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A LOCALIZATION PRINCIPLE FOR CLASSES OF MEANS

Abstract. Several families of continuous means defined on a square $I \times I$ have the remarkable property of being entirely determined when their values in an arbitrary small neigborhood of the diagonal $\{(x,x) : x \in I\}$ of the square are known. Some examples are given of application of this property in solving functional equations.

1. Introduction

Let I be a real interval and denote by Q the square $I \times I$. This paper is mainly concerned with families of *continuous symmetric means* on I; i.e., with classes of *continuous* functions $\mu : Q \to I$ satisfying the following three properties ([1], [2]):

i) $\mu(x,x) = x,\ x \in I$;
ii) $\mu(x,y) = \mu(y,x),\ x, y \in I$, (symmetry);
iii) μ is strictly increasing in both variables x and y.

Important examples of classes of continuous symmetric means are provided by $\mathcal{QA}(I)$, the family of *quasiarithmetic means* on I,

$$(1)\qquad \mathcal{QA}(I) = \left\{ f^{-1}\left(\frac{f(x)+f(y)}{2}\right) : f : I \to \mathbb{R} \text{ is continuous and strictly monotone} \right\},$$

and $\mathcal{L}(I)$, the family of Lagrangian means ([3],[5],[6]). The *Lagrangian mean* on I generated by a continuous and strictly monotone function $f : I \to \mathbb{R}$ is defined through

$$(2)\qquad \mu(x,y) = \begin{cases} f^{-1}\left(\frac{1}{y-x}\int_x^y f(\xi)\,d\xi\right), & x \neq y \\ x, & x = y. \end{cases}$$

By their connection to the Lagrange mean value theorem, these means were called *mean-value means* in [7], pg. 343 and ss., and even other names were given them in the classic literature. The concise denomination we presently use come from [5].

Another family of means we have interested in this paper is the family of anti-Lagrangian means on an interval I. For a given continuous and strictly monotone function $f : I \to \mathbb{R}$, the *anti-Lagrangian mean* μ *generated by* f is

$$\mu(x,y) = \begin{cases} \dfrac{1}{f(y)-f(x)} \int_x^y \xi \, df(\xi), & x \neq y \\ x, & x = y, \end{cases} \tag{3}$$

where the integral is understood in the sense of Riemann-Stieltjes. The class $\mathcal{AL}(I)$ of anti-Lagrangian means on I was introduced and studied in [6] as a remarkable instance of means resulting from the Cauchy mean value theorem.

This paper addresses the question of localization for classes of symmetric continuous means. Consider a family $\mathcal{F}$ of functions defined on a set X and taking their values in another set Y; i.e., $\mathcal{F} \subseteq Y^X$. Along this paper, the family $\mathcal{F}$ is said to satisfy a *localization principle on* X when there exists a proper subset X_0 of X such that, for every pair of functions $f, g \in \mathcal{F}$, the equality $f = g$ holds provided that $f(x) = g(x)$, $x \in X_0$. To emphasize the role of the *set of localization* X_0 often we say that the family $\mathcal{F}$ is X_0*-localizable*. For instance, the family $P_n \subset \mathbb{R}[x]$ of all polynomials p with $\deg p \leq n$ is X_0-localizable for any $X_0 \subset \mathbb{R}$ with $n+1$ points at least. A less trivial example of localization is furnished by the class $\mathcal{H}(\Omega)$ of harmonic functions on a domain Ω of $\mathbb{R}^n$: $\mathcal{H}(\Omega)$ is X_0-localizable on every non-empty open subset X_0 of Ω. As it is well known, this fact is a direct consequence of the real analyticity of harmonic functions.

