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ORTHOGONAL STABILITY OF THE CAUCHY EQUATION ON BALLS

Abstract. We deal with stability of some functional equations postulated for orthogonal vectors in a ball centered at the origin. The maps considered are defined on a finite dimensional inner product space and take their values in a real sequentially complete linear topological space. The main result establishes the stability of the corresponding conditional Cauchy functional equation and as a consequence we obtain some other stability results. Results which do not involve the orthogonality relation are considered in more general structures.

1. Introduction

R. Ger and J. Sikorska [2] considered the stability of the Cauchy functional equation postulated for orthogonal vectors only and defined on the whole space. F. Skof [7], [8] and F. Skof & S. Terracini [9] dealt with stability of the Cauchy and quadratic equations on the interval. Z. Kominek [3] studied stability of the Cauchy equation on the N-dimensional cube in the space $\mathbb{R}^N$.

In the present paper we unify all these investigations by considering the stability of the Cauchy equation postulated only for orthogonal vectors (orthogonal stability) from a ball centered at the origin. Because of methods used in proofs we restrict ourselves to the orthogonality in a finite dimensional inner product space.

In what follows let $(X, (\cdot|\cdot))$ be a real inner product space and $\dim X = N$ for some integer $N \geq 2$. Let Y be a real sequentially complete linear topological space and V be a nonempty bounded convex and symmetric with respect to zero subset of Y. Let, further, for some positive number r, the set $B_r := \{x \in X : \|x\| < r\}$ denote the open ball in X centered at the origin and having radius r, where $\|\cdot\|$ stands for a usual norm in the inner

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product space. Unless explicitely stated we shall permanently use the just introduced notation.

We shall say that two vectors $x, y \in X$ are orthogonal ($x \perp y$) if and only if $(x|y) = 0$. Moreover, the symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ will stand for the sets of positive integers, nonnegative integers, real numbers, positive and nonnegative real numbers, respectively.

2. Auxiliary results

To show the orthogonal stability of the Cauchy functional equation on a ball centered at the origin we have to prove first several lemmas. We say that a function $f : B \to Y$ is *additive* (*on a ball* B) if and only if for all $x, y \in B$ such that $x+y \in B$ we have $f(x+y) = f(x)+f(y)$, and a function $f : B \to Y$ is *quadratic* (*on a ball* B) if and only if for all $x, y \in B$ such that $x+y,\ x-y \in B$ we have $f(x+y)+f(x-y) = 2f(x)+2f(y)$. We say that a function $f : B \to Y$ is *orthogonally additive* (*on a ball* B) if and only if for all $x, y \in B$ such that $x+y \in B$ and $x \perp y$ we have $f(x+y) = f(x)+f(y)$.

LEMMA 1. *If $f : B_r \to Y$ is additive (odd orthogonally additive, quadratic, even orthogonally additive), then there exists exactly one additive (odd orthogonally additive, quadratic, even orthogonally additive) mapping $F : X \to Y$ such that $F|_{B_r} = f$.*

Proof. We give the proof for an odd orthogonally additive function. In the remaining cases the proofs are similar.

Assume that $f : B_r \to Y$ is odd orthogonally additive. For an arbitrary $x \in B_r$ there exists a $y \in B_r$ such that $x \perp y$ and $x+y \perp x-y$. Moreover, since $x \perp -y$ and f is odd, we have

$$f(x) = \left(f(x) - f\left(\frac{x+y}{2}\right) - f\left(\frac{x-y}{2}\right)\right) + \left(f\left(\frac{x+y}{2}\right) - f\left(\frac{x}{2}\right) - f\left(\frac{y}{2}\right)\right) + \left(f\left(\frac{x-y}{2}\right) - f\left(\frac{x}{2}\right) - f\left(-\frac{y}{2}\right)\right) + 2f\left(\frac{x}{2}\right) = 2f\left(\frac{x}{2}\right).$$

Hence, for an arbitrary $x \in B_r$, the following condition is satisfied

$$f(x) = 2f\left(\frac{x}{2}\right).$$

Observe that for every $m, n \in \mathbb{N}_0$, if $\frac{1}{2^n}x \in B_r$ then

$$2^{n+m} f\left(\frac{1}{2^{n+m}}x\right) = 2^n \cdot 2^m f\left(\frac{1}{2^m} \cdot \frac{1}{2^n}x\right) = 2^n f\left(\frac{1}{2^n}x\right). \tag{1}$$

Define a function $F : X \to Y$ by the formula

$$F(x) := 2^n f\left(\frac{1}{2^n}x\right) \quad \text{for all } x \in X,$$

where n is an integer such that $\frac{1}{2^n}x \in B_r$. Equality (1) guarantees that the function F is well defined.

We show that F is odd orthogonally additive. Fix $x, y \in X$ such that $x \perp y$. There exist $n_1, n_2 \in \mathbb{N}_0$ such that $\frac{1}{2^{n_1}}x,\ \frac{1}{2^{n_2}}y \in B_r$. Let $n := \max\{n_1, n_2\} + 1$. Then $\frac{1}{2^n}x,\ \frac{1}{2^n}y,\ \frac{1}{2^n}(x+y),\ \frac{1}{2^n}(x-y) \in B_r$ and

$$\begin{aligned} F(x) + F(y) &= 2^n f\left(\frac{1}{2^n}x\right) + 2^n f\left(\frac{1}{2^n}y\right) \\ &= 2^n f\left(\frac{1}{2^n}x + \frac{1}{2^n}y\right) = F(x+y). \end{aligned}$$

To show that F is unique, assume that there exist two functions $F_1,\ F_2 : X \to Y$ such that $F_1|_{B_r} = F_2|_{B_r} = f$. Fix arbitrary $x \in X$. Let $n \in \mathbb{N}_0$ is such that $\frac{1}{2^n}x \in B_r$. Then

$$F_1(x) = 2^n F_1\left(\frac{1}{2^n}x\right) = 2^n f\left(\frac{1}{2^n}x\right) = 2^n F_2\left(\frac{1}{2^n}x\right) = F_2(x),$$

hence $F_1 = F_2$.

