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SOME SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS

Abstract. The object of this paper is to introduce a new concept of lacunary strong convergence with respect to an Orlicz function and examine some properties of the resulting sequence spaces. We establish some elementary connections between lacunary strong convergence and lacunary strong convergence with respect to an Orlicz function which satisfies Δ_2-condition. It is also shown that if a sequence is lacunary strongly convergent with respect to an Orlicz function then it is lacunary statistically convergent. In addition, lacunary strong convergence with respect to an Orlicz function is compared to other summability methods.

1. Introduction

By a lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, \ldots$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and we let $h_r = k_r - k_{r-1}$. The ratio k_r/k_{r-1} will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [5] as follows

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} |x_k - \ell| = 0 \text{ for some } \ell\Big\}.$$

The space N_θ is a BK-space with the norm

$$\|x\|_\theta = \sup_r \Big(h_r^{-1} \sum_{k\in I_r} |x_k|\Big).$$

N_θ^0 denotes the subset of those sequences in N_θ for which $\ell = 0$. $(N_\theta^0, \|\cdot\|_\theta)$ is also a BK-space. There is a strong connection [5] between N_θ and the

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space ω of strongly Cesáro summable sequences, which is defined by

$$\omega = \Big\{x = (x_k) : \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} |x_k - \ell| = 0 \text{ for some } \ell\Big\}.$$

In the special case where $\theta = (2^r)$, we have $N_\theta = \omega$.

Recall [9]–[12] that an Orlicz function M is a continuous, convex, non-decreasing function defined for $x \geq 0$ such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$. Lindenstrauss and Tzafriri [12] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \{x = (x_k) : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty \text{ for some } \rho > 0\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \leq 1\Big\}$$

becomes a Banach space which is called an Orlicz sequence space. Lindenstrauss and Tzafriri proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p for some $p \geq 1$, thereby answering a general conjecture that every infinite dimensional Banach space contains a closed subspace isomorphic to c_0 or some ℓ_p, positively for a class of spaces (see [11] and [18] for discussion of this and related conjectures). For $M(x) = x^p$; $1 \leq p < \infty$, the spaces ℓ_M coincide with the classical sequence spaces ℓ_p.

Recently, Parashar and Choudhary [20] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function M, which generalized the well-known Orlicz sequence space ℓ_M and strongly summable sequence spaces $[C, 1, p]$, $[C, 1, p]_0$ and $[C, 1, p]_\infty$. It may be noted here that the spaces of strongly summable sequences were discussed by Maddox [14]. Nuray and Gülcü [19], Demirci [3] and others have also used an Orlicz function to construct some sequence spaces.

In the present paper we introduce a new concept of lacunary strong convergence with respect to an Orlicz function and examine some properties of the resulting spaces. We establish some elementary connections between lacunary strong convergence and lacunary strong convergence with respect to an Orlicz function which satisfies Δ_2-condition. It is shown that if a sequence is lacunary strongly convergent with respect to an Orlicz function then it is lacunary statistically convergent. Also, lacunary strong convergence with respect to an Orlicz function is compared to other summability methods.

We now introduce the generalizations of the spaces of lacunary strongly convergent sequences.

DEFINITION 1.1. Let M be an Orlicz function and $p = (p_k)$ be any sequence of strictly positive real numbers. We define the following sequence spaces

$$[N_\theta, M, p] = \Big\{ x = (x_k) : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} \Big[M\Big(\frac{|x_k - \ell|}{\rho}\Big)\Big]^{p_k} = 0 \text{ for some } \ell, \text{ and } \rho > 0 \Big\},$$

$$[N_\theta, M, p]_0 = \Big\{ x = (x_k) : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} \Big[M\Big(\frac{|x_k|}{\rho}\Big)\Big]^{p_k} = 0 \text{ for some } \rho > 0 \Big\},$$

$$[N_\theta, M, p]_\infty = \Big\{ x = (x_k) : \sup_r h_r^{-1} \sum_{k\in I_r} \Big[M\Big(\frac{|x_k|}{\rho}\Big)\Big]^{p_k} < \infty \text{ for some } \rho > 0 \Big\}.$$

We denote $[N_\theta, M, p]$, $[N_\theta, M, p]_0$ and $[N_\theta, M, p]_\infty$ as $[N_\theta, M]$, $[N_\theta, M]_0$ and $[N_\theta, M]_\infty$ when $p_k = 1$ for all k. If $x \in [N_\theta, M]$ we say that x is lacunary strongly convergent with respect to the Orlicz function M.