Existence of a localization principle supposedly satisfied by the entire class of symmetric continuous means is a hopeless question. We show, however, that the three aforementioned classes of means are $\mathcal{U}$-localizable for every neighborhood $\mathcal{U}$ of the diagonal $\Delta(Q) = \{(x,x) : x \in I\}$ of the square Q. Before our main result in this connection be formally stated and proved, we would like to convey the reader some intuition of this fact. To this end, we consider the family of $\mathcal{C}^2$ quasiarithmetic means on I and we suppose that μ is a member of this family. From (1) we see that

$$2f(\mu(x,y)) = f(x) + f(y), \qquad x, y \in I, \tag{4}$$

for a certain continuous and strictly monotone f defined on I. If we assume, in addition, that $f \in \mathcal{C}^2(I)$, then a simple application of the implicit function theorem to (4) shows that $\mu \in \mathcal{C}^2(Q)$ and so, identity (4) can be twice implicitly differentiated. In fact, by taking $\partial^2/\partial x \partial y$ on both members of

(4), we obtain

$$(5)\quad 0=\frac{\partial^2}{\partial x\partial y}(f(x)+f(y))=2\frac{\partial^2}{\partial x\partial y}\left(f(\mu(x,y))\right)$$
$$=2\frac{\partial}{\partial x}\left(f'(\mu(x,y))\frac{\partial\mu}{\partial y}\right)=2\left(f''(\mu(x,y))\frac{\partial\mu}{\partial y}\frac{\partial\mu}{\partial x}+f'(\mu(x,y))\frac{\partial^2\mu}{\partial x\partial y}\right).$$

Since $\mu(x,x)\equiv x$, we have

$$(6)\qquad \frac{\partial\mu}{\partial y}(x,x)+\frac{\partial\mu}{\partial x}(x,x)=1,\qquad x\in I;$$

furthermore, the symmetry of μ implies

$$\frac{\partial\mu}{\partial y}(x,x)=\frac{\partial\mu}{\partial x}(x,x),$$

whence we see that

$$(7)\qquad \frac{\partial\mu}{\partial y}(x,x)\equiv\frac{1}{2}\equiv\frac{\partial\mu}{\partial x}(x,x).$$

By making $y\to x$ in (5) and using (7) we finally arrive at the second order linear differential equation

$$(8)\qquad f''(x)+4A(x)f'(x)=0,$$

where $A(x)=(\partial^2\mu/\partial x\partial y)(x,x)$.

Given a non-constant particular solution ϕ to equation (8), its general solution f is computed as

$$(9)\qquad f=\alpha\phi+\beta,$$

with α,β real constants. Moreover, a non-constant solution to (8) satisfy

$$\phi'(x)=\exp\Big(-4\int^{x}A(\xi)\,d\xi\Big)>0,\qquad x\in I,$$

so that every pair ϕ, f of monotone solutions to equation (8) are related by (9) with a non zero constant α. In this way, if two $\mathcal{C}^2$ quasiarithmetic means μ_1 and μ_2 coincide in a neighborhood $\mathcal{U}$ of the diagonal $\Delta(Q)$ of the square Q, then

$$\frac{\partial^2\mu_1}{\partial x\partial y}\equiv\frac{\partial^2\mu_2}{\partial x\partial y}\quad\text{on }\mathcal{U},$$

and therefore $(\partial^2\mu_1/\partial x\partial y)(x,x)=(\partial^2\mu_2/\partial x\partial y)(x,x)$, $x\in I$. By assuming that μ_i is generated by f_i $(i=1,2)$, we see that f_1 and f_2 are monotone solutions to the same equation (8) and so, there exist two constants α,β, $\alpha\neq 0$, such that $f_2\equiv\alpha f_1+\beta$. But then, for every $x,y\in I$, we have

$$\begin{aligned}\mu_2(x,y) &= f_2^{-1}\left(\frac{f_2(x)+f_2(y)}{2}\right)\\ &= f_1^{-1}\left(\frac{\frac{(\alpha f_1(x)+\beta)+(\alpha f_1(y)+\beta)}{2}-\beta}{\alpha}\right)\\ &= f_1^{-1}\left(\frac{f_1(x)+f_1(y)}{2}\right) = \mu_1(x,y).\end{aligned}$$

