*\beta-1}\mathbb{L}_{-a_2^*,\beta})(\mathfrak{L}_{1-\nu,r}). \tag{4.12}$$

When $1-\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^1(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (4.12).

THEOREM 4.8. *Let* $a^* > 0$, $a_1^* > 0$, $a_2^* < 0$, $\alpha < \nu < \beta$ *and let* $1 < r < \infty$.

(a) *If* $1-\nu \notin \mathcal{E}_{\mathcal{H}}$, *or if* $1 < r \leqq 2$, *then the transform* $\boldsymbol{H}^1$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$.

(b) *Let*

$$\omega = a^*\eta - \mu - \frac{1}{2} \tag{4.13}$$

and let η *and* ξ *be chosen as*

$$a^*\mathrm{Re}(\eta) \geqq \gamma(r) - 2a_2^*\nu + \mathrm{Re}(\mu), \quad \mathrm{Re}(\eta) > -\nu, \tag{4.14}$$

$$\mathrm{Re}(\xi) < \nu. \tag{4.15}$$

If $1-\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\begin{aligned}&\boldsymbol{H}^1(\mathfrak{L}_{\nu,r}) \\ &= (M_{1/2+\omega/(2a_2^*)}\mathbb{H}_{-2a_2^*,2a_2^*\xi+\omega-1}\mathbb{L}_{-a^*,1/2+\eta-\omega/(2a_2^*)})(\mathfrak{L}_{\nu-1/2-\mathrm{Re}(\omega)/(2a_2^*),r}).\end{aligned} \tag{4.16}$$

When $1-\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^1(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (4.16).

THEOREM 4.9. *Let* $a^* > 0$, $a_1^* < 0$, $a_2^* > 0$, $\alpha < \nu < \beta$ *and let* $1 < r < \infty$.

(a) *If* $1-\nu \notin \mathcal{E}_{\mathcal{H}}$, *or if* $1 < r \leqq 2$, *then the transform* $\boldsymbol{H}^1$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$.

(b) *Let*

$$\omega = a^*\eta - \Delta - \mu - \frac{1}{2}, \tag{4.17}$$

and let η *and* ξ *be chosen as*

$$a^*\mathrm{Re}(\eta) \geqq \gamma(r) + 2a_1^*(\nu-1) + \Delta + \mathrm{Re}(\mu), \quad \mathrm{Re}(\eta) > \nu - 1, \tag{4.18}$$

$$\mathrm{Re}(\xi) < 1-\nu. \tag{4.19}$$

If $1-\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\begin{aligned}&\boldsymbol{H}^1(\mathfrak{L}_{\nu,r}) \\ &= (M_{-1/2-\omega/(2a_1^*)}\mathbb{H}_{2a_1^*,2a_1^*\xi+\omega-1}\mathbb{L}_{a^*,1/2-\eta+\omega/(2a_1^*)})(\mathfrak{L}_{\nu+1/2+\mathrm{Re}(\omega)/(2a_1^*),r}).\end{aligned} \tag{4.20}$$

When $1-\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^1(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (4.20).

The results in Theorems 4.5–4.9 follow from the relation (2.1) and the corresponding assertions for the $\boldsymbol{H}$-transform [7, Theorems 4.1, 4.2, 5.1 and 5.2] by taking into account the isometric property (4.6).

5. Boundedness, representation and range of $\boldsymbol{H}^2$-transform in $\mathfrak{L}_{\nu,r}$

Similarly to the previous section the boundedness, the representation and the range of the $\boldsymbol{H}^2$-transform (1.2) can be established by virtue of the relation (2.2), Lemma 2.1(a) and the corresponding results for the $\boldsymbol{H}$-transform (1.7) given in [12]–[13] and [6]–[7].

Let first state the results in the case $a^* = 0$.

THEOREM 5.1. *Let* $a^* = \Delta = 0$, $\mathrm{Re}(\mu) = 0$, $\alpha < 1 - \nu < \beta$ *and let* $1 < r < \infty$.

(a) *The transform* $\boldsymbol{H}^2$ *defined on* $\mathfrak{L}_{\nu,2}$ *can be extended to* $\mathfrak{L}_{\nu,r}$ *as an element of* $[\mathfrak{L}_{\nu,r}]$.

