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EXISTENCE AND UNIQUENESS THEOREMS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

Abstract. The paper is devoted to study the Cauchy-type problem for the nonlinear differential equation of fractional order $\alpha \in \mathbf{C}$, $\mathrm{Re}(\alpha) > 0$,

$$(D_{a+}^{\alpha}y)(x) = f[x, y(x)] \ (n-1 < \mathrm{Re}(\alpha) \leq n, \ n = -[-\mathrm{Re}(\alpha)]),$$
$$(D_{a+}^{\alpha-k}y)(a+) = b_k, \ b_k \in \mathbf{C} \ (k = 1, 2, \ldots, n),$$

containing the Riemann-Liouville fractional derivative $D_{a+}^{\alpha}y$, on a finite interval $[a, b]$ of the real axis $\mathbf{R} = (-\infty, \infty)$ in the space of summable functions $L(a, b)$. The equivalence of this problem and the nonlinear Volterra integral equation is established. The existence and uniqueness of the solution $y(x)$ of the above Cauchy-type problem is proved by using the method of successive approximations. The corresponding assertions for the ordinary differential equations are presented. Examples are given.

1. Introduction

Our paper is devoted to study the existence and uniqueness of the solution $y(x)$ of the nonlinear differential equation of fractional order $\alpha \in \mathbf{C}$, $\mathrm{Re}(\alpha) > 0$ ($\mathbf{C}$ being the set of complex numbers):

$$(1.1) \qquad (D_{a+}^{\alpha}y)(x) = f[x, y(x)] \ (n-1 < \mathrm{Re}(\alpha) \leq n, \ n = -[-\mathrm{Re}(\alpha)]),$$

on a finite interval $[a, b]$ of the real axis $\mathbf{R} = (-\infty, \infty)$, with the initial conditions

$$(1.2) \qquad (D_{a+}^{\alpha-k}y)(a+) = b_k, \ b_k \in \mathbf{C} \ (k = 1, 2, \ldots, n = -[-\mathrm{Re}(\alpha)]),$$

in the space $L(a, b)$ of summable functions.

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Here $D^{\alpha}_{a+}y$ is the Riemann-Liouville fractional derivative defined for $\alpha \in \mathbf{C}$, $\mathrm{Re}(\alpha) > 0$, by [45, Section 2.2 and 2.4]

$$(1.3) \qquad (D^{\alpha}_{a+}y)(x) = \left(\frac{d}{dx}\right)^{n} \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y(t)dt}{(x-t)^{\alpha-n+1}}, \; n = [\mathrm{Re}(\alpha)] + 1,$$

where $\Gamma(n-\alpha)$ is the Gamma-function and $[\mathrm{Re}(\alpha)]$ is the integral part of $\mathrm{Re}(\alpha)$. The notation $(D^{\alpha-k}_{a+}y)(a+)$ means

$$(1.4) \qquad (D^{\alpha-k}_{a+}y)(a+) = \lim_{x\to a+} (D^{\alpha-k}_{a+}y)(x) \quad (1 \le k \le n),$$

where the limit is taken in almost all points of the right-sided neigbourhood $(a, a+\epsilon)$ $(\epsilon > 0)$ of a. The condition in (1.2) for $k = n$ is understood in the sense

$$(1.5) \; (D^{\alpha-n}_{a+}y)(a+) = \lim_{x\to a+} (I^{n-\alpha}_{a+}y)(x) \; (\alpha \ne n); \; (D^{0}_{a+}y)(a+) = y(a) \; (\alpha = n)$$

where I^{α}_{a+} is the Riemann-Liouville fractional integration operator of order α defined by

$$(1.6) \qquad (I^{\alpha}_{a+}g)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g(t)dt}{(x-t)^{1-\alpha}} \; (\alpha \in \mathbf{C}, \; \mathrm{Re}(\alpha) > 0).$$

In particular, if $\alpha = n \in \mathbf{N} = \{1, 2, \ldots\}$, (1.1)-(1.2) is reduced to the usual Cauchy problem for the ordinary differential equation of order n

$$(1.7) \qquad y^{(n)}(x) = f[x, y(x)], \; y^{(n-k)}(a) = b_k, \; b_k \in \mathbf{C} \; (k = 1, 2, \ldots, n),$$

if we take into account (1.3) with $\alpha = n$. Therefore, by analogy, the problem (1.1)-(1.2) is called Cauchy-type problem [45, Section 42].

The existence and uniqueness of the solution of the problem (1.1)-(1.2) and some of its modifications were studied by many authors in some, basically continuous, spaces of functions. However, they have not completed these investigations. The most of the authors obtained some results not for the initial value problems, but for the corresponding Volterra integral equations. Some authors considered only particular cases. Moreover, some of their results contain mistakes in the proof of the equivalence of Cauchy-type problems and the Volterra integral equations and in the proof of the uniqueness theorem.

We study the Cauchy-type problem (1.1)-(1.2) in the space $L(a, b)$ of summable functions on a finite interval $[a, b]$, and suppose that the relations (1.1) and (1.2) are satisfied in almost every point $x \in [a, b]$ for $y(x) \in L(a, b)$. First we give the conditions for the equivalence of the Cauchy-type problem (1.1)-(1.2) and the nonlinear Volterra integral equation of the second kind

$$y(x) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)}(x-a)^{\alpha-j} + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y(t)]dt}{(x-t)^{1-\alpha}}. \tag{1.8}$$

in the sense that if $y(x) \in L(a,b)$ satisfies (1.1) and (1.2), then it satisfies (1.8), and inverse almost everywhere. Then using method of successive approximations, we give the conditions for the existence and uniqueness of a solution $y(x)$ of the Cauchy-type problem (1.1)-(1.2). In particular, we consider this problem in the case $0 < \mathrm{Re}(\alpha) \leq 1$:

$$(D_{a+}^{\alpha} y)(x) = f[x, y(x)] \ (0 < \mathrm{Re}(\alpha) \leq 1), \ (I_{a+}^{1-\alpha} y)(a+) = b_1, \ b_1 \in \mathbf{C}, \tag{1.9}$$

being arisen in applications, and the weighted Cauchy-type problem:

$$(D_{a+}^{\alpha} y)(x) = f[x, y(x)] \ (0 < \mathrm{Re}(\alpha) < 1), \ \lim_{x \to a+} (x-a)^{1-\alpha} y(x) = c, \ c \in \mathbf{C}. \tag{1.10}$$

We present the corresponding assertions for ordinary differential equations and give some examples in conclusion.