In short, we have proved that whenever two $\mathcal{C}^2$ quasiarithmetic means coincide in a neighborhood of the diagonal $\Delta(Q)$ of the square Q, then they will coincide over the entire square Q. This localization principle for regular quasiarithmetic means is shared by other families of continuous symmetric means. By following a procedure like the previous one, it is not difficult to see that the equation

$$f''(x) + \lambda A(x) f'(x) = 0, \tag{10}$$

with $A(x) \equiv (\partial^2\mu/\partial x\partial y)(x,x)$ and $\lambda = 12$ is satisfied by the monotone function $f \in \mathcal{C}^2$ generating a regular Lagrangian mean μ and the same equation holds with $\lambda = 6$ when f is the generating function $f \in \mathcal{C}^2$ of a regular anti-Lagrangian mean μ. Then, we conclude that the families of *regular* quasiarithmetic, Lagrangian or anti-Lagrangian means all are $\mathcal{U}$-localizable for any neighborhood $\mathcal{U}$ of the diagonal $\Delta(Q)$. But we will see that the requirement of regularity can be relaxed so that we can state the following result.

THEOREM 1. *Let I be a real interval and $\mathcal{U}$ be a neighborhood of the diagonal $\Delta(Q)$ of the square $Q = I \times I$. Then, the families $\mathcal{QA}(I)$, $\mathcal{L}(I)$ and $\mathcal{AL}(I)$ of continuous symmetric means are $\mathcal{U}$-localizable.*

To prove this theorem is devoted our next section. Some examples of its applications in solving functional equations and a few remarks on possible generalizations are given in Section 3 of the paper.

2. Proof of Theorem 1

Let $\mathcal{M}(I)$ denote any one of the three classes of means $\mathcal{QA}(I)$, $\mathcal{L}(I)$ or $\mathcal{AL}(I)$. In view of definitions (1), (2) and (3), a member μ of $\mathcal{M}(I)$ is generated by a continuous and strictly monotone function $f : I \to \mathbb{R}$, which we generally indicate by writing $\mu = [f]$. The proof we give of Theorem 1 is supported by the following basic result of representation of means in the classes $\mathcal{QA}(I)$, $\mathcal{L}(I)$ and $\mathcal{AL}(I)$.

THEOREM 2. *If $[f], [g] \in \mathcal{M}(I)$, then $[f] = [g]$ if and only if there exist two real constants α, β, $\alpha \neq 0$, such that $g = \alpha f + \beta$.*

The fact that $[\alpha f+\beta] = [f]$ for $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, turns out to be a simple computation with (1), (2) and (3). At least in the case in which f and g are regular, say $f, g \in \mathcal{C}^2(I)$, a proof of the converse can be given along the lines at the end of previous section: if $[f] = [g]$, then f and g are two non-constant solution to the same differential equation (10), hence $g = \alpha f + \beta$ with $\alpha \neq 0$. For the proof of the general case when $\mathcal{M}(I) = \mathcal{QA}(I)$ the reader is remitted to Theorem 2, pg. 290, of [1] or Corollary 5, pg. 246, of [2]. See Corollary 7 in [5] (also, in a slightly different version, Theorem 1, pg. 344, of [7]) for a proof in the case $\mathcal{M}(I) = \mathcal{L}(I)$. As for the final case in which $\mathcal{M}(I) = \mathcal{AL}(I)$, a proof can be found in [6]. ∎