(b) *If* $1 < r \leqq 2$, *the transform* $\boldsymbol{H}^2$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$ *and the equality* (3.12) *holds for* $f \in \mathfrak{L}_{\nu,r}$ *and* $\mathrm{Re}(s) = \nu$.

(c) *If* $f \in \mathfrak{L}_{\nu,r}$ *and* $g \in \mathfrak{L}_{1-\nu,r'}$ *with* $r' = r/(r-1)$, *then the relation* (3.13) *holds.*

(d) *If* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *then the transform* $\boldsymbol{H}^2$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$ *and*

$$\boldsymbol{H}^2(\mathfrak{L}_{\nu,r}) = \mathfrak{L}_{\nu,r}. \tag{5.1}$$

(e) *If* $f \in \mathfrak{L}_{\nu,r}, \lambda \in \mathbb{C}$ *and* $h > 0$, *then* $\boldsymbol{H}^2 f$ *is given by* (3.14) *for* $\mathrm{Re}(\lambda) > (1-\nu)h - 1$, *while by* (3.15) *for* $\mathrm{Re}(\lambda) < (1-\nu)h - 1$.

THEOREM 5.2. *Let* $a^* = \Delta = 0$, $\mathrm{Re}(\mu) < 0$, $\alpha < 1 - \nu < \beta$, *and let either* $m > 0$ *or* $n > 0$. *Let* $1 < r < \infty$.

(a) *The transform* $\boldsymbol{H}^2$ *defined on* $\mathfrak{L}_{\nu,2}$ *can be extended to* $\mathfrak{L}_{\nu,r}$ *as an element of* $[\mathfrak{L}_{\nu,r}, \mathfrak{L}_{\nu,s}]$ *for all* $s \geqq r$ *such that* $1/s > 1/r + \mathrm{Re}(\mu)$.

(b) *If* $1 < r \leqq 2$, *then the transform* $\boldsymbol{H}^2$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$ *and the equality* (3.12) *holds for* $f \in \mathfrak{L}_{\nu,r}$ *and* $\mathrm{Re}(s) = \nu$.

(c) *If* $f \in \mathfrak{L}_{\nu,r}$ *and* $g \in \mathfrak{L}_{1-\nu,s}$ *with* $1 < s < \infty$ *and* $1 \leqq 1/r + 1/s < 1 - \mathrm{Re}(\mu)$, *then the relation* (3.13) *holds.*

(d) *If* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *then the transform* $\boldsymbol{H}^2$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$ *and*

$$\boldsymbol{H}^2(\mathfrak{L}_{\nu,r}) = I^{-\mu}_{0+;k,(1-\alpha)/k-1}(\mathfrak{L}_{\nu,r}) \tag{5.2}$$

for $k \geqq 1$ *and* $m > 0$, *and*

$$\boldsymbol{H}^2(\mathfrak{L}_{\nu,r}) = I^{-\mu}_{-;k,(\beta-1)/k}(\mathfrak{L}_{\nu,r}) \tag{5.3}$$

for $0 < k \leqq 1$ *and* $n > 0$. *When* $\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^2(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (5.2) *and* (5.3) *in respective cases.*

(e) *If* $f \in \mathfrak{L}_{\nu,r}, \lambda \in \mathbb{C}$ *and* $h > 0$, *then* $\boldsymbol{H}^2 f$ *is given by* (3.14) *for* $\mathrm{Re}(\lambda) > (1-\nu)h-1$, *while by* (3.15) *for* $\mathrm{Re}(\lambda) < (1-\nu)h-1$. *If furthemore* $\mathrm{Re}(\mu) < -1$, *then* $\boldsymbol{H}^2 f$ *is given by* (1.2).

THEOREM 5.3. *Let* $a^* = 0$, $\Delta \neq 0$, $\alpha < 1-\nu < \beta$, $1 < r < \infty$ *and* $\Delta(1-\nu)+\mathrm{Re}(\mu) \leqq 1/2-\gamma(r)$, *where* $\gamma(r)$ *is defined by* (4.4). *Assume that* $m > 0$ *if* $\Delta > 0$ *and* $n > 0$ *if* $\Delta < 0$.