Some well-known spaces are obtained by specializing θ, M and p.

(i) If $M(x) = x$, $p_k = 1$ for all k, then $[N_\theta, M, p] = N_\theta$, $[N_\theta, M, p]_0 = N_\theta^0$ (Freedman et al. [5]).

(ii) If $M(x) = x$, $\theta = (2^r)$, then $[N_\theta, M, p] = [C, 1, p]$, $[N_\theta, M, p]_0 = [C, 1, p]_0$, $[N_\theta, M, p]_\infty = [C, 1, p]_\infty$ (Maddox [14]).

(iii) If $M(x) = x$, $p_k = 1$ for all k, $\theta = (2^r)$ then $[N_\theta, M, p] = \omega$, $[N_\theta, M, p]_0 = \omega_0$, $[N_\theta, M, p]_\infty = \omega_\infty$ (Freedman et al. [5], Maddox [14], [15], [17]).

(iv) If $\theta = (2^r)$ then $[N_\theta, M, p] = W(M, p)$, $[N_\theta, M, p]_0 = W_0(M, p)$, $[N_\theta, M, p]_\infty = W_\infty(M, p)$ (Parashar and Choudhary [20]).

2. Linear topological structure of $[N_\theta, M, p]$ spaces and inclusion theorems

In this section we examine some topological properties of $[N_\theta, M, p]$ spaces and investigate some inclusion relations between these spaces.

THEOREM 2.1. *For any Orlicz function M and a bounded sequence $p = (p_k)$ of strictly positive real numbers, $[N_\theta, M, p]$, $[N_\theta, M, p]_0$ and $[N_\theta, M, p]_\infty$ are linear spaces over the set of complex numbers.*

Proof. We shall prove only for $[N_\theta, M, p]_0$. The others can be treated similarly. Let $x, y \in [N_\theta, M, p]_0$ and $\alpha, \beta \in C$. In order to prove the result we need to find some $\rho_3 > 0$ such that $\lim_r h_r^{-1} \sum_{k\in I_r} [M(\frac{|\alpha x_k + \beta y_k|}{\rho_3})]^{p_k} = 0$.

Since $x, y \in [N_\theta, M, p]_0$, there exist a positive ρ_1 and ρ_2 such that

$$\lim_r h_r^{-1} \sum_{k\in I_r} \left[M\left(\frac{|x_k|}{\rho_1}\right)\right]^{p_k} = 0$$

and

$$\lim_r h_r^{-1} \sum_{k\in I_r} \left[M\left(\frac{|y_k|}{\rho_2}\right)\right]^{p_k} = 0.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non-decreasing and convex,

$$\begin{aligned} &h_r^{-1} \sum_{k\in I_r} \left[M\left(\frac{|\alpha x_k + \beta y_k|}{\rho_3}\right)\right]^{p_k} \\ &\quad \le h_r^{-1} \sum_{k\in I_r} \left[M\left(\frac{|\alpha x_k|}{\rho_3} + \frac{|\beta y_k|}{\rho_3}\right)\right]^{p_k} \\ &\quad \le h_r^{-1} \sum_{k\in I_r} \frac{1}{2^{p_k}} \left[M\left(\frac{|x_k|}{\rho_1}\right) + M\left(\frac{|y_k|}{\rho_2}\right)\right]^{p_k} \\ &\quad < h_r^{-1} \sum_{k\in I_r} \left[M\left(\frac{|x_k|}{\rho_1}\right) + M\left(\frac{|y_k|}{\rho_2}\right)\right]^{p_k} \\ &\quad \le C h_r^{-1} \sum_{k\in I_r} \left[M\left(\frac{|x_k|}{\rho_1}\right)\right]^{p_k} + C h_r^{-1} \sum_{k\in I_r} \left[M\left(\frac{|y_k|}{\rho_2}\right)\right]^{p_k} \to 0 \text{ as } r \to \infty, \end{aligned}$$

where $C = \max(1, 2^{H-1})$, $H = \sup p_k$; so that $\alpha x + \beta y \in [N_\theta, M, p]_0$. This proves that $[N_\theta, M, p]_0$ is linear.