To prove Theorem 1 we first assume that I is a *compact* real interval. Let μ_i, $(i = 1, 2)$, be two means in $\mathcal{M}(I)$ such that $\mu_1(x, y) = \mu_2(x, y)$, $(x, y) \in \mathcal{U}$, being $\mathcal{U}$ any neighborhood of the diagonal $\Delta(Q)$ of the square $Q = I \times I$. We can suppose that every μ_i is generated by a strictly monotone real function f_i defined and continuous on I; i.e., $\mu_i = [f_i]$, $i = 1, 2$. Corresponding to every $x \in I$, there exists a $\delta > 0$ such that the square $Q(x; \delta) = ((x-\delta, x+\delta) \times (x-\delta, x+\delta)) \cap Q \subseteq \mathcal{U}$ and therefore, an application of Theorem 2 shows that, calling $I(x) = (x - \delta, x + \delta) \cap I$, the equality

$$f_2|_{I(x)}(\xi) = \alpha_x \, f_1|_{I(x)}(\xi) + \beta_x, \qquad \xi \in I(x), \tag{11}$$

holds for two constants $\alpha_x \neq 0$ and β_x. Now, by compactness, there are a finite number of points $x_1 < x_2 < \cdots < x_n$ of the interval I such that $I \subseteq \bigcup_{k=1}^{n} I(x_k)$. For every $\xi \in I(x_k) \cap I(x_{k+1})$, we obviously have

$$\alpha_{x_k} \, f_1|_{I(x_k)}(\xi) + \beta_{x_k} = \alpha_{x_{k+1}} \, f_1|_{I(x_{k+1})}(\xi) + \beta_{x_{k+1}},$$

whence, taking into account that $I(x_k) \cap I(x_{k+1})$ is a non-void open subset of I and the strict monotonicity of f_1, we deduce

$$\alpha_{x_k} = \alpha_{x_{k+1}}, \; \beta_{x_k} = \beta_{x_{k+1}}. \tag{12}$$

Since these equalities hold for $k = 1, 2, \ldots, n-1$, we conclude from (11) that

$$f_2(\xi) = \alpha f_1(\xi) + \beta, \qquad \xi \in I, \tag{13}$$

being α and β the common value of α_{x_k} and β_{x_k}, respectively. As a consequence of (13), for every $x, y \in I$ we have

$$\mu_2(x, y) = [f_2](x, y) = [\alpha f_1 + \beta](x, y) = [f_1](x, y) = \mu_1(x, y),$$

which proves Theorem 1 when I is a compact interval.

Now suppose that I is not compact. Then we can choose a sequence $\{I_n\}$ of compact intervals such that $I_{n+1} \subseteq I_n \subseteq I$, $n \in \mathbb{N}$, and $\bigcup_{n=1}^{\infty} I_n = I$. The just proved part of the theorem shows that, for every $n \in \mathbb{N}$, $\mu_2(x, y) = \mu_1(x, y)$ for $x, y \in I_n$; hence, $\mu_2(x, y) = \mu_1(x, y)$, $x, y \in I$. This completes the proof of Theorem 1. ∎

3. Examples and remarks

A list follows of examples of application of Theorem 1 to solve some functional equations. Our first example has to do with the Jensen functional equation

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}, \quad x,y\in I, \tag{14}$$

where I is a real interval. That the general continuous solution to equation (14) is given by

$$\{f(x)=\alpha x+\beta : \alpha,\beta\in\mathbb{R}\},$$

the family of affine function, is a classical result (see, for example, [1], pg. 43, or [2], pg. 242). Now let us consider a neighborhood $\mathcal{U}$ of the diagonal $\Delta(Q)$ of the form $\{(x,y)\in Q : |x-y|<\delta\}$, $(\delta>0)$, and the Jensen equation (14) with domain $\mathcal{U}$; i.e.,

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}, \quad (x,y)\in\mathcal{U}. \tag{15}$$