The paper is organized as follows. In Section 2 we give a short survey of the known results in the theory of fractional differential equations (1.1). Section 3 contains preliminary results from the theory of absolutely continuous functions and fractional calculus operators in the space $L(a, b)$. Section 4 is devoted to the equivalence of the Cauchy-type problem (1.1)-(1.2) and the nonlinear integral equation (1.8). Section 5 deals with the existence and uniqueness of the Cauchy-type problem (1.1)-(1.2), while the problems (1.9) and (1.10) are studied in Section 6. Examples are presented in Section 7.

2. Some results in the theory of fractional differential equations

Pitcher and Sewell [42] first considered the nonlinear fractional differential equation (1.1) with $0 < \alpha < 1$ provided that $f(x, y)$ is bounded in the special region G lying in $\mathbf{R} \times \mathbf{R}$ and Lipschitzian with respect to y. They proved the existence of the continuous solution $y(x)$ for the corresponding nonlinear integral equation of the form (1.8). But the main result of Pitcher and Sewell given in [42, Theorem 4.2] for the equation (1.1) is not true because they have used only the particular case of the composition relation (3.14) between fractional integral and differential operators I_{a+}^{α} and D_{a+}^{α}. However, the paper of Pitcher and Sewell contained the idea of reduction of the solution of fractional differential equation (1.1) to the solution of Volterra integral equation (1.8).

Barrett [7] first considered the Cauchy-type problem for the linear differential equation

$$(D_{a+}^{\alpha} y)(x) - \lambda y(x) = f(x) \ (n - 1 \leq \mathrm{Re}(\alpha) < n, \ \alpha \neq n - 1), \tag{2.1}$$

where n is the smallest positive integer such that $n > \mathrm{Re}(\alpha) > 0$, with the initial conditions (1.2). He reduced this problem to the linear Volterra integral equation of the form (1.8) and proved [7, Theorem 2.1] that if $f(x)$ belongs to $L(a,b)$, then such a problem has the unique solution $y(x)$ in some subspace of $L(a,b)$ given by

$$(2.2)\qquad y(x) = \sum_{k=1}^{n} b_k x^{\alpha-k} E_{\alpha,\alpha-k+1}\left(\lambda(x-a)^{\alpha}\right) + \\ + \int_a^x (x-t)^{\alpha-1} E_{\alpha,\alpha}\left[\lambda(x-t)^{\alpha}\right] f(t)dt.$$

Here $E_{\alpha,\beta}(z)$ is an entire function, called the Mittag-Leffler function, defined by

$$(2.3)\quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \ (\alpha \in \mathbf{C},\ \mathrm{Re}(\alpha) > 0;\ \beta \in \mathbf{C},\ \mathrm{Re}(\beta) > 0),$$

see the books by Erdelyi, Magnus, Oberhettinger and Tricomi [22, Chapter 18] and Dzhrbashyan [15].

Al-Bassam [3], Leskovskii [34], Arora and Alshamani [6] and El-Sayed [18] investigated the Cauchy-type problem (1.1)-(1.2). The simplest case of this problem (1.9) with $0 < \alpha \leq 1$ and $a = 0$ was studied by Al-Abedeen [1], Al-Abedeen and Arora [2], Tazali [49], Semenchuk [46], Hadid and Al-shamani [25], Luszczki and Rzepecki [37], El-Sayed [17], Hadid [24], Hadid, Masaedeh and Momani [26], Hadid, Ta'ani and Momani [27], and systems of these equations—by Tazali and Karim [50]. But, as was indicated in Introduction, their investigations were not complete. In particular, in the space $L(a,b)$ the existence result for the Cauchy-type problem (1.1)-(1.2) was proved only for the correponding integral equation.

We also mention other results in the theory of fractional differential equations. For more general, than (1.1), fractional differential equations the Cauchy type problem with the initial data (1.2) was considered by El-Sayed [19], and with the initial data

$$(2.4)\qquad y^{(k)}(a) = d_k,\ d_k \in \mathbf{R}\ (k = 1, 2, \dots, n)$$

- by El-Sayed and Ibragim [21], El-Sayed and Gafar [20]. Delbosco and Rodino [11] investigated the Cauchy-type problem

$$(2.5)\qquad (D_{0+}^{\alpha} y)(x) = f[y(x)],\ y(a) = b\ (0 < \alpha < 1,\ a > 0,\ b \in \mathbf{R})$$

and the weighted Cauchy-type problem

$$(2.6)\quad (D_{0+}^{\alpha} y)(x) = f(y(x)),\ \lim_{x\to 0} x^{1-\alpha} y(x) = b\ (0 < \alpha < 1,\ a > 0,\ b \in \mathbf{R})$$

Hayek, Trujillo, Rivero, Bonilla and Moreno [28] studied the Cauchy-type problem

$$(2.7)\qquad (D^{\alpha}_{0+}y)(x) = f[x, y(x)]\ (x > 0),\ y(x_0) = y_0\ (x_0 > 0,\ y_0 \in \mathbf{R})$$

and the corresponding vectorial case.

Dzhrbashyan and Nersasyan [16] established the existence and uniqueness theorem of the Cauchy-type problem in the space of continuous functions for the linear fractional diferential equation of the form (2.1)-(1.2), in which the fractional derivative D^{α}_{0+} is replaced by the composition of several Riemann-Liouville derivatives, see [45, Section 42.2] in this connection. Podlubny [43, Section 3.1] considered such a problem by using Laplace transform. He have also studied the nonlinear Cauchy-type problem (1.1)-(1.2) with such fractional sequential derivative in the space of continuous functions, but his proof of the uniqueness result in [43, Theorem 3.4] contains mistake.