(a) *The transform* $\boldsymbol{H}^2$ *defined on* $\mathfrak{L}_{\nu,2}$ *can be extended to* $\mathfrak{L}_{\nu,r}$ *as an element of* $[\mathfrak{L}_{\nu,r}, \mathfrak{L}_{\nu,s}]$ *for all* s *with* $r \leqq s < \infty$ *such that* $s' \geqq [1/2-\Delta(1-\nu) - \mathrm{Re}(\mu)]^{-1}$ *with* $s' = s/(s-1)$.

(b) *If* $1 < r \leqq 2$, *the transform* $\boldsymbol{H}^2$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$ *and the equality* (3.12) *holds for* $f \in \mathfrak{L}_{\nu,r}$ *and* $\mathrm{Re}(s) = 1-\nu$.

(c) *If* $f \in \mathfrak{L}_{\nu,r}$ *and* $g \in \mathfrak{L}_{1-\nu,s}$ *with* $1 < s < \infty$, $1/r+1/s \geqq 1$ *and* $\Delta(1-\nu)+\mathrm{Re}(\mu) \leqq 1/2-\max[\gamma(r),\gamma(s)]$, *then the relation* (3.13) *holds.*

(d) *If* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *then the transform* $\boldsymbol{H}^2$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$. *If we set* $\eta = -\Delta\alpha-\mu-1$ *if* $\Delta > 0$ *and* $\eta = -\Delta\beta-\mu-1$ *if* $\Delta < 0$, *then* $\mathrm{Re}(\eta) > -1$ *and*

$$\boldsymbol{H}^2(\mathfrak{L}_{\nu,r}) = (M_{-\mu/\Delta-1/2}\mathbb{H}_{-\Delta,\eta})(\mathfrak{L}_{3/2-\nu+\mathrm{Re}(\mu)/\Delta,r}). \tag{5.4}$$

When $\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^2(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (5.4).

(e) *If* $f \in \mathfrak{L}_{\nu,r}$, $\lambda \in \mathbb{C}$, $h > 0$ *and* $\Delta(1-\nu)+\mathrm{Re}(\mu) \leqq 1/2-\gamma(r)$, *then* $\boldsymbol{H}^2 f$ *is given by* (3.14) *for* $\mathrm{Re}(\lambda) > (1-\nu)h-1$, *while by* (3.15) *for* $\mathrm{Re}(\lambda) < (1-\nu)h-1$. *If furthemore* $\Delta(1-\nu)+\mathrm{Re}(\mu) < -1$, *then* $\boldsymbol{H}^2 f$ *is given by* (1.2).

The results in Theorems 5.1–5.3 follow from (2.2), Lemma 2.1(a) and the corresponding assertions for the $\boldsymbol{H}$-transform (1.7) [7, Theorems 4.1, 4.2, 5.1 and 5.2]. The relation (5.1) is derived by using the isometric property (4.6). The relations (5.2) and (5.3) follow by employing the directly verified formulas

$$RI^{\alpha}_{0+;\sigma,\eta} = I^{\alpha}_{-;\sigma,\eta+1-1/\sigma}R, \qquad RI^{\alpha}_{-;\sigma,\eta} = I^{\alpha}_{0+;\sigma,\eta-1+1/\sigma}R. \tag{5.5}$$

The relation (5.4) is obtained by the relation [7, Theorems 7 and 8]

$$\boldsymbol{H}(\mathfrak{L}_{\nu,r}) = (M_{\mu/\Delta+1/2}\mathbb{H}_{\Delta,\eta})(\mathfrak{L}_{\nu-1/2-\mathrm{Re}(\mu)/\Delta,r})$$

for the $\boldsymbol{H}$-transform (1.7) and by using the relations (2.2), (4.6) and the easily verified formulas

$$RM_{\zeta} = M_{-\zeta}R, \qquad R\mathbb{H}_{k,\eta} = \mathbb{H}_{-k,\eta}R. \tag{5.6}$$

In fact,

$$\begin{aligned}\boldsymbol{H}^1(\mathfrak{L}_{\nu,r}) &= R\boldsymbol{H}(\mathfrak{L}_{\nu,r}) = (RM_{\mu/\Delta+1/2}\mathbb{H}_{\Delta,\eta})(\mathfrak{L}_{\nu-1/2-\mathrm{Re}(\mu)/\Delta,r}) \\ &= (M_{-\mu/\Delta-1/2}R\mathbb{H}_{\Delta,\eta})(\mathfrak{L}_{\nu-1/2-\mathrm{Re}(\mu)/\Delta,r}) \\ &= (M_{-\mu/\Delta-1/2}\mathbb{H}_{-\Delta,\eta}R)(\mathfrak{L}_{\nu-1/2-\mathrm{Re}(\mu)/\Delta,r}) \\ &= (M_{-\mu/\Delta-1/2}\mathbb{H}_{-\Delta,\eta})(\mathfrak{L}_{3/2-\nu+\mathrm{Re}(\mu)/\Delta,r})\end{aligned}$$