Continuous solutions to this equation must be locally affine functions and so, a compactness argument shows that they really are affine. To the same conclusion we will alternatively arrive by using Theorem 1. As a first step, we show that every non-constant continuous solution to equation (15) must be strictly monotone in I. Indeed, if the equality $f(x_0)=f(y_0)$ were true for two points $x_0,y_0\in I$, $x_0<y_0$; then, due to the continuity of f, there exist $x_1,y_1\in I$, $x_1<y_1$, with $|x_1-y_1|<\delta$ and $f(x_1)=f(y_1)=c$. From (15) we conclude that $f((x_1+y_1)/2)=(c+c)/2=c$ or, after an inductive reasoning, $f((1-d)x_1+dy_1)=c$ for every dyadic rational $d\in[0,1]$. Since f is continuous, we can ensure that $f(x)=c$, $x\in[x_1,y_1]$. Define $[x_1^*,y_1^*]$ to be the maximal subinterval of $[x_0,y_0]$ containing $[x_1,y_1]$ and such that $f\equiv c$ on $[x_1^*,y_1^*]$. We will see that $y_1^*=y_0$. In fact, if $y_1^*<y_0$, then by eventually replacing f by $-f$ we can suppose that there exists a sequence $\{y_n\}$ such that $y_1^*<y_n<y_{n+1}<y_0$, $n\in\mathbb{N}$, $y_n\downarrow y_1^*$ and $f(y_n)>c$, $n\in\mathbb{N}$. Hence, fixed a small enough $\varepsilon>0$ an $n\in\mathbb{N}$ can be found such that $y_n<y_1^*+\varepsilon$ and we would attain the following contradiction:

$$c=f\left(\frac{(y_1^*-\varepsilon)+y_n}{2}\right)=\frac{f(y_1^*-\varepsilon)+f(y_n)}{2}>c.$$

Thus $y_1^*=y_0$ and we can similarly prove that $x_1^*=x_0$; that is, $f\equiv c$ on $[x_0,y_0]$. Finally, an argument like the previous one shows that the maximal subinterval I^* of I containing $[x_0,y_0]$ and such that $f\equiv c$ on I^* must coincide with I; thus, $f\equiv c$ on I.

Now, if f is a continuous non-constant solution to equation (15), then we can write

$$\frac{x+y}{2} = f^{-1}\left(\frac{f(x)+f(y)}{2}\right), \qquad (x,y) \in \mathcal{U};$$

i.e., the quasiarithmetic mean $[f]$ coincides with the arithmetic mean in the neighborhood $\mathcal{U}$ of the diagonal $\Delta(Q)$. Since the arithmetic mean is generated by $g(x) = x$, $x \in I$, Theorem 1 shows that

$$f(x) = \alpha x + \beta, \qquad x \in I, \tag{16}$$

with $\alpha \neq 0$, β constants. The argument finishes by observing that constant solutions to (15) are included in (16) when $\alpha = 0$.

The ideas just expounded for the Jensen equation can be suitably extended to the more general equation

$$f(M(x,y)) = N(f(x), f(y)), \qquad x, y \in I, |x-y| < \delta, \tag{17}$$

where M and N are quasiarithmetic means defined on intervals I and J, respectively. As before, a non-constant continuous solution $f : I \to J$ to this equation is proved to be strictly monotone in I (see [4]), so that by writing $M = [\phi]$, $N = [\psi]$, equation (17) becomes

$$[\phi] = (f^{-1} \circ [\psi] \circ (f \times f))(x,y) = [\psi \circ f](x,y), \qquad x, y \in I, |x-y| < \delta.$$

By Theorem 1 we then conclude

$$(\psi \circ f)(x) = \alpha\phi(x) + \beta, \qquad x \in I,$$

whence

$$f(x) = \psi^{-1}(\alpha\phi(x) + \beta), \qquad x \in I. \tag{18}$$

Of course, the constants α and β are not completely arbitrary here: the condition $\alpha\phi(I) + \beta \subseteq \psi(J)$ must be fulfilled in order that (18) be meaningful. Briefly, the general continuous solution to equation (17) is given by (18) with α, β real constants such that $\alpha\phi(I) + \beta \subseteq \psi(J)$.