Several approaches were developed to obtain explicit solutions of some fractional differential equations. For linear diferential equations of fractional order with constant coeficients the Laplace transform method was discussed by many authors, see [23], [38], [39] and [43], while the operational method—by Luchko and Srivastava [36], Luchko and Gorenflo [35]. Connections of the solutions of the corresponding homogeneous equations with the roots of their characteristic quasipolynomials were discussed by Campos [9] and Miller and Ross [39]. The exact solution of a certain nonlinear fractional differential equation via solution of some transcendental equation was given by Kilbas and Saigo [30]. Formal power series solutions of some linear fractional differential equations were obtained by Al-Bassam [4], [5] and by Miller and Ross [39, Section IV.3]. The methods based on compositions of fractional differentiation operators with special functions were developed by Kilbas and Saigo [31]-[32], Saigo and Kilbas [44] and Kilbas, Bonilla, Rodriguez, Trujillo and Rivero [29], and were applied to obtain solutions of certain linear fractional differential equations with nonconstant coefficients. We also indicate that Srivastava, Owa and Nishimoto [48] and Campos [9] gave solutions of some fractional differential equations with fractional derivatives in the complex plane, see [45, Sections 22 and 43] in this connection.

We also mention that Dzherbashyan [14] and Nakhushev [40] studied some boundary value problems for a Sturm-Liouville fractional differential operators, Delbosco [10]—the existence of solution of Dirichlet type problem for a particular linear fractional differential equation, and numerical treatment of solutions of fractional differential equations began to develop recently by Shkhanukov [47], Blank [8], Diethelm [12], Diethelm and Walz [13] and Podlubny [43].

3. Absolutely continuous functions and fractional integro-differentiation of summable functions

Let $[a,b]$ $(-\infty < a < b < \infty)$ be a finite interval of the real axis $\mathbf{R}$, let $AC[a,b]$ be the space of functions absolutely continuous on $[a,b]$ and let $L(a,b)$ be space of Lebesgue measurable (real or complex valued) functions $\varphi(x)$:

$$L(a,b) = \{\varphi : \ \|\varphi\|_1 = \int_a^b |\varphi(t)|dt < \infty\}. \tag{3.1}$$

It is known (for example, see [33, p. 338]) that $AC[a,b]$ coincides with the space of primitives of Lebesgue summable functions:

$$g(x) \in AC[a,b] \Leftrightarrow g(x) = c + \int_a^x \varphi(t)dt, \ \varphi(x) \in L(a,b), \tag{3.2}$$

and therefore absolutely continuous function $g(x)$ have a summable derivative $g'(x) = \varphi(x)$ almost everywhere on $[a,b]$.

For $n \in \mathbf{N} = \{1,2,\ldots\}$ we denote by $AC^n[a,b]$ the space of functions $g(x)$ which have continuous derivatives up to order $n-1$ on $[a,b]$ with $g^{(n-1)}(x) \in AC[a,b]$. It is clear that $AC^1[a,b] = AC[a,b]$. Similarly to (3.2) the following property holds [45, Lemma 2.4]:

PROPERTY 1. *The space $AC^n[a,b]$ consists of those and only those functions $g(x)$, which are represented in the form*

$$g(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\varphi(t)dt + \sum_{k=0}^{n-1} c_k(x-a)^k, \ \varphi(x) \in L(a,b), \tag{3.3}$$

where c_k are arbitrary constants. It follow from (3.3) *that*

$$\varphi(x) = g^{(n)}(x), \ c_k = \frac{g^{(k)(a)}}{k!} \ (k = 0,1,\ldots,n-1). \tag{3.4}$$

Let $\alpha \in \mathbf{C}$ $(\mathrm{Re}(\alpha) > 0)$, let D^{α}_{a+} be the Riemann-Liouville fractional diferentiation operators defined by (1.3). If $0 < \mathrm{Re}(\alpha) \leq 1$, then

$$(D^{\alpha}_{a+}y)(x) = \frac{1}{\Gamma(1-\alpha+[\mathrm{Re}(\alpha)])}\frac{d}{dx}\int_a^x \frac{y(t)dt}{(x-t)^{\alpha-[\mathrm{Re}(\alpha)]}}, \tag{3.5}$$

and when $\alpha > 0$,

$$(D^{\alpha}_{a+}y)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} \frac{1}{\Gamma(1-\{\alpha\})}\int_a^x \frac{y(t)dt}{(x-t)^{\{\alpha\}}} \ (\alpha > 0), \tag{3.6}$$

in particular for $0 < \alpha < 1$:

$$(D^{\alpha}_{a+}y)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{y(t)dt}{(x-t)^{\alpha}} \ (0 < \alpha < 1). \tag{3.7}$$

Now we present some properties of the fractional calculus operators D_{a+}^{α} and I_{a+}^{α} defined by (1.3) and (1.6). Properties 2-4 and 6-7 are given in [45, Section 2], while Property 5 follows from Properties 3 and 4.

PROPERTY 2. *The fractional integration operator I_{a+}^{α} is bounded in $L(a,b)$*

$$\|I_{a+}^{\alpha} g\|_1 \leq \frac{(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|}\|g\|_1 \ (\alpha \in \mathbf{C},\ \mathrm{Re}(\alpha) > 0). \tag{3.8}$$

PROPERTY 3. *The fractional integration I_{a+}^{α} has the semigroup property*

$$(I_{a+}^{\alpha} I_{a+}^{\beta} g)(x) = (I_{a+}^{\alpha+\beta} g)(x) \ (\alpha \in \mathbf{C},\ \mathrm{Re}(\alpha) > 0;\ \beta \in \mathbf{C},\ \mathrm{Re}(\beta) > 0), \tag{3.9}$$

which is satisfied in almost every point $x \in [a,b]$ for $g(x) \in L(a,b)$.

PROPERTY 4. *If $\alpha \in \mathbf{C}$ $(\mathrm{Re}(\alpha) > 0)$ and $g(x) \in L(a,b)$, then the equality*

$$(D_{a+}^{\alpha} I_{a+}^{\alpha} g)(x) = g(x) \tag{3.10}$$

holds almost everywhere (a.e.) on $[a,b]$.

PROPERTY 5. *If $\alpha \in \mathbf{C}$ and $\beta \in \mathbf{C}$ be such that $\mathrm{Re}(\alpha) > \mathrm{Re}(\beta) > 0$, then for $g(x) \in L(a,b)$ the relation*

$$(D_{a+}^{\beta} I_{a+}^{\alpha} g)(x) = I_{a+}^{\alpha-\beta} g(x) \tag{3.11}$$

holds a.e. on $[a,b]$.