THEOREM 2.2. *For any Orlicz function M and a bounded sequence $p = (p_k)$ of strictly positive real numbers, $[N_\theta, M, p]_0$ is a topological linear space, totally paranormed by*

$$g(x) = \inf\left\{\rho^{p_r/H} : \left(h_r^{-1} \sum_{k\in I_r} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k}\right)^{1/H} \le 1, \quad r = 1, 2, \dots\right\},$$

where $H = \max(1, \sup_k p_k)$.

Proof. Clearly $g(x) = g(-x)$. By using Theorem 2.1 for a $\alpha = \beta = 1$, we get $g(x+y) \le g(x) + g(y)$. Since $M(0) = 0$, we get $\inf\{\rho^{p_r/H}\} = 0$ for $x = 0$. Conversely, suppose $g(x) = 0$, then

$$\inf\left\{\rho^{p_r/H} : \left(h_r^{-1} \sum_{k\in I_r} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k}\right)^{1/H} \le 1\right\} = 0.$$

This implies that for a given $\epsilon > 0$, there exists some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_k|}{\rho_\epsilon}\right)\right]^{p_k}\right)^{1/H} \leq 1.$$

Thus

$$\left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_k|}{\epsilon}\right)\right]^{p_k}\right)^{1/H} \leq \left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_k|}{\rho_\epsilon}\right)\right]^{p_k}\right)^{1/H} \leq 1.$$

Suppose $x_m \neq 0$ for some $m \in I_r$. Let $\epsilon \to 0$, then $\left(\frac{|x_m|}{\epsilon}\right) \to \infty$. It follows that

$$\left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_k|}{\rho_\epsilon}\right)\right]^{p_k}\right)^{1/H} \to \infty$$

which is a contradiction. Therefore $x_m = 0$ for each m. Finally, we prove that scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = \inf \left\{\rho^{p_r/H} : \left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k}\right)^{1/H} \leq 1, \quad r = 1, 2, \dots \right\}.$$

Then

$$g(\lambda x) = \inf \left\{(|\lambda| s)^{p_r/H} : \left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_k|}{s}\right)\right]^{p_k}\right)^{1/H} \leq 1, \quad r = 1, 2, \dots \right\},$$

where $s = \rho/|\lambda|$. Since $|\lambda|^{p_r} \leq \max(1, |\lambda|^{\sup p_r})$, we have

$$g(\lambda x) \leq (\max(1, |\lambda|^{\sup p_r}))^{1/H} \inf \left\{s^{p_r/H} : \left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_k|}{s}\right)\right]^{p_k}\right)^{1/H} \leq 1, \quad r = 1, 2, \dots \right\}$$

which converges to zero as x converges to zero in $[N_\theta, M, p]_0$.

Now suppose $\lambda_n \to 0$ and x is fixed in $[N_\theta, M, p]_0$. For arbitrary $\epsilon > 0$, let N be a positive integer such that

$$h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k} < (\epsilon/2)^H \quad \text{for some } \rho > 0 \text{ and all } r > N.$$

This implies that

$$\left(h_r^{-1}\sum_{k\in I_r}\left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k}\right)^{1/H} < \epsilon/2 \quad \text{for some } \rho > 0 \text{ and all } r > N.$$

Let $0 < |\lambda| < 1$, using convexity of M, for $r > N$, we get

$$h_r^{-1}\sum_{k\in I_r}\left[M\left(\frac{|\lambda x_k|}{\rho}\right)\right]^{p_k} < h_r^{-1}\sum_{k\in I_r}\left[|\lambda| M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k} < (\epsilon/2)^H.$$