Last we look for continuous solutions to the integral-functional equations

$$\int_x^y f(\xi)\, d\xi = (y-x) f\left(\frac{x+y}{2}\right), \qquad (x,y) \in \mathcal{U}, \tag{19}$$

and

$$\int_x^y \xi\, df(\xi) = (f(y) - f(x))\frac{x+y}{2}, \qquad (x,y) \in \mathcal{U}, \tag{20}$$

where $\mathcal{U}$ is a neighborhood of the diagonal $\Delta(Q)$. Note that equations (19) and (20) can be interpreted as prescriptions of the mean value in the Mean Value Theorem for Riemann and for Riemann-Stieltjes integrals, respectively. Let x_0 be an interior point of I; then, from (19) we obtain

$$\frac{1}{2\varepsilon} \int_{x_0-\varepsilon}^{x_0+\varepsilon} f(\xi)\, d\xi = f(x_0), \tag{21}$$

for small enough ε's; i.e., solutions to equation (19) (locally) have the Mean Value Property and therefore, they are real analytic functions on I. It follows that a non-constant continuous solution f to (19) is strictly monotone on I. In fact, if it were not so, then there would exist two points $x_0, y_0 \in I$, $x_0 < y_0$, such that $f(x_0) = f(y_0)$, whence function f would admit an extremum $x^* \in (x_0, y_0)$. From (19) we then deduce that $f \equiv f(x^*)$ in a neighborhood of x^* or, since f is real analytic, $f \equiv f(x^*)$ in I, which violates the initial assumption on f. Now, if f is a non-constant solution f, we can rewrite equation (19) as follows

$$f^{-1}\left(\frac{1}{y-x}\int_x^y f(\xi)\,d\xi\right) = \frac{x+y}{2}, \qquad (x,y) \in \mathcal{U}. \tag{22}$$

The right hand side of (22) is the Lagrangian mean generated by f while the second one is that one generated by $g(x) \equiv x$, so that Theorem 1 applies to derive $f(x) = \alpha x + \beta$, $x \in I$. Summarizing, the family of affine functions is the general continuous solution to equation (19).

Equation (20) can be similarly studied but details on its treatment will be omitted. We only observe that every non-constant continuous solution to (20) is also proved to be strictly monotone on I, whence equation (20) can be transformed in

$$\frac{1}{f(y)-f(x)}\int_x^y \xi\,df(\xi) = \frac{x+y}{2}, \qquad (x,y) \in \mathcal{U}.$$

In this form, equation (20) expresses that the anti-Lagrangian mean $[f]$ is equal to $[x]$ on $\mathcal{U}$ and, as before, Theorem 1 then implies that $f(x) \equiv \alpha x + \beta$.

With regard to possible generalizations of Theorem 1, we realize that means in the three classes $\mathcal{QA}(I)$, $\mathcal{L}(I)$ and $\mathcal{AL}(I)$ are of the form

$$\mu(x,y) = f^{-1}\left(\int_I f(\xi)\,p(x,y;d\xi)\right), \tag{23}$$

where f is a strictly monotone and continuous function on I and $\{p(x,y,\cdot) : x, y \in I\}$ is a given two-parameter family of Borel probability measures on I satisfying the following properties:

GM1) if $x \le y$, then $p(x,y;\cdot)$ is supported on $[x,y]$;

GM2) $p(x,y;\cdot)$ depends continuously on $(x,y) \in Q$ in a weak sense: $(x,y) \to \int_I f(\xi)\,p(x,y;d\xi)$ is continuous on Q whatever be the continuous function f.