PROPERTY 6. *Let $\alpha \in \mathbf{C}$ $(\mathrm{Re}(\alpha) > 0)$, $n = -[-\mathrm{Re}(\alpha)]$ and $(I_{a+}^{n-\alpha} g)(x)$ be the fractional integral (1.6) of order $n-\alpha$. If $g(x) \in L(a,b)$ and $(I_{a+}^{n-\alpha} g)(x) \in AC^n[a,b]$, then the equality*

$$(I_{a+}^{\alpha} D_{a+}^{\alpha} g)(x) = g(x) - \sum_{j=1}^{n} \frac{g_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j},\ g_{n-\alpha}(x) = (I_{a+}^{n-\alpha} g)(x), \tag{3.12}$$

holds a.e. on $[a,b]$.

In particular, if $\alpha = n \in \mathbf{N}$

$$(I_{a+}^{n} D_{a+}^{n} g)(x) = g(x) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!}(x-a)^k, \tag{3.13}$$

and if $0 < \mathrm{Re}(\alpha) < 1$,

$$(I_{a+}^{\alpha} D_{a+}^{\alpha} g)(x) = g(x) - \frac{g_{1-\alpha}(a)}{\Gamma(\alpha}(x-a)^{\alpha-1},\ g_{1-\alpha}(x) = (I_{a+}^{1-\alpha} g)(x). \tag{3.14}$$

PROPERTY 7. *If $\alpha \in \mathbf{C}$ $(\mathrm{Re}(\alpha) > 0)$ and $\beta \in \mathbf{C}$ $(\mathrm{Re}(\beta) > 0)$, then*

$$\left(I_{a+}^{\alpha}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1} \tag{3.15}$$

and

$$\left(D_{a+}^{\alpha}(t-a)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1}. \tag{3.16}$$

In particular,

$$\left(D_{a+}^{\alpha}(t-a)^{\alpha-j}\right)(x)=0 \ (j=1,2,\dots,[\mathrm{Re}(\alpha)]+1). \tag{3.17}$$

4. Equivalence of the Cauchy-type problem and the Volterra nonlinear integral equation

In this section we prove that the Cauchy-type problem (1.1)-(1.2) and the nonlinear Volterra integral equation (1.8) are equivalent in the sense that if $y(x)\in L(a,b)$ satisfies one of these relations that it satisfies another one also. We prove such a result under the following assumption:

$$f[x,y(x)]\in L(a,b),\ \|f[x,y(x)]\|_1=M<\infty. \tag{4.1}$$

Below and later we shall understand all relations almost everywhere (a.e.) on $[a,b]$.

THEOREM 1. *Let* $\alpha\in\mathbf{C}$, $\mathrm{Re}(\alpha)>0$, *and let* $y(x)$ *and* $f[x,y(x)]$ *be Lebesgue measurable functions on* $[a,b]$ *such that the condition in* (4.1) *holds.*

If $y(x)\in L(a,b)$, *then* $y(x)$ *satisfies a.e. the relations* (1.1) *and* (1.2) *if and only if* $y(x)$ *satisfies a.e. the equation* (1.8).

Proof. Firts we prove the necessity. Let $y(x)\in L(a,b)$ satisfies a.e. the relations (1.1) and (1.2). Then (1.1) means that there exists a.e. on $[a,b]$ the fractional derivative

$$(D_{a+}^{\alpha}y)(x)\in L_1[a,b]. \tag{4.2}$$

According to (1.3)

$$(D_{a+}^{\alpha}y)(x)=\left(\frac{d}{dx}\right)^{n}(I_{a+}^{n-\alpha}y)(x),\ n=[\mathrm{Re}(\alpha)]+1, \tag{4.3}$$

and hence by Property 1 $(I_{a+}^{n-\alpha}y)(x)\in AC^n[a,b]$. By (4.1) we can apply Property 6 (with $g(x)$ being replaced by $y(x)$) and in accordance with (3.12) we have

(4.4)

$$(I_{a+}^{\alpha}D_{a+}^{\alpha}y)(x)=y(x)-\sum_{j=1}^{n}\frac{y_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j},\ y_{n-\alpha}(x)=(I_{a+}^{n-\alpha}y)(x).$$

By (1.3) and (4.4)

$$y_{n-\alpha}^{(n-j)}(x)=\left(\frac{d}{dx}\right)^{n-j}(I_{a+}^{(n-j)-(\alpha-j)}y)(x)=(D_{a+}^{\alpha-j}y)(x). \tag{4.5}$$

Using (4.5) and (1.2) we rewrite (4.4) in the form

$$(4.6)\qquad (I_{a+}^{\alpha}D_{a+}^{\alpha}y)(x) = y(x) - \sum_{j=1}^{n} \frac{(D_{a+}^{\alpha-j}y)(a)}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j} = y(x) - \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j}.$$

By (4.1) and Property 2 the integral $(I_{a+}^{\alpha}f[t,y(t)])(x) \in L(a,b)$ exists a.e. on $[a,b]$ and there holds the estimate

$$(4.7)\qquad \|I_{a+}^{\alpha}f[x,y(x)]\|_1 \le M\frac{(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|}.$$

Applying the operator I_{a+}^{α} to the both sides of (1.1) and using (4.6), we obtain the equation (1.8), and hence necessity is proved.

Now we prove the sufficiency. Let $y(x) \in L(a,b)$ satisfies a.e. the equation (1.8). Applying the operator D_{a+}^{α} to the both sides of (1.8), we have

$$(D_{a+}^{\alpha}y)(x) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha-j+1)}\left(D_{a+}^{\alpha}(t-a)^{\alpha-j}\right)(x) + \left(D_{a+}^{\alpha}I_{a+}^{\alpha}f[t,y(t)]\right)(x).$$

From here in accordance with the formula (3.17) and Property 4 (with $g(x)$ being replaced by $f[x,y(x)]$) we came to the equation (1.1).

