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Celina Rom

REMARKS ON DECOMPOSABLE SETS

Abstract. In this paper we present some properties of decomposable sets, which are analogous to known properties of convex sets. In particular, an analogy of the classical theorem of Kakutani is given.

Introduction

Let $(X, \|\cdot\|)$ be a Banach space, (Ω, Σ, μ) be a measure space and $L_1 = L_1(\Omega, \Sigma, \mu)$ be Banach space of equivalence classes $[f]$ (with respect to the relation of equality -a.e. in Ω) of Σ-measurable function $f : \Omega \to X$ with the norm $\|f\|_L = \int_\Omega \|f\| d\mu < +\infty$. A set $K \subset L_1$ is called *decomposable* if for any $[u], [v] \in K$ and each $\mathbf{A} \in \Sigma$, $[\chi_{\mathbf{A}} u + \chi_{\Omega \setminus \mathbf{A}} v] \in K$, where $\chi_{\mathbf{A}}$ denotes the characteristic function of $\mathbf{A}$. Cz. Olech in [1] has showed, that certain properties of convex sets can be carryed on decomposable sets. Described in [1] properties are analogues of Krein–Milman and Carathéodory theorems (for $X = \mathbb{R}^n$, $\Omega = [a, b]$, μ–the Lebesgue measure).

The purpose of this note is to examine some other similarities between properties of convex and decomposable sets.

Properties of decomposable sets

The main result of this note is the following analogy of the classical Kakutani theorem.

THEOREM 1. *If $A, B \subset L_1$ are decomposable and disjoint sets, then there is a decomposable set K such that:*

a) *$L_1 \setminus K$ is also decomposable,*
b) *$A \subset K$ and $B \subset L_1 \setminus K$.*

In the proof of the Theorem 1 we use the following lemma.

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LEMMA 1. *Let T be an arbitary set of indexes. Assume that for every $\alpha \in T$ we have a family $\mathbb{A}_\alpha$ of sets in Σ such that $\mu(\bigcap_{i=1}^n \mathbf{A}_i) > 0$ for each $n \in \mathbb{N}$ and $\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathbb{A}_\alpha$. Let $(\mathbf{C}, \leq)$ be the ordered set of all families $\mathbb{A}_\alpha$, $\alpha \in T$, with the ordering $\mathbb{A}_\alpha \leq \mathbb{A}_\beta$ iff $\mathbb{A}_\alpha \subset \mathbb{A}_\beta$. Then for every $\mathbb{A}_\alpha \in \mathbf{C}$ there exists a maximal family $\mathbb{A}_0$ in $\mathbf{C}$ such that $\mathbb{A}_\alpha \subset \mathbb{A}_0$.*

Proof. Note first, that every chain $\mathbf{R} \subset \mathbf{C}$ has an upper bound $\mathbb{A} = \bigcup_{\mathbb{A}_\alpha \in \mathbf{R}} \mathbb{A}_\alpha$. If $\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathbb{A}$, then there are $\mathbb{A}_{\alpha_1}, \ldots, \mathbb{A}_{\alpha_n} \in \mathbf{R}$ such that $\mathbf{A}_1 \in \mathbb{A}_{\alpha_1}, \ldots, \mathbf{A}_n \in \mathbb{A}_{\alpha_n}$. Because $\mathbf{R}$ is a chain, we receive that $\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathbb{A}_{\alpha_i}$ for certain α_i. Hence $\mu\Big(\bigcap_{k=1}^n \mathbf{A}_k\Big) > 0$ and $\mathbb{A} \in \mathbf{C}$. To end the proof, it suffices to use the Kuratowski–Zorn lemma. ∎

REMARK 1. Immediately from the definition we have following properties of the maximal family $\mathbb{A}_0$ in $\mathbf{C}$:

(W 1.) $\Omega \in \mathbb{A}_0$,
(W 2.) $\bigcap_{k=1}^n \mathbf{A}_k \in \mathbb{A}_0$ for each $n \in \mathbb{N}$ and $\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathbb{A}_0$,
(W 3.) if $\mathbf{A} \in \Sigma$ and $\mu(\mathbf{B} \cap \mathbf{A}) > 0$ for every $\mathbf{B} \in \mathbb{A}_0$, then $\mathbf{A} \in \mathbb{A}_0$.