The extension of the $\boldsymbol{H}^2$-transform (1.2) from $\mathfrak{L}_{\nu,2}$ to $\mathfrak{L}_{\nu,r}$ for the case $a^* > 0$ follows from (2.2), Lemma 2.1(a) and [7, Theorem 6.1].

THEOREM 5.4. *Let $a^* > 0$, $\alpha < 1-\nu < \beta$ and $1 \leqq r \leqq s \leqq \infty$.*

(a) *The transform $\boldsymbol{H}^2$ defined on $\mathfrak{L}_{\nu,2}$ can be extended to $\mathfrak{L}_{\nu,r}$ as an element of $[\mathfrak{L}_{\nu,r}, \mathfrak{L}_{\nu,s}]$. If $1 \leqq r \leqq 2$, then the transform $\boldsymbol{H}^2$ is one-to-one from $\mathfrak{L}_{\nu,r}$ onto $\mathfrak{L}_{\nu,s}$.*

(b) *If $f \in \mathfrak{L}_{\nu,r}$ and $g \in \mathfrak{L}_{1-\nu,s'}$ with $s' = s/(s-1)$, then the relation* (3.13) *holds.*

When $a^* > 0$ by various combinations of signs of a_1^*, a_2^*, the one-to-one property and the range for the $\boldsymbol{H}^2$-transform (1.2) in the space $\mathfrak{L}_{\nu,r}$ are obtained on the basis of (2.2) and Lemma 2.1(a) in view of the corresponding results for the $\boldsymbol{H}$-transform (1.7).

THEOREM 5.5. *Let $a_1^* > 0$, $a_2^* > 0$, $m > 0$, $n > 0$, $\alpha < 1-\nu < \beta$, $\omega = \mu + a_1^*\alpha - a_2^*\beta + 1$ and let $1 < r < \infty$.*

(a) *If $\nu \notin \mathcal{E}_{\mathcal{H}}$, or if $1 < r \leqq 2$, then the transform $\boldsymbol{H}^2$ is one-to-one on $\mathfrak{L}_{\nu,r}$.*

(b) *If* $\mathrm{Re}(\omega) \geqq 0$ *and $\nu \notin \mathcal{E}_{\mathcal{H}}$, then*

$$\boldsymbol{H}^2(\mathfrak{L}_{\nu,r}) = (\mathbb{L}_{-a_1^*,1-\alpha}\mathbb{L}_{-a_2^*,\beta+\omega/a_2^*})(\mathfrak{L}_{\nu,r}). \tag{5.7}$$

When $\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^2(\mathfrak{L}_{\nu,r})$ is a subset of the right hand side of (5.7).

(c) *If* $\mathrm{Re}(\omega) < 0$ *and $\nu \notin \mathcal{E}_{\mathcal{H}}$, then*

$$\boldsymbol{H}^2(\mathfrak{L}_{\nu,r}) = (I^{-\omega}_{0+;1/a_1^*,(1-\alpha)a_1^*-1}\mathbb{L}_{-a_1^*,1-\alpha}\mathbb{L}_{-a_2^*,\beta})(\mathfrak{L}_{\nu,r}). \tag{5.8}$$

When $\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^2(\mathfrak{L}_{\nu,r})$ is a subset of the right hand side of (5.8).

THEOREM 5.6. *Let $a_1^* > 0$, $a_2^* = 0$, $m > 0$, $\alpha < 1-\nu < \beta$, $\omega = \mu + a_1^*\alpha + 1/2$ and let $1 < r < \infty$.*

(a) *If $\nu \notin \mathcal{E}_{\mathcal{H}}$, or if $1 < r \leqq 2$, then the transform $\boldsymbol{H}^2$ is one-to-one on $\mathfrak{L}_{\nu,r}$.*

(b) *If* $\mathrm{Re}(\omega) \geqq 0$ *and $\nu \notin \mathcal{E}_{\mathcal{H}}$, then*

$$\boldsymbol{H}^2(\mathfrak{L}_{\nu,r}) = \mathbb{L}_{-a_1^*,1-\alpha+\omega/a_1^*}(\mathfrak{L}_{1-\nu,r}). \tag{5.9}$$

When $\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^2(\mathfrak{L}_{\nu,r})$ is a subset of the right hand side of (5.9).