Now we show that the relations in (1.2) are also hold. If $1 \le k \le n-1$, then in accordance with (3.16)-(3.17) and Property 5 (with $g(x)$ being replaced by $f[x,y(x)]$) we have

$$(D_{a+}^{\alpha-k}y)(x) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha-j+1)}\left(D_{a+}^{\alpha-k}(t-a)^{\alpha-j}\right)(x) + (D_{a+}^{\alpha-k}I_{a+}^{\alpha}f[t,y(t)])(x) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(k-j+1)}(x-a)^{k-j} + I_{a+}^{k}f[t,y(t)])(x)$$

and hence

$$(4.8)\quad (D_{a+}^{\alpha-k}y)(x) = \sum_{j=1}^{k} \frac{b_j}{(k-j)!}(x-a)^{k-j} + \frac{1}{(k-1)!}\int_a^x (x-t)^{k-1}f[t,y(t)]dt.$$

If $k = n$, then in accordance with (1.5) and (3.15) similarly to (4.8) we obtain

$$(4.9)\quad (D_{a+}^{\alpha-n}y)(x) = \sum_{j=1}^{n} \frac{b_j}{(n-j)!}(x-a)^{n-j} + \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}f[t,y(t)]dt.$$

Taking in (4.8) and (4.9) a limit as $x \to a+$ a.e., we obtain the relations in (1.2). Thus sufficiency is proved which completes the proof of theorem.

COROLLARY 1. *Let $\alpha \in \mathbf{C}$, $0 < \mathrm{Re}(\alpha) \leq 1$, and let $y(x)$ and $f[x, y(x)]$ be Lebesgue measurable functions on $[a, b]$ such that the condition in* (4.1) *holds.*

If $y(x) \in L(a, b)$, then $y(x)$ satisfies a.e. the relations in (1.9) *if and only if $y(x)$ satisfies a.e. the equation*

$$y(x) = \frac{b_1}{\Gamma(\alpha)}(x-a)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y(t)]dt}{(x-t)^{1-\alpha}}. \tag{4.10}$$

When $\alpha = n \in \mathbf{N}$, the problem (1.1)-(1.2) is equivalent to the problem (1.7), while the integral equation (1.8)—to the equation

$$y(x) = \sum_{j=1}^{n} \frac{b_j}{(n-j)!}(x-a)^{n-j} + \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f[t, y(t)]dt. \tag{4.11}$$

Therefore from Theorem 1 we obtain the corresponding statement for the Cauchy problem (1.7).

THEOREM 2. *Let $n \in \mathbf{N}$ and let $y(x)$ and $f[x, y(x)]$ be Lebesgue measurable functions on $[a, b]$ such that the condition in* (4.1) *holds.*

If $y(x) \in L(a, b)$, then $y(x)$ satisfies a.e. the relations in (1.7) *if and only if $y(x)$ satisfies a.e. the equation* (4.11).

COROLLARY 2. *Let $y(x)$ and $f[x, y(x)]$ be Lebesgue measurable functions on $[a, b]$ such that the condition in* (4.1) *holds.*

If $y(x) \in L(a, b)$, then $y(x)$ satisfies a.e. the relations

$$y'(x) = f[x, y(x)], \; y(a) = b_1 \in \mathbf{C} \tag{4.12}$$

if and only if $y(x)$ satisfies a.e. the equation

$$y(x) = b_1 + \int_a^x f[t, y(t)]dt. \tag{4.13}$$

5. Existence and uniqueness of the solution of the Cauchy-type problem

In this section we establish the existence and uniqueness of the solution of the Cauchy-type problem (1.1)-(1.2) in $L(a, b)$ under the conditions of Theorem 4.1 and two additional conditions, namely Lipschitzian of $f[x, y(x)]$ with respect to y:

$$\|f[x, y(x)] - f[x, Y(x)]\|_1 \leq A\|y(x) - Y(x)\|_1 \; (A > 0), \tag{5.1}$$

and

$$A \frac{(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|} < 1. \tag{5.2}$$

We denote by G_n ($n \in \mathbf{N}$) the folowing set of the points $(x, y) \in \mathbf{R}^2$:

(5.3) $G_n =$

$$\left\{(x,y) \in \mathbf{R}^2 : \ a \le x \le b, \ \|y(x) - \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)}(x-a)^{\alpha-j}\|_1 \le d\right\},$$

$$d \ge \frac{M(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|}, \tag{5.4}$$

where the constant b_j $(j = 1, \ldots, n)$ and M are given in (1.2) and (4.1), respectively.

THEOREM 3. *Let* $\alpha \in \mathbf{C}$, $\mathrm{Re}(\alpha) > 0$, *and let* $y(x)$ *and* $f[x, y(x)]$ *be Lebesgue measurable functions on* $[a, b]$ *such that the conditions in* (4.1), (5.1) *and* (5.2) *hold.*

Then there exists a unique solution $y(x) \in L(a, b)$ *of the Cauchy-type problem* (1.1)-(1.2) *in the region* G_n *defined by* (5.3), (5.4).

Proof. According to Theorem 1 it is sufficient to prove the existence of an unique solution $y(x) \in L(a, b)$ of the nonlinear Volterra integral equation (1.8) in G_n. First we use the method of successive approximation to prove the existence of the solution $y(x) \in L(a, b)$ of the equation (1.8) in G_n. Let

$$y_0(x) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)}(x-a)^{\alpha-j}, \tag{5.5}$$

$$y_n(x) = y_0(x) + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y_{n-1}(t)]dt}{(x-t)^{1-\alpha}} \quad (n = 1, 2, \ldots).$$

First of all we show that the points $(x, y_n(x))$ to be lie in G_n. Using (5.5), the condition (4.1) and Property 2, we have

$$\left\|y_n(x) - y_0(x)\right\|_1 = \left\|\frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y_{n-1}(t)]dt}{(x-t)^{1-\alpha}}\right\|_1 \le \frac{M(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|}. \tag{5.6}$$

Since the condition (5.4) holds, $(x, y_n(x)) \in G_n$.