Since M is continuous everywhere in $[0,\infty)$, then for $r \leq N$,

$$f(t) = h_r^{-1}\sum_{k\in I_r}\left[M\left(\frac{|tx_k|}{\rho}\right)\right]^{p_k}$$

is continuous at 0. So there is $1 > \delta > 0$ such that $|f(t)| < (\epsilon/2)^H$ for $0 < t < \delta$. Let K be such that $|\lambda_n| < \delta$ for $n > K$, then for $n > K$ and $r \leq N$,

$$\left(h_r^{-1}\sum_{k\in I_r}\left[M\left(\frac{|\lambda_n x_k|}{\rho}\right)\right]^{p_k}\right)^{1/H} < \epsilon/2.$$

Thus

$$\left(h_r^{-1}\sum_{k\in I_r}\left[M\left(\frac{|\lambda_n x_k|}{\rho}\right)\right]^{p_k}\right)^{1/H} < \epsilon$$

for $n > K$ and all r, so that $g(\lambda x) \to 0$ $(\lambda \to 0)$.

DEFINITION 2.3 [10]. An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$.

It is easy to see that always $K > 2$. The Δ_2-condition is equivalent to the satisfaction of inequality $M(\ell u) \leq K(\ell)M(u)$ for all values of u and for $\ell > 1$.

LEMMA 2.4. *Let M be a an Orlicz function which satisfies Δ_2-condition and let $0 < \delta < 1$. Then for each $x \geq \delta$ we have $M(x) < Kx\delta^{-1}M(2)$ for some constant $K > 0$.*

Proof. Since M is non-decreasing and convex, and $x < \delta^{-1}x < 1+\delta^{-1}x$ for $x \geq \delta$, it follows that $M(x) < M(1+\delta^{-1}x) = M(\frac{1}{2}\cdot 2+\frac{1}{2}\cdot 2\delta^{-1}x) < \frac{1}{2}M(2)+\frac{1}{2}M(2\delta^{-1}x)$. Since M satisfies Δ_2-condition, there is a constant $K > 2$ such that $M(2\delta^{-1}x) \leq \frac{1}{2}K\delta^{-1}xM(2)$, therefore $M(x) < \frac{1}{2}K\delta^{-1}xM(2) + \frac{1}{2}K\delta^{-1}xM(2) = K\delta^{-1}xM(2)$ and hence the lemma.

THEOREM 2.5. *For any Orlicz function M which satisfies Δ_2-condition, we have $N_\theta \subseteq [N_\theta, M]$.*

Proof. Let $x \in N_\theta$ so that

$$A_r \equiv h_r^{-1} \sum_{k \in I_r} |x_k - \ell| \to 0 \text{ as } r \to \infty, \text{ for some } \ell.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \epsilon$ for $0 \le t \le \delta$. We can write

$$h_r^{-1} \sum_{k \in I_r} M(|x_k - \ell|) = h_r^{-1} \sum_{k \in I_r,\, |x_k - \ell| \le \delta} M(|x_k - \ell|)$$

$$+ h_r^{-1} \sum_{k \in I_r,\, |x_k - \ell| > \delta} M(|x_k - \ell|) < h_r^{-1}(h_r \epsilon) + h_r^{-1} K \delta^{-1} M(2) h_r A_r,$$

by Lemma 2.4. Letting $r \to \infty$, it follows that $x \in [N_\theta, M]$.

The method of the proof of Theorem 2.5 shows that, for any Orlicz function M which satisfies Δ_2-condition, we have $N_\theta^0 \subseteq [N_\theta, M]_0$ and $N_\theta^\infty \subseteq [N_\theta, M]_\infty$.