(c) *If* $\mathrm{Re}(\omega) < 0$ *and* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\boldsymbol{H}^2(\mathfrak{L}_{\nu,r}) = (I^{-\omega}_{0+;1/a_1^*,(1-\alpha)a_1^*-1}\mathbb{L}_{-a_1^*,1-\alpha})(\mathfrak{L}_{1-\nu,r}). \tag{5.10}$$

When $\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^2(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (5.10).

THEOREM 5.7. *Let* $a_1^* = 0$, $a_2^* > 0$, $n > 0$, $\alpha < 1-\nu < \beta$, $\omega = \mu - a_2^*\beta + 1/2$ *and let* $1 < r < \infty$.

(a) *If* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *or if* $1 < r \leqq 2$, *then the transform* $\boldsymbol{H}^2$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$.

(b) *If* $\mathrm{Re}(\omega) \geqq 0$ *and* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\boldsymbol{H}^2(\mathfrak{L}_{\nu,r}) = \mathbb{L}_{a_2^*,1-\beta-\omega/a_2^*}(\mathfrak{L}_{1-\nu,r}). \tag{5.11}$$

When $\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^2(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (5.11).

(c) *If* $\mathrm{Re}(\omega) < 0$ *and* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\boldsymbol{H}^2(\mathfrak{L}_{\nu,r}) = (I^{-\omega}_{-;1/a_2^*,a_2^*(\beta-1)}\mathbb{L}_{a_2^*,1-\beta})(\mathfrak{L}_{1-\nu,r}). \tag{5.12}$$

When $\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^2(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (5.12).

THEOREM 5.8. *Let* $a^* > 0$, $a_1^* > 0$, $a_2^* < 0$, $\alpha < 1-\nu < \beta$ *and let* $1 < r < \infty$.

(a) *If* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *or if* $1 < r \leqq 2$, *then the transform* $\boldsymbol{H}^2$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$.

(b) *Let* ω *be given by* (4.13) *and* η *be chosen as*

$$a^*\mathrm{Re}(\eta) \geqq \gamma(r) - 2a_2^*(1-\nu) + \mathrm{Re}(\mu), \quad \mathrm{Re}(\eta) > \nu - 1, \tag{5.13}$$

and let ξ *be chosen as in* (4.19). *If* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\begin{aligned} &\boldsymbol{H}^2(\mathfrak{L}_{\nu,r}) \\ &= (M_{-1/2-\omega/(2a_2^*)}\mathbb{H}_{2a_2^*,2a_2^*\xi+\omega-1}\mathbb{L}_{a^*,1/2-\eta+\omega/(2a_2^*)})(\mathfrak{L}_{\nu-1/2-\mathrm{Re}(\omega)/(2a_2^*),r}). \end{aligned} \tag{5.14}$$

When $\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^2(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (5.14).

THEOREM 5.9. *Let* $a^* > 0$, $a_1^* < 0$, $a_2^* > 0$, $\alpha < 1-\nu < \beta$ *and let* $1 < r < \infty$.

(a) *If* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *or if* $1 < r \leqq 2$, *then the transform* $\boldsymbol{H}^2$ *is one-to-one on* $\mathfrak{L}_{\nu,r}$.

(b) *Let* ω *be given by* (4.17) *and* η *be chosen as*

$$a^*\mathrm{Re}(\eta) \geqq \gamma(r) - 2a_1^*\nu + \Delta + \mathrm{Re}(\mu), \quad \mathrm{Re}(\eta) > -\nu, \tag{5.15}$$

and let ξ *be chosen as in* (4.15). *If* $\nu \notin \mathcal{E}_{\mathcal{H}}$, *then*

$$\begin{aligned} &\boldsymbol{H}^2(\mathfrak{L}_{\nu,r}) \\ &= (M_{1/2+\omega/(2a_1^*)}\mathbb{H}_{-2a_1^*,2a_1^*\xi+\omega-1}\mathbb{L}_{-a^*,1/2+\eta-\omega/(2a_1^*)})(\mathfrak{L}_{\nu+1/2+\mathrm{Re}(\omega)/(2a_1^*),r}). \end{aligned} \tag{5.16}$$

When $\nu \in \mathcal{E}_{\mathcal{H}}$, $\boldsymbol{H}^2(\mathfrak{L}_{\nu,r})$ *is a subset of the right hand side of* (5.16).