Now we estimate $\|y_n(x) - y_{n-1}(x)\|_1$ for $n \in \mathbf{N}$. For $n = 1$ in accordance with (5.6) the estimate

$$\|y_1(x) - y_0(x)\|_1 \le \frac{M(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|} \tag{5.7}$$

holds. Using (5.5), (3.8) and apply the Lipschitzian condition (5.1) and (5.7), for $n = 2$ we have

$$\|y_2(x) - y_1(x)\|_1 = \left\|\frac{1}{\Gamma(\alpha)} \int_a^x \frac{[f[t, y_1(t) - f[t, y_0(t)]]dt}{(x-t)^{1-\alpha}}\right\|_1$$

$$\leq \frac{(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|}\|f[t,y_1(t)-f[t,y_0(t)]\|_1 \leq A\frac{(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|}\|y_1(t)-y_0(t)\|_1$$
$$\leq M\frac{(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|}\left(A\frac{(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|}\right).$$

Repeating such an estimate n times we arrive at the inequality

$$\|y_n(x)-y_{n-1}(x)\|_1 \leq M\frac{(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|}\left(A\frac{(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|}\right)^{n-1}. \tag{5.8}$$

By (5.2), it follows from here that the sequence $y_n(x)$ tends to a certain limit funtion $y(x)$ in $L(a,b)$:

$$\lim_{n\to\infty}\|y_n(x)-y(x)\|_1 = 0. \tag{5.9}$$

Using (3.8) and the Lipschitzian condition (5.1), we have

$$\left\|\frac{1}{\Gamma(\alpha)}\int_a^x \frac{f[t,y_n(t)]dt}{(x-t)^{1-\alpha}} - \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f[t,y(t)]dt}{(x-t)^{1-\alpha}}\right\|_1$$
$$\leq \frac{(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|}\|f[t,y_n(t)-f[t,y(t)]\|_1 \leq A\frac{(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|}\|y_n(x)-y(x)\|_1,$$

and hence

$$\lim_{n\to\infty}\|\frac{1}{\Gamma(\alpha)}\int_a^x \frac{f[t,y_n(t)]dt}{(x-t)^{1-\alpha}} - \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f[t,y(t)]dt}{(x-t)^{1-\alpha}}\|_1 = 0. \tag{5.10}$$

It follows from (5.9) and (5.10) that $y(x)$ is the solution of the equation (1.8) in the space $L(a,b)$.

Now we show that this solution $y(x)$ is an unique. We suppose that there exist two solutions $y(x)$ and $Y(x)$ of the equation (1.8). Substituting them into (1.8) and subtracting one from the other we have

$$\|y(x)-Y(x)\|_1 = \|\frac{1}{\Gamma(\alpha)}\int_a^x \frac{(f[t,y(t)]-f[t,Y(t)])\,dt}{(x-t)^{1-\alpha}}\|_1.$$

Applying (3.8) and (5.1), we obtain the estimate

$$\|y(x)-Y(x)\|_1 \leq A\frac{(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|}\|y(x)-Y(x)\|_1,$$

and hence

$$1 \leq A\frac{(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|},$$

which contradicts with the assumption (5.2). Thus there exist a one solution $y(x) \in L(a,b)$ of the equations (1.8), and hence the Cauchy-type problem (1.1)-(1.2). This completes the proof of theorem.

COROLLARY 3. *Let $\alpha > 0$ and let $y(x)$ and $f[x, y(x)]$ be Lebesgue measurable functions on $[a, b]$ such that the conditions in* (4.1), (5.1) *hold and*

$$A\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} < 1. \tag{5.11}$$

Then there exists a unique solution $y(x) \in L(a, b)$ of the Cauchy-type problem (1.1)-(1.2) *in the region G_n defined by* (5.3) *and*

$$d \geq \frac{M(b-a)^{\alpha}}{\Gamma(\alpha+1)}. \tag{5.12}$$

When $\alpha = n \in \mathbf{N}$, the condition (5.11) takes the form

$$A\frac{(b-a)^{n}}{n!} < 1, \tag{5.13}$$

and the set G_n in (5.3), (5.12) is given by

$$G_n = \left\{(x, y) \in \mathbf{R}^2 : \ a \leq x \leq b, \ \left\| y(x) - \sum_{j=1}^{n} \frac{b_j}{(n-j)!}(x-a)^{n-j} \right\|_1 \leq d \right\}, \tag{5.14}$$

$$d \geq \frac{M(b-a)^{n}}{n!}, \tag{5.15}$$

where the constants b_j $(j = 1, \ldots, n)$ are given in (1.7). Then from Corollary 3 we obtain the corresponding statement for the Cauchy problem (1.7).

THEOREM 4. *Let $n \in \mathbf{N}$ and let $y(x)$ and $f[x, y(x)]$ be Lebesgue measurable functions on $[a, b]$ such that the conditions in* (4.1), (5.1) *and* (5.13) *hold.*

Then there exists a unique solution $y(x) \in L(a, b)$ of the Cauchy problem (1.7) *in the region G_n defined by* (5.14), (5.15).

COROLLARY 4. *Let $y(x)$ and $f[x, y(x)]$ be Lebesgue measurable functions on $[a, b]$ such that the conditions in* (4.1), (5.1) *hold and $A(b-a) < 1$.*

Then there exists a unique solution $y(x) \in L(a, b)$ of the Cauchy problem (4.12) *in the region G_1 defined by*

$$G_1 = \{(x, y) \in \mathbf{R}^2 : \ a \leq x \leq b, \ \|y(x) - b_1\|_1 \leq d\}, \ d \geq M(b-a). \tag{5.16}$$

6. Existence and uniqueness of the solution of the Cauchy-type problems in the case $0 < \mathrm{Re}(\alpha) \leq 1$

When $0 < \mathrm{Re}(\alpha) \leq 1$, the set G_1 takes the form

$$G_1 = \{(x, y) \in \mathbf{R}^2 : \ a \leq x \leq b, \ \|y(x) - \frac{b_1}{\Gamma(\alpha)}(x-a)^{\alpha-1}\|_1 \leq d\}, \tag{6.1}$$

$$d \geq \frac{M(b-a)^{\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|}, \tag{6.2}$$

From Theorem 3 we obtain the existence and uniquenes result for the Cauchy-type problem (1.9).

THEOREM 5. *Let* $\alpha \in \mathbf{C}$, $0 < \text{Re}(\alpha) \leq 1$, *and let* $y(x)$ *and* $f[x, y(x)]$ *be Lebesgue measurable functions on* $[a, b]$ *such that the conditions in* (4.1), (5.1) *and* (5.2) *hold.*

Then there exists a unique solution $y(x) \in L(a, b)$ *of the Cauchy-type problem* (1.9) *in the region* G_1 *defined by* (6.1), (6.2).