THEOREM 2.6. *Let $0 < p_k \le q_k$ and (q_k/p_k) be bounded. Then $[N_\theta, M, q] \subseteq [N_\theta, M, p]$.*

Proof. Let $x \in [N_\theta, M, q]$. We write $w_k = \left[M\left(\frac{|x_k - \ell|}{\rho}\right)\right]^{q_k}$, $p_k/q_k = \lambda_k$, so that $0 < \lambda < \lambda_k \le 1$, with λ constant. Now define $u_k = w_k (w_k \ge 1)$, $u_k = 0 (w_k < 1)$, $v_k = 0 (w_k \ge 1)$, $v_k = w_k (w_k < 1)$, so that $w_k = u_k + v_k$, $w_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. It follows that $u_k^{\lambda_k} \le u_k \le w_k$ and $v_k^{\lambda_k} \le v_k^{\lambda}$. Therefore $h_r^{-1} \sum_{k \in I_r} \omega_k^{\lambda_k} \le h_r^{-1} \sum_{k \in I_r} \omega_k + h_r^{-1} \sum_{k \in I_r} v_k^{\lambda}$.

Now, for each r,

$$h_r^{-1} \sum_{k \in I_r} v_k^{\lambda} = \sum_{k \in I_r} (h_r^{-1} v_k)^{\lambda} (h_r^{-1})^{1-\lambda}$$

$$\le \Big(\sum_{k \in I_r} [(h_r^{-1} v_k)^{\lambda}]^{1/\lambda} \Big)^{\lambda} \Big(\sum_{k \in I_r} [(h_r^{-1})^{1-\lambda}]^{1/1-\lambda} \Big)^{1-\lambda},$$

by Hölder's inequality

$$= (h_r^{-1} \sum_{k \in I_r} v_k)^{\lambda},$$

and so,

$$h_r^{-1} \sum_{k \in I_r} \omega_k^{\lambda_k} \le h_r^{-1} \sum_{k \in I_r} \omega_k + [h_r^{-1} \sum_{k \in I_r} v_k]^{\lambda}$$

and hence $x \in [N_\theta, M, p]$.

3. Comparison with other summability methods

In this section lacunary strong convergence with respect to an Orlicz function is compared to lacunary statistical convergence and other summability methods.

We first study the inclusions $[\omega, M, p] \subset [N_\theta, M, p]$ and $[N_\theta, M, p] \subset [\omega, M, p]$ under certain restrictions on $\theta = (k_r)$.

LEMMA 3.1. *Let $\theta = (k_r)$ be a lacunary sequence with $\liminf_r q_r > 1$, then for any Orlicz function M, $[\omega, M, p] \subset [N_\theta, M, p]$, where*

$$[\omega, M, p] = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[M\left(\frac{|x_k - \ell|}{\rho} \right) \right]^{p_k} = 0 \right.$$
$$\left. \text{for some } \ell, \text{ and } \rho > 0 \right\}$$

(we write $[\omega, M, p] = [\omega, M, p]_0$ in the case when $\ell = 0$).

Proof. It sufficient to show that $[\omega, M, p]_0 \subset [N_\theta, M, p]_0$; the general inclusion follows by linearity. Suppose $\liminf_r q_r > 1$, then there exists $\delta > 0$ such that $q_r = (k_r/k_{r-1}) \geq 1 + \delta$ for all $r \geq 1$. Then for $x \in [\omega, M, p]_0$, we write

$$\begin{aligned} A_r &\equiv h_r^{-1} \sum_{k\in I_r} \left[M\left(\frac{|x_k|}{\rho} \right) \right]^{p_k} \\ &= h_r^{-1} \sum_{k=1}^{k_r} \left[M\left(\frac{|x_k|}{\rho} \right) \right]^{p_k} - h_r^{-1} \sum_{k=1}^{k_{r-1}} \left[M\left(\frac{|x_k|}{\rho} \right) \right]^{p_k} \\ &= \frac{k_r}{h_r} \left(k_r^{-1} \sum_{k=1}^{k_r} \left[M\left(\frac{|x_k|}{\rho} \right) \right]^{p_k} \right) - \frac{k_{r-1}}{h_r} \left(k_{r-1}^{-1} \sum_{k=1}^{k_{r-1}} \left[M\left(\frac{|x_k|}{\rho} \right) \right]^{p_k} . \end{aligned}$$

Since $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta} \quad \text{and} \quad \frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}.$$

The terms

$$k_r^{-1} \sum_{k=1}^{k_r} \left[M\left(\frac{|x_k|}{\rho} \right) \right]^{p_k} \quad \text{and} \quad k_{r-1}^{-1} \sum_{k=1}^{k_{r-1}} \left[M\left(\frac{|x_k|}{\rho} \right) \right]^{p_k}$$

both converge to zero, and it follows that A_r converges to 0 as $r \to \infty$, that is, $x \in [N_\theta, M, p]_0$.