Proof of Theorem 1. Let $A, B \subset L_1$ be disjoint decomposable sets. Put

$$\tilde{A} = \bigcup_{[f] \in A} [f], \quad \tilde{B} = \bigcup_{[g] \in B} [g], \quad \text{and } \mathbf{A}_{f,g} = \{t \in \Omega : \ f(t) \neq g(t)\}.$$

We will show that the family $\mathbb{A}_{f,g} = \{\mathbf{A}_{f,g} : f \in \tilde{A}, g \in \tilde{B}\}$ is such that $\mu(\bigcap_{i=1}^n \mathbf{A}_{f_i,g_i}) > 0$ for each $n \in \mathbb{N}$ and each $\{\mathbf{A}_{f_1,g_1}, \ldots, \mathbf{A}_{f_n,g_n}\} \subset \mathbb{A}_{f,g}$.

Let us take $f_1, f_2 \in \tilde{A}$, $g_1, g_2 \in \tilde{B}$ and put

$$f = f_2 \chi_{\mathbf{A}_{f_1,g_1}} + f_1(1 - \chi_{\mathbf{A}_{f_1,g_1}}) \in \tilde{A}, \qquad g = g_2 \chi_{\mathbf{A}_{f_1,g_1}} + g_1(1 - \chi_{\mathbf{A}_{f_1,g_1}}) \in \tilde{B}.$$

Then we have that $\mathbf{A}_{f,g} = \mathbf{A}_{f_1,g_1} \cap \mathbf{A}_{f_2,g_2}$, and hence $\mu(\mathbf{A}_{f_1,g_1} \cap \mathbf{A}_{f_2,g_2}) > 0$. Repeating this process for $\mathbf{A}_{f,g}$ and the next sets $\mathbf{A}_{f_i,g_i}$, we obtain $\mu(\bigcap_{i=1}^n \mathbf{A}_{f_i,g_i}) > 0$. By Lemma 1 there exists a maximal family $\mathbb{A}_0 \in \mathbf{C}$ such that $\mathbb{A}_{f,g} \subset \mathbb{A}_0$. Consider a set $K \subset L_1$ such that

$$[h] \in K \iff \exists_{\mathbf{A} \in \mathbb{A}_0} \exists_{f \in \tilde{A}} \forall_{t \in \mathbf{A}} \quad f(t) = h(t). \tag{1}$$

We will show that K satisfies assumptions of Theorem 1.

Because $\Omega \in \mathbb{A}_0$ we have that $A \subset K$. To prove that K is a decomposable set let us take $[u], [v] \in K$, $\mathbf{D} \in \Sigma$. Then there exists $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{A}_0$, $f_1, f_2 \in \tilde{A}$ such that $u(t) = f_1(t)$ for $t \in \mathbf{A}_1$ and $v(t) = f_2(t)$ for $t \in \mathbf{A}_2$. The functions $\chi_{\mathbf{D}} u + \chi_{\Omega \setminus \mathbf{D}} v$ and $\chi_{\mathbf{D} \cap \mathbf{A}} f_1 + \chi_{\Omega \setminus (\mathbf{D} \cap \mathbf{A})} f_2$ coinside on the set $\mathbf{A} = \mathbf{A}_1 \cap \mathbf{A}_2$. Since the second function belongs to $\tilde{A}$ and $\mathbf{A} \in \mathbb{A}_0$ (by **W 2**), then $[\chi_{\mathbf{D}} u + \chi_{\Omega \setminus \mathbf{D}} v] \in K$.