COROLLARY 5. *Let* $0 < \alpha \leq 1$ *and let* $y(x)$ *and* $f[x, y(x)]$ *be Lebesgue measurable functions on* $[a, b]$ *such that the conditions in* (4.1), (5.1) *and* (5.11) *hold.*

Then there exists a unique solution $y(x) \in L(a, b)$ *of the Cauchy-type problem*

$$(D_{a+}^{\alpha} y)(x) = f[x, y(x)] \ (0 < \alpha \leq 1), \ (I_{a+}^{1-\alpha} y)(a+) = b_1, \ b_1 \in \mathbf{C}, \tag{6.3}$$

in the region G_1 *defined by* (6.1), (5.12).

When $0 < \text{Re}(\alpha) < 1$, the result of Theorem 5 stays true for the weighted Cauchy-type problem (1.10). Its proof is based on two preliminary assertions.

LEMMA 1. *Let* $\alpha \in \mathbf{C}$, $0 < \text{Re}(\alpha) < 1$, *and let* $y(x)$ *be Lebesgue measurable functions on* $[a, b]$. *If there exists a.e. a limit*

$$\lim_{x \to a+} (x - a)^{1-\alpha} y(x) = c, \ c \in \mathbf{C}, \tag{6.4}$$

then also there exists a.e. a limit

$$(I_{a+}^{1-\alpha} y)(a+) \equiv \lim_{x \to a+} (I_{a+}^{1-\alpha} y)(x) = c\Gamma(\alpha). \tag{6.5}$$

Proof. Choose arbitrary $\epsilon > 0$. By (6.4) there exists $\delta = \delta(\epsilon) > 0$ such that

$$|(t - a)^{1-\alpha} y(t) - c| < \epsilon \frac{|\Gamma(1 - \alpha)|}{\Gamma[\text{Re}(\alpha)]\Gamma[1 - \text{Re}(\alpha)]} \tag{6.6}$$

for $a < t < a + \delta$. According to (3.15),

$$\Gamma(\alpha) = (I_{a+}^{1-\alpha} (t - a)^{\alpha-1})(x) \ (0 < \alpha < 1). \tag{6.7}$$

Using (6.7) and (1.6), we have

$$\begin{aligned} |(I_{a+}^{1-\alpha} y)(x) - c\Gamma(\alpha)| &= |(I_{a+}^{1-\alpha} y)(x) - c(I_{a+}^{1-\alpha} (t - a)^{\alpha-1})(x)| \\ &\leq \frac{1}{|\Gamma(1 - \alpha)|} \int_a^x (x - t)^{-\text{Re}(\alpha)} |y(t) - c(t - a)^{\alpha-1}| dt \\ &\leq \frac{1}{|\Gamma(1 - \alpha)|} \int_a^x (x - t)^{-\text{Re}(\alpha)} (t - a)^{\text{Re}(\alpha)-1} |(t - a)^{1-\alpha} y(t) - c| dt. \end{aligned}$$

If we choose $a < x < a + \delta$, then $a < t < x < a + \delta$ and we can apply the estimate (6.6) and the relation (3.15) to obtain

$$|(I_{a+}^{1-\alpha}y)(x) - c\Gamma(\alpha)| < \frac{\epsilon}{\Gamma[\mathrm{Re}(\alpha)]}\left(I_{a+}^{1-\mathrm{Re}(\alpha)}(t-a)^{\mathrm{Re}(\alpha)-1}\right)(x) = \epsilon,$$

which proves (6.5).

LEMMA 2. *Let $\alpha \in \mathbf{C}$, $0 < \mathrm{Re}(\alpha) < 1$, and let $y(x)$ be Lebesgue measurable functions on $[a,b]$ such that there exists a.e. a limit*

$$(I_{a+}^{1-\alpha}y)(a+) \equiv \lim_{x\to a+}(I_{a+}^{1-\alpha}y)(x) = b, \ b \in \mathbf{C}. \tag{6.8}$$

If there exist a.e. a limit $\lim_{x\to a+}(x-a)^{1-\alpha}y(x)$, then

$$\lim_{x\to a+}(x-a)^{1-\alpha}y(x) = \frac{b}{\Gamma(\alpha)}. \tag{6.9}$$

Proof. Suppose that the limit in (6.9) is equal to c:

$$\lim_{x\to a+}(x-a)^{1-\alpha}y(x) = c.$$

Then by Lemma 1

$$(I_{a+}^{1-\alpha}y)(a+) \equiv \lim_{x\to a+}(I_{a+}^{1-\alpha}y)(x) = c\Gamma(\alpha),$$

and hence, in accordance with (6.8), $c = b/\Gamma(\alpha)$, which proves (6.9).

From Theorem 5 and Lemmas 1 and 2 we obtain the existence and uniquenes result for the weightd Cauchy-type problem (1.10).

THEOREM 6. *Let $\alpha \in \mathbf{C}$, $0 < \mathrm{Re}(\alpha) < 1$, and let $y(x)$ and $f[x,y(x)]$ be Lebesgue measurable functions on $[a,b]$ such that the conditions in* (4.1), (5.1) *and* (5.2) *hold.*

Then there exists a unique solution $y(x) \in L(a,b)$ of the weighted Cauchy-type problem (1.10) *in the region G_1 defined by* (6.1), (6.2).

Proof. If $y(x)$ satisfies the conditions (1.10), then by Lemma 1 $y(x)$ also satisfies the conditions (1.9) with $b = c\Gamma(\alpha)$:

$$(D_{a+}^{\alpha}y)(x) = f[x,y(x)], \ (I_{a+}^{1-\alpha}y)(a+) = c\Gamma(\alpha). \tag{6.10}$$

According to Theorem 5, there exists an unique solution $y(x) \in L(a,b)$ of this problem. By Lemma 2 $y(x)$ is also the solution of the weighted Cauchy-type problem (1.10). This $y(x)$ will be unique solution of (1.10). Indeed, if we suppose that the weighted problem (1.10) has two different solutions in $L(a,b)$, then by Lemma 1 they will be also two different solutions of the Cauchy-type problem (6.10) in $L(a,b)$, which contradicts its uniqueness.

COROLLARY 6. *Let $0 < \alpha < 1$ and let $y(x)$ and $f[x,y(x)]$ be Lebesgue measurable functions on $[a,b]$ such that the conditions in* (4.1), (5.1) *and* (5.11) *hold.*

Then there exists a unique solution $y(x) \in L(a,b)$ *of the weighted Cauchy-type problem*

(6.11) $(D^{\alpha}_{a+}y)(x) = f[x, y(x)]$ $(0 < \alpha < 1)$, $\lim_{x\to a+}(x-a)^{1-\alpha}y(x) = c$, $c \in \mathbf{C}$,

in the region G_1 *defined by* (6.1), (5.12).