The results in Theorems 5.4–5.9 are obtained from the relation (2.2) and the corresponding assertions for the $\boldsymbol{H}$-transform [7, Theorems 4.1, 4.2, 5.1 and 5.2]. Especially, the relations in (5.7)–(5.12), (5.14) and (5.16) follow from directly varified formulas

$$R\mathbb{L}_{k,\alpha} = \mathbb{L}_{-k,1-\alpha}R, \qquad W_\delta \mathbb{L}_{k,\alpha} = \delta \mathbb{L}_{k,\alpha} W_{1/\delta}, \tag{5.17}$$

the relations in (5.5), (5.6) and the isometric property (4.6). In these relations, if δ or its power appears as a multiplier, it can be canceled, since such a range does not depend on a constant multiplier.

We note that in the above arguments we have found the ranges by the aid of mapping properties of the operators R, M_ζ and W_δ, of the Erdélyi-Kober operators and of modified Hankel and Laplace transforms in $\mathfrak{L}_{\nu,r}$ (see Lemma 2.1 and [20, Theorem 5.1]).

6. Inversion of the transforms H^1 and H^2 in $\mathfrak{L}_{\nu,r}$

In Sections 4 and 5 we have used the relations (2.1) and (2.2) to obtain the boundednes, the representations and the ranges on the space $\mathfrak{L}_{\nu,r}$ for the modified $\boldsymbol{H}$-transforms (1.1) and (1.2) due to the corresponding results for the $\boldsymbol{H}$-transform (1.7). Here we also apply (2.1) and (2.2) again to give the inversion formulas for the transforms $\boldsymbol{H}^1$ and $\boldsymbol{H}^2$ in $\mathfrak{L}_{\nu,r}$ by using the already known inversion relations for the $\boldsymbol{H}$-transform (1.7).

It is known [24] that, if $a^* = 0$ under some additional conditions, the inversion of the $\boldsymbol{H}$-transform in $\mathfrak{L}_{\nu,r}$-space in the respective form (3.7) or (3.10) can be found. Namely

$$\begin{aligned} f(x) &\\ &= hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}\int_0^\infty H^{q-m,p-n+1}_{p+1,q+1}[xt](\boldsymbol{H}f)(t)dt \quad (x>0) \end{aligned} \tag{6.1}$$

for $\mathrm{Re}(\lambda) < \nu h - 1$, where

$$\begin{aligned} &H^{q-m,p-n+1}_{p+1,q+1}[z] \\ &= H^{q-m,p-n+1}_{p+1,q+1}\left[z \left| \begin{matrix} (-\lambda,h),(1-a_i-\alpha_i,\alpha_i)_{n+1,p},(1-a_i-\alpha_i,\alpha_i)_{1,n} \\ (1-b_j-\beta_j,\beta_j)_{m+1,q},(1-b_j-\beta_j,\beta_j)_{1,m},(-\lambda-1,h) \end{matrix} \right.\right], \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} f(x) &\\ &= -hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}\int_0^\infty H^{q-m+1,p-n}_{p+1,q+1}[xt](\boldsymbol{H}f)(t)dt \quad (x>0) \end{aligned} \tag{6.3}$$

for $\mathrm{Re}(\lambda) > \nu h - 1$, where

(6.4) $H^{q-m+1,p-n}_{p+1,q+1}[z]$

$$= H^{q-m+1,p-n}_{p+1,q+1}\left[z \,\middle|\, \begin{matrix} (1-a_i-\alpha_i,\alpha_i)_{n+1,p}, (1-a_i-\alpha_i,\alpha_i)_{1,n}, (-\lambda, h) \\ (-\lambda-1,h), (1-b_j-\beta_j,\beta_j)_{m+1,q}, (1-b_j-\beta_j,\beta_j)_{1,m} \end{matrix}\right].$$