LEMMA 3.2. *Let $\theta = (k_r)$ be a lacunary sequence with $\limsup_r q_r < \infty$, then for any Orlicz function M, $[N_\theta, M, p] \subset [\omega, M, p]$.*

Proof. If $\limsup_r q_r < \infty$, there exists $B > 0$ such that $q_r < B$ for all $r \geq 1$. Let $x \in [N_\theta, M, p]_0$ and $\epsilon > 0$. There exists $R > 0$ such that for every $j \geq R$

$$A_j \equiv h_r^{-1} \sum_{k \in I_j} [M(\frac{|x_k|}{\rho})]^{p_k} < \epsilon.$$

We can also find $K > 0$ such that $A_j \leq K$ for all $j = 1, 2, \ldots$. Now let m be any integer with $k_{r-1} < m \leq k_r$, where $r > R$. Then

$$m^{-1} \sum_{k=1}^{m} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k}$$

$$\leq k_{r-1}^{-1} \sum_{k=1}^{k_r} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k}$$

$$= k_{r-1}^{-1} \left\{ \sum_{k \in I_1} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k} + \sum_{k \in I_2} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k} + \ldots \sum_{k \in I_r} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k} \right\}$$

$$= \frac{k_1}{k_{r-1}} k_1^{-1} \sum_{k \in I_1} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k} + \frac{k_2 - k_1}{k_{r-1}} (k_2 - k_1)^{-1} \sum_{k \in I_2} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k} + \ldots$$

$$+ \frac{k_R - k_{R-1}}{k_{r-1}} (k_R - k_{R-1})^{-1} \sum_{k \in I_R} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k} + \ldots$$

$$+ \frac{k_r - k_{r-1}}{k_{r-1}} (k_r - k_{r-1})^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k}$$

$$= \frac{k_1}{k_{r-1}} A_1 + \frac{k_2 - k_1}{k_{r-1}} A_2 + \ldots + \frac{k_R - k_{R-1}}{k_{r-1}} A_R$$

$$+ \frac{k_{R+1} - k_R}{k_{r-1}} A_{R+1} + \ldots \frac{k_r - k_{r-1}}{k_{r-1}} A_r$$

$$\leq (\sup_{j \geq 1} A_j) \frac{k_R}{k_{r-1}} + (\sup_{j \geq R} A_j) \frac{k_r - k_R}{k_{r-1}}$$

$$< K \frac{k_R}{k_{r-1}} + \epsilon B.$$

Since $k_{r-1} \to \infty$ as $m \to \infty$, it follows that

$$m^{-1} \sum_{k=1}^{m} \left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k} \to 0 \quad \text{and, consequently } x \in [\omega, M, p]_0.$$

The next result follows from Lemmas 3.1 and 3.2.

THEOREM 3.3. *Let $\theta = (k_r)$ be a lacunary sequence with* $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$. *Then for any Orlicz function M,* $[\omega, M, p] = [N_\theta, M, p]$.