REMARK 1. The results in Sections 4-5 and 6 can be extended to systems of Cauchy-type problems (1.1)-(1.2), (1.9) and (1.10).

7. Examples

In this section we give the examples of solution $y(x) \in L(a,b)$ of the Cauchy-type problem (1.1)-(1.2). All relations below are understood almost everywhere on a finite interval $[a,b]$. We note that the uniqueness of the solutions $y(x) \in L(a,b)$ of the Cauchy-type problems for linear differential equations of fractional order in Examples 1-4 follows from the uniqueness of the solutions $y(x) \in L(a,b)$ of the corresponding linear Volterra integral equations of the form (1.8).

EXAMPLE 1. The Cauchy-type problem for the homogeneous linear differential equation

$$(D^{\alpha}_{a+}y)(x) = \lambda y(x) \ \ (a \leq x \leq b;\ \alpha \in \mathbf{C},\ \mathrm{Re}(\alpha) > 0);\ \lambda \in \mathbf{C}), \tag{7.1}$$

with the initial conditions (1.2) has the unique solution $y(x) \in L(a,b)$:

$$y(x) = \sum_{k=1}^{n} b_k (x-a)^{\alpha-k} E_{\alpha,\alpha+1-k}\left(\lambda(x-a)^{\alpha}\right), \tag{7.2}$$

where the Mittag-Leffler functions $E_{\alpha,\alpha+1-k}(z)$ $(k = 1, \ldots, n)$ are given by (2.3).

EXAMPLE 2. The Cauchy-type problem for the inhomogeneous linear differential equation

$$(D^{\alpha}_{a+}y)(x) = \lambda y(x) + f(x) \ \ (a \leq x \leq b;\ \alpha \in \mathbf{C},\ \mathrm{Re}(\alpha) > 0);\ \lambda \in \mathbf{C}), \tag{7.3}$$

with $f(x) \in L(a,b)$ and the initial conditions (1.2) has the unique solution $y(x) \in L(a,b)$:

$$y(x) = \sum_{k=1}^{n} b_k (x-a)^{\alpha-k} E_{\alpha,\alpha+1-k}\left(\lambda(x-a)^{\alpha}\right) + \int_a^x (x-t)^{\alpha-1} E_{\alpha,\alpha}\left[\lambda(x-t)^{\alpha}\right] f(t)dt. \tag{7.4}$$

REMARK 2. The uniqueness of the solutions (7.4) of the Cauchy-type problem (7.3), (1.2) in some subspace of $L(a,b)$ was proved by Barrett [7, Theorem 2.1].

EXAMPLE 3. The Cauchy-type problem for the equation in the theory of voltammetry at expanding electrodes [41, p. 159]

(7.5) $$x^{1/2}(D_{0+}^{1/2}y)(x) + x^w y(x) = 1 \ (0 < x \leq b), \ (D_{0+}^{1/2}y)(0+) = b$$

with $0 < w \leq 1/2$ has the unique solution $y(x) \in L(a,b)$:

(7.6) $$y(x) = b\pi^{-1/2}x^{-1/2}E_{1/2,2w,2w-2}(-x^w) + \sqrt{\pi}E_{1/2,2w,2w-1}(-x^w).$$

Here $E_{1/2,2w,2w-2}(z)$ is the so-called Mittag-Leffler type function defined for $\alpha > 0$, $m > 0$ and $l \in \mathbf{R}$ by [31]

(7.7)
$$E_{\alpha,m,l}(z) = \sum_{k=0}^{\infty} c_k z^k, \ c_0 = 1, \ c_k = \prod_{i=0}^{k-1} \frac{\Gamma[\alpha(im+l)+1]}{\Gamma[\alpha(im+l+1)+1]} \ (k = 1, 2, \ldots).$$

For $m = 1$, this entire funtion coincides with the Mittag-Leffler function (2.4) with the exactness to the multiplier $\Gamma(\alpha l + 1)$:

(7.8) $$E_{\alpha,1,l}(z) = \Gamma(\alpha l + 1)E_{\alpha,\alpha l+1}(z).$$

EXAMPLE 4. The Cauchy-type problem for the equation of the polarography theory [51]

(7.9) $$(D_{0+}^{1/2}y)(x) = ax^\beta y(x) + x^{-1/2} \ (0 \leq x \leq b), \ (D_{0+}^{1/2}y)(0+) = b$$

with $a \in \mathbf{R}$ and $-1/2 < \beta \leq 0$ has the unique solution $y(x) \in L(a,b)$:

(7.10)
$$y(x) = b\pi^{-1/2}x^{-1/2}E_{1/2,2\beta+1,2\beta-1}\left(ax^{\beta+1/2}\right) + \sqrt{\pi}E_{1/2,2\beta+1,2\beta}\left(ax^{\beta+1/2}\right).$$

EXAMPLE 5. We denote by $B(a,b) \subset L(a,b)$ the space of Lebesgue measurable functions on $[a,b]$ bounded almost everywhere on $[a,b]$:

(7.11) $$K = \|y(x)\|_\infty < \infty.$$

If $f(x) \in L(a,b)$, $n \in \mathbf{R}$ $(n \neq 1)$, $\beta \in \mathbf{R}$ and $\lambda \in \mathbf{C}$, then the Cauchy-type problem for the nolinear fractional differential equation

(7.12) $$(D_{a+}^\alpha y)(x) = \lambda(x-a)^\beta y^n(x) + f(x) \ (a \leq x \leq b; \ \alpha \in \mathbf{C}, \ \mathrm{Re}(\alpha) > 0),$$

has the unique solution $y(x) \in B(a,b)$ in the region G_n defined by (5.3), (5.4) with $M = |\lambda|K^n(b-a)^{\beta+1} + \|f\|_1$ provided that

(7.13) $$n \geq 1; \ \beta \geq 0; \ n|\lambda|\frac{K^{n-1}(b-a)^{\mathrm{Re}(\alpha)+\beta}}{\mathrm{Re}(\alpha)|\Gamma(\alpha)|} < 1.$$

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