First we deduce the inversion relations for the transform $\boldsymbol{H}^1$. From (2.1) we have

$$\boldsymbol{H}g = \boldsymbol{H}^1 f, \quad g = Rf \in \mathfrak{L}_{1-\nu,r}. \tag{6.5}$$

Therefore if $a^* = 0$, from (6.1), (6.3) and (6.5) we come to the inversion formula for $g = Rf$ in the form

(6.6) $(Rf)(x)$

$$= hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}\int_0^\infty H^{q-m,p-n+1}_{p+1,q+1}[xt](\boldsymbol{H}^1 f)(t)dt \quad (x>0)$$

or

(6.7) $(Rf)(x)$

$$= -hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}\int_0^\infty H^{q-m+1,p-n}_{p+1,q+1}[xt](\boldsymbol{H}^1 f)(t)dt \quad (x>0),$$

If we notice that the inverse operator R^{-1} of R coincides with itself:

$$R^{-1} = R \tag{6.8}$$

and the relation (3.16), the inversion formulas for the transform $\boldsymbol{H}^1$ can be represented in terms of (6.2) and (6.4) in the form:

(6.9) $f(x)$

$$= -hx^{(\lambda+1)/h}\frac{d}{dx}x^{1-(\lambda+1)/h}\int_0^\infty H^{q-m,p-n+1}_{p+1,q+1}\left[\frac{t}{x}\right](\boldsymbol{H}^1 f)(t)\frac{dt}{x} \quad (x>0)$$

or

(6.10) $f(x)$

$$= hx^{(\lambda+1)/h}\frac{d}{dx}x^{1-(\lambda+1)/h}\int_0^\infty H^{q-m+1,p-n}_{p+1,q+1}\left[\frac{t}{x}\right](\boldsymbol{H}^1 f)(t)\frac{dt}{x} \quad (x>0).$$

Now let us find the inversion relations for the second modified $\boldsymbol{H}$-transform (1.2). Since from the isometric property (4.6) and (6.8) the relation (2.2) is equivalent to

$$\boldsymbol{H}f = R\boldsymbol{H}^2 f \tag{6.11}$$

for $f \in \mathfrak{L}_{\nu,r}$, then, if $a^* = 0$, (6.1) and (6.3) imply the inversion relations

(6.12) $f(x)$

$$= hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}\int_0^\infty H_{p+1,q+1}^{q-m,p-n+1}[xt](R\boldsymbol{H}^2 f)(t)dt \quad (x>0)$$

or

(6.13) $f(x)$

$$= -hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}\int_0^\infty H_{p+1,q+1}^{q-m+1,p-n}[xt](R\boldsymbol{H}^2 f)(t)dt \quad (x>0),$$

and thus

(6.14) $f(x)$

$$= hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}\int_0^\infty H_{p+1,q+1}^{q-m,p-n+1}\left[\frac{x}{t}\right](\boldsymbol{H}^2 f)(t)\frac{dt}{t} \quad (x>0)$$

or

(6.15) $f(x)$

$$= -hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}\int_0^\infty H_{p+1,q+1}^{q-m+1,p-n}\left[\frac{x}{t}\right](\boldsymbol{H}^2 f)(t)\frac{dt}{t} \quad (x>0).$$

The conditions for the validity of these formulas depend on the inversion formulas (6.1) and (6.2) for the $\boldsymbol{H}$-transform (1.7) given in [24]. These conditions are different in the cases when $a^* = 0$, and $\Delta = 0$ or $\Delta \neq 0$.

According to (6.5) and (4.6), we can use the results in [24, Theorems 3.1, 3.2, 4.1 and 4.2] with ν being replaced by $1-\nu$ for the transform $\boldsymbol{H}^1$.

THEOREM 6.1. *Let* $a^* = 0$, $\alpha < \nu < \beta$ *and* $\alpha_0 < 1-\nu < \beta_0$ *and let* $\lambda \in \mathbb{C}$, $h > 0$.

(a) *If* $\Delta\nu + \mathrm{Re}(\mu) = 0$ *and* $f \in \mathfrak{L}_{\nu,2}$, *then the inversion formula* (6.9) *holds for* $\mathrm{Re}(\lambda) > (1-\nu)h - 1$ *and* (6.10) *for* $\mathrm{Re}(\lambda) < (1-\nu)h - 1$.