The famous space $\hat{c}$ of all almost convergent sequences was defined by Lorentz [13]. The space of strongly almost convergent sequence $[\hat{c}]$ was introduced by Maddox [16] and also independently by Freedman et. al. [5] as follows:

$$[\hat{c}] = \Big\{x = (x_i) : \lim_{n\to\infty} n^{-1} \sum_{i=p+1}^{p+n} |x_i - \ell| = 0 \text{ uniformly in } p, \text{ for some } \ell\Big\}.$$

For any Orlicz function M and a bounded sequence $p = (p_k)$ of strictly positive real numbers, we extend the space $[\hat{c}]$ to $[\hat{c}, M, p]$ as defined below:

$$[\hat{c}, M, p] = \Big\{x = (x_k) : \lim_{n\to\infty} \sum_{k=m+1}^{m+n} \Big[M\Big(\frac{|x_k - \ell|}{\rho}\Big)\Big]^{p_k} = 0$$

$$\text{uniformly in } m \text{ for some } \ell, \text{ and } \rho > 0\Big\}.$$

Note that if we take $M(x) = x$ and $p_k = 1$ for all k, then $[\hat{c}, M, p] = [\hat{c}]$.

THEOREM 3.4. *Let M be any Orlicz function and $p = (p_k)$ be any bounded sequence of strictly positive real numbers, then $[\hat{c}, M, p] \subset [N_\theta, M, p]$ for every lacunary sequence θ.*

Proof. Let $x \in [\hat{c}, M, p]$ and $\epsilon > 0$. There exists a positive integer n_0, a number ℓ, and $\rho > 0$ such that $n^{-1}\sum_{k=m+1}^{m+n}[M(\frac{|x_k-\ell|}{\rho})]^{p_k} < \epsilon$ for $n > n_0$, $m = 0, 1, 2, \ldots$. Since θ is lacunary, we can choose $R > 0$ such that $r \geq R$ implies $h_r > n_0$ and consequently, $A_r \equiv h_r^{-1}\sum_{k\in I_r}[M(\frac{|x_k-\ell|}{\rho})]^{p_k} < \epsilon$. Thus $x \in [N_\theta, M, p]$.

To show that $[N_\theta, M, p]$ strictly contains $[\hat{c}, M, p]$, we proceed as in [5, p. 513]. We define $x = (x_k)$ by $x_k = 1$ if $k_{r-1} < k \leq k_{r+1} + [\sqrt{h_r}]$ for some r and $x_k = 0$ otherwise. Then there are arbitrarily long strings of 0's in the coordinates of x, as well as arbitrarily long strings of consecutive 1's, from which it follows that $x \notin [\hat{c}, M, p]$. However, $x \in [N_\theta, M]_0$ since $h_r^{-1}\sum_{k\in I_r} M(|x_k|) = h_r^{-1}[\sqrt{h_r}]M(1) \to 0$ as $r \to \infty$ where $[\ ]$ denotes the greatest integer function.

We now introduce a natural relationship between lacunary strong convergence with respect to an Orlicz function and lacunary statistical convergence. The notion of statistical convergence was given in earlier works [1], [4], [7], [21]. Recently, Fridy and Orhan [8] introduced the concept of lacunary statistical convergence as follows.

DEFINITION 3.5 [8]. Let θ be a lacunary sequence. Then a sequence $x = (x_k)$ is said to be lacunary statistically convergent to a number ℓ if for every

$\epsilon > 0$, $\lim h_r^{-1}|K_\theta(\epsilon)| = 0$, where $K_\theta(\epsilon) = \{k\epsilon I_r : |x_k - \ell| \geq \epsilon\}$ and $|K_\theta(\epsilon)|$ denotes cardinality of $K_\theta(\epsilon)$. The set of all lacunary statistically convergent sequences is denoted by S_θ.

We now establish an inclusion relation between $[N_\theta, M]$ and S_θ.

THEOREM 3.6. *For any Orlicz function M, $[N_\theta, M] \subset S_\theta$.*

Proof. Let $x \in [N_\theta, M]$ and $\epsilon > 0$. Then

$$h_r^{-1} \sum_{k \in I_r} M\left(\frac{|x_k - \ell|}{\rho}\right) \geq h_r^{-1} \sum_{k \in I_r,\, |x_k - \ell| \geq \epsilon} M\left(\frac{|x_k - \ell|}{\rho}\right)$$
$$> h_r^{-1} M(\epsilon/\rho)|K_\theta(\epsilon)|$$

from which it follows that $x \in S_\theta$.

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