The next step of our proof is to show that $B \subset L_1 \setminus K$. If $[g] \in B$ and $[h] \in K$ then there exist $\mathbf{A} \in \mathbb{A}_0$ and $f \in \tilde{A}$ such that $h(t) = f(t)$

for each $t \in \mathbf{A}$. Because $\mu(\mathbf{A} \cap \mathbf{A}_{f,g}) > 0$ (the definition of $\mathbb{A}_0$) we receive that $[g] \neq [h]$. To show that $L_1 \setminus K$ is a decomposable set let us consider $[\gamma_1], [\gamma_2] \in L_1 \setminus K$ and $\mathbf{D} \in \Sigma$. Then

$$(2) \qquad \forall_{\mathbf{A}\in\mathbb{A}_0}\, \forall_{f\in\tilde{A}}\, \exists_{t_{\gamma_1},t_{\gamma_2}\in\mathbf{A}} \quad \gamma_1(t_{\gamma_1}) \neq f(t_{\gamma_1}) \quad \text{and} \quad \gamma_2(t_{\gamma_2}) \neq f(t_{\gamma_2}).$$

Let us take a function h such that $[h] \in K$. Now let us fix a set $\mathbf{A}_0 \in \mathbb{A}_0$ and a function $f \in \tilde{A}$ such that $f(t) = h(t)$ for $t \in \mathbf{A}_0$. Existence of such $\mathbf{A}_0$ and f is guaranted by the condition (1).

Let us denote $\mathbf{B}_1 = \{t \in \Omega : \gamma_1(t) \neq f(t)\}$ and $\mathbf{B}_2 = \{t \in \Omega : \gamma_2(t) \neq f(t)\}$. By the maximality of $\mathbb{A}_0$, property (W 3.) and condition (2) we obtain $\mu(\mathbf{B}_1 \cap \mathbf{A}) > 0$ and $\mu(\mathbf{B}_2 \cap \mathbf{A}) > 0$ for every $\mathbf{A} \in \mathbb{A}_0$. Useing again property (W 3) we receive that $\mathbf{B}_1, \mathbf{B}_2 \in \mathbb{A}_0$ and $\mu(\mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{A}_0) > 0$. Let us consider the function $\gamma = \chi_{\mathbf{D}}\gamma_1 + \chi_{\Omega\setminus\mathbf{D}}\gamma_2$. Notice that $\gamma(t) \neq f(t)$ for every $t \in \mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{A}_0$. Because $\mu(\mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{A}_0) > 0$, we have $[\gamma] \neq [h]$, and because $[h] \in K$ has been chosen arbitraily, $[\gamma]$ cannot belong to K. This completes the proof. ∎

The same result has been obtained independently by W. Kubiś [2] as a consequence of a theorem characterizing spaces having the Kakutani separation property. However, the method presented here is direct and completely differen than the one used in [2].

As an immediate consequence of Theorem 1 we obtain the following corollaries.

COROLLARY 1. *If $[f], [g] \in L_1$ and $[f] \neq [g]$, then there exist decomposable sets K and $L_1 \setminus K$ such that $[f] \in K$ and $[g] \in L_1 \setminus K$.*

COROLLARY 2. *If $A \subset L_1$ is a decomposable set, then $A = \bigcap_{K\subset L_1, A\subset K} K$, where K and $L_1 \setminus K$ are decomposable sets.*

Notice that not all properties of convex sets have their substitutes for decomposable sets. For example an analogy of Helly theorem does not hold for decomposable sets. Namely, for every $n \in \mathbb{N}$ we can create a family of closed and decomposable sets $A_1, \ldots, A_n, A_{n+1}$ such that $\bigcap_{i=1}^{n} A_{k_i} \neq \emptyset$ for each $\{k_i\}_{i=1}^{n} \subset \{1, \ldots, n+1\}$ and $\bigcap_{i=1}^{n+1} A_k = \emptyset$. For instance to construct such a family let us take $x_0, \ldots, x_n \in X$ such that $x_k \neq x_l$ for $k \neq l$ and for $k = 1, \ldots, n$ let us define functions $g_k : \Omega \to X$ by the formula $g_k(t) = x_k$. Now for $k = 1, \ldots, n$ let A_k be the smallest decomposable set containing a fixed combination of $n-1$ elements of $\{[g_1], \ldots, [g_n]\}$ and $[g_0]$ and let A_{n+1} be the smallest decomposable set such that $\{[g_1], \ldots, [g_n]\} \subset A_{n+1}$. Similarly one can prove that the Rådström cancellation law does not hold for Minkowski's sums of decomposable closed and bounded sets.

References

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DEPARTMENT OF MATHEMATICS
DIVISION OF ŁÓDŹ TECHNICAL UNIVERSITY
Willowa 2
43-309 BIELSKO-BIAŁA, POLAND

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