(b) *If* $\Delta = \mathrm{Re}(\mu) = 0$ *and* $f \in \mathfrak{L}_{\nu,r}$ $(1 < r < \infty)$, *then the inversion formula* (6.9) *holds for* $\mathrm{Re}(\lambda) > (1-\nu)h - 1$ *and* (6.10) *for* $\mathrm{Re}(\lambda) < (1-\nu)h - 1$.

THEOREM 6.2. *Let* $a^* = 0$, $1 < r < \infty$ *and* $\Delta\nu + \mathrm{Re}(\mu) \leqq 1/2 - \gamma(r)$, *and let* $\lambda \in \mathbb{C}$, $h > 0$.

(a) *If* $\Delta > 0$, $m > 0$, $\alpha < \nu < \beta$, $\alpha_0 < 1-\nu < \min[\beta_0, \{\mathrm{Re}(\mu + 1/2)/\Delta\} + 1]$ *and if* $f \in \mathfrak{L}_{\nu,r}$, *then the inversion formulas* (6.9) *and* (6.10) *hold for* $\mathrm{Re}(\lambda) > (1-\nu)h - 1$ *and* $\mathrm{Re}(\lambda) < (1-\nu)h - 1$, *respectively.*

(b) *If* $\Delta < 0$, $n > 0$, $\alpha < \nu < \beta$, $\max[\alpha_0, \{\mathrm{Re}(\mu + 1/2)/\Delta\} + 1] < 1-\nu < \beta_0$ *and if* $f \in \mathfrak{L}_{\nu,r}$, *then the inversion formulas* (6.9) *and* (6.10) *hold for* $\mathrm{Re}(\lambda) > (1-\nu)h - 1$ *and* $\mathrm{Re}(\lambda) < (1-\nu)h - 1$, *respectively.*

By virtue of (6.11) and (4.6) the results in [24, Theorems 3.1, 3.2, 4.1 and 4.2] can be applied, similarly to the above, to obtain the statements giving conditions for the validity of the inversion formulas (6.14) and (6.15) for the transform $\boldsymbol{H}^2$.

THEOREM 6.3. *Let* $a^* = 0$, $\alpha < 1 - \nu < \beta$ *and* $\alpha_0 < \nu < \beta_0$ *and let* $\lambda \in \mathbb{C}$, $h > 0$.

(a) *If* $\Delta(1-\nu) + \mathrm{Re}(\mu) = 0$ *and* $f \in \mathfrak{L}_{\nu,2}$, *then the inversion formula* (6.14) *holds for* $\mathrm{Re}(\lambda) > \nu h - 1$ *and* (6.15) *for* $\mathrm{Re}(\lambda) < \nu h - 1$.

(b) *If* $\Delta = \mathrm{Re}(\mu) = 0$ *and* $f \in \mathfrak{L}_{\nu,r}$ $(1 < r < \infty)$, *then the inversion formula* (6.14) *holds for* $\mathrm{Re}(\lambda) > \nu h - 1$ *and* (6.15) *for* $\mathrm{Re}(\lambda) < \nu h - 1$.

THEOREM 6.4. *Let* $a^* = 0$, $1 < r < \infty$ *and* $\Delta(1-\nu) + \mathrm{Re}(\mu) \leqq 1/2 - \gamma(r)$ *and let* $\lambda \in \mathbb{C}$, $h > 0$.

(a) *If* $\Delta > 0$, $m > 0$, $\alpha < 1 - \nu < \beta$, $\alpha_0 < \nu < \min[\beta_0, \{\mathrm{Re}(\mu + 1/2)/\Delta\} + 1]$ *and if* $f \in \mathfrak{L}_{\nu,r}$, *then the inversion formulas* (6.14) *and* (6.15) *hold for* $\mathrm{Re}(\lambda) > \nu h - 1$ *and* $\mathrm{Re}(\lambda) < \nu h - 1$, *respectively.*

(b) *If* $\Delta < 0$, $n > 0$, $\alpha < 1 - \nu < \beta$, $\max[\alpha_0, \{\mathrm{Re}(\mu + 1/2)/\Delta\} + 1] < \nu < \beta_0$ *and if* $f \in \mathfrak{L}_{\nu,r}$, *then the inversion formulas* (6.14) *and* (6.15) *hold for* $\mathrm{Re}(\lambda) > \nu h - 1$ *and* $\mathrm{Re}(\lambda) < \nu h - 1$, *respectively.*

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