Observe that $p(x,x;\cdot)$ is supported on $\{x\}$ by **GM2)**, so that $p(x,x;\cdot) = \delta(x-\cdot)$, the Dirac measure concentrated at x. Under conditions **GM1)**, **GM2)**, (23) is a continuous mean on I; i.e., in addition to continuity, (23) verifies the *internality property*:

$$\min\{x,y\} < \mu(x,y) < \max\{x,y\}, \qquad x, y \in I,\ x \neq y.$$

As it occurred with quasiarithmetic and Lagrangian means, different families of means are generated by expression (23) when p is fixed and f varies on the set of continuous and strictly monotone functions on I. The reciprocal situation in which f is fixed while it is the two-parameter family of measures what varies in (23) is also of interest: this is the case that corresponds to anti-Lagrangian means. Localization properties of the so generated families of means may considerably differ from that one specified in Theorem 1. Take for instance the family $\{\mu_\lambda : \lambda \in (0,1)\}$ with

$$\mu_\lambda(x,y) = f^{-1}((1-\lambda)f(x) + \lambda f(y)),$$

which is derived from the class of Borel measures $p_\lambda(x,y;\xi) = (1-\lambda)\delta(x-\xi) + \lambda\delta(y-\xi)$, $\lambda \in (0,1)$, $x,y \in I$, and from a fixed function f. If the value of μ_λ at a point (x_0,y_0) out of the diagonal $\Delta(Q)$ is known, then the corresponding value of λ can be computed:

$$\lambda = \frac{f(\mu_\lambda(x_0,y_0)) - f(x_0)}{f(y_0) - f(x_0)}.$$

In consequence, the singleton $X_0 = \{(x_0,y_0)\}$ serves as a localization set for this family of means whenever $(x_0,y_0) \notin \Delta(Q)$.

It should be added that the symmetry condition $p(x,y;\cdot) = p(y,x;\cdot)$, $x,y \in I$, does not play any important role here. On one hand we could have stated Theorem 1 by taking $\mathcal{U} \cap \{(x,y) \in Q : x < y\}$ instead of a full neighborhood $\mathcal{U}$ of the diagonal $\Delta(Q)$ but, on the other hand, Theorem 1 admits a simple extension to the case of weighted quasiarithmetic means

$$\mu(x,y) = f^{-1}(pf(x) + qf(y)),$$

where p,q are positive numbers such that $p+q=1$.

As an instance of this general viewpoint, we give a passing glance to the family of means derived from (23) by choosing

$$p(x,y;d\xi) = \begin{cases} 2(y-\xi)(y-x)^{-2}\chi_{[x,y]}(\xi)\,d\xi, & x<y \\ \delta(\xi-x), & x=y \\ 2(\xi-x)(y-x)^{-2}\chi_{[y,x]}(\xi)\,d\xi, & x>y. \end{cases}$$

These means are related to the mean value arising in the Lagrange form of the complementary term in the second order Taylor expansion of a function. It is easy to see that the equality

$$(y-x)\,[f(\mu(x,y)) + f(\mu(y,x))] = 2\int_x^y f(\xi)\,d\xi, \tag{24}$$

holds for the mean $\mu = [f]$ generated by the function f. Thus, by applying $\partial^2/\partial x\partial y$ on both members of (24), then dividing by $(y-x)$ and taking limits for $y \to x$, we obtain

$$f''(x) + \frac{27}{2}\mu_{xy}(x,x)f'(x) = 0. \tag{25}$$

In deriving (25) we have used relation (6) from the Introduction and also the following ones

$$\mu_x(x,x) \equiv \frac{2}{3},\ \mu_y(x,x) \equiv \frac{1}{3},\ \mu_{xx}(x,x) \equiv -\mu_{xy}(x,x) \equiv \mu_{yy}(x,x).$$

Now, the argument at the end of the Introduction works for equation (25) as well: at least for a regular mean μ, the knowledge of μ in a neighborhood of the diagonal $\Delta(Q)$ suffices to determine the generating function f and this, in turn, determine μ in the whole square Q.

Finally, we remark that another direction of generalization of Theorem 1 consists in considering means in several variables. For example, a suitable version of this result holds for quasiarithmetic means in n variables:

$$\mu(x_1, x_2, \dots, x_n) = f^{-1}\left(\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}\right),\ x_1, x_2, \dots, x_n \in I.$$

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