LEMMA 2. *Let $f : B_r \to Y$ be odd orthogonally additive. Then f is additive.*

Proof. On account of Lemma 1 there exists an odd orthogonally additive extension $F : X \to Y$ of function f. Hence, from J. Rätz's paper [6, Corollary 7], F is additive, and so is $f = F|_{B_r}$.

LEMMA 3. *Let $f : B_r \to Y$ be even orthogonally additive. Then f is quadratic. More precisely, there exists an additive function $b : \mathbb{R}_0^+ \to Y$ such that $f(x) = b(\|x\|^2)$ for all $x \in B_r$.*

Proof. Follows from Lemma 1 and from J. Rätz's paper [6, Corollaries 7 and 10].

As an immediate consequence of Lemma 2 and Lemma 3 we obtain the following

COROLLARY 1. *Let $f : B_r \to Y$ be orthogonally additive. Then there exist additive mappings $a : X \to Y$ and $b : \mathbb{R}_0^+ \to Y$ such that $f(x) = a(x) + b(\|x\|^2)$ for all $x \in B_r$.*

The following lemmas establish some stability results concerning odd and even orthogonally additive mappings, respectively.

LEMMA 4. *Let $f : B_r \to Y$ be an odd function satisfying condition:*

(2) $\quad (x, y, x+y \in B_r,\ x \perp y) \quad$ *implies* $f(x+y) - f(x) - f(y) \in V.$

Then for each two linearly dependent vectors x and y we have

$$x, y, x+y \in B_r \quad \text{implies} \quad f(x+y) - f(x) - f(y) \in 3V.$$

Proof. Fix an $x \in B_r$. There exists a $y \in B_r$ such that $x \perp y$ and $\|x\| = \|y\|$. Then $\frac{x+y}{2} \perp \frac{x-y}{2}$ and

$$f(x) - f\left(\frac{x+y}{2}\right) - f\left(\frac{x-y}{2}\right) \in V,$$
$$f\left(\frac{x+y}{2}\right) - f\left(\frac{x}{2}\right) - f\left(\frac{y}{2}\right) \in V,$$
$$f\left(\frac{x-y}{2}\right) - f\left(\frac{x}{2}\right) + f\left(\frac{y}{2}\right) \in V.$$

Consequently, we infer that

$$f(x) - 2f\left(\frac{x}{2}\right) \in 3V.$$

Now, we are going to show that for each real λ and each $x \in B_r$ such that λx, $(\lambda + 1)x \in B_r$ the following relationship

$$f(x + \lambda x) - f(x) - f(\lambda x) \in 3V \tag{3}$$

holds. To show this let us distinguish four cases:
(i) $\lambda > 0$, (ii) $\lambda = 0$, (iii) $-1 < \lambda < 0$, (iv) $\lambda \leq -1$.

(i) Take an $x \in B_r$ such that $(\lambda + 1)x \in B_r$. There exists a vector $y \in X$ such that $x \perp y$ and $x + y \perp \lambda x - y$. It is easy to check that $y, x + y, \lambda x - y \in B_r$. Hence

$$f(x + \lambda x) - f(x + y) - f(\lambda x - y) \in V,$$
$$f(x + y) - f(x) - f(y) \in V,$$
$$f(\lambda x - y) - f(\lambda x) + f(y) \in V,$$

whence (3) immediately follows.

(ii) Then (3) is obviously fulfilled, because $f(0) \in V \subset 3V$.

(iii) Fix an $x \in B_r$ such that $\lambda x \in B_r$. Then, using (i) and the oddness of f, we infer that

$$f(x + \lambda x) - f(x) - f(\lambda x) = f(x + \lambda x) + f(-\lambda x) - f(x)$$
$$= f(x+\lambda x)+f\left(\left(-\frac{\lambda}{1+\lambda}\right)(1+\lambda)x\right)-f\left((1+\lambda)x+\left(-\frac{\lambda}{1+\lambda}\right)(1+\lambda)x\right) \in 3V.$$

(iv) Fix an $x \in B_r$ such that $\lambda x \in B_r$. Using (i) again and the oddness of f we obtain

$$f(x + \lambda x) - f(x) - f(\lambda x) = f((-1 - \lambda)(-x)) + f(-x) - f((-\lambda)(-x)) \in 3V.$$

This completes the proof of the lemma.

LEMMA 5. *Let $f : B_r \to Y$ be an odd function satisfying* (2). *Then there exists an additive function $a : X \to Y$ such that*

$$a(x) - f(x) \in k_1 \operatorname{seq} \operatorname{cl} V \quad \text{for all } x \in B_r, \tag{4}$$

where

$$k_1 = \begin{cases} 25 & \text{for } N = 2, \\ (10N+8) & \text{for } N \geq 3. \end{cases}$$

Proof. Without loss of generality we can assume that B_r is the unit ball ($r = 1$) and put $B := B_1$. Let $u_1, \ldots, u_N$ be vectors in the space X such that $u_i \perp u_j$ for $i \neq j$, $i, j \in \{1, \ldots, N\}$, $\|u_i\| = \frac{1}{2}$, $i \in \{1, \ldots, N\}$ and $X = \operatorname{lin}\{u_1, \ldots, u_N\}$. An arbitrary $x \in X$ can be written as $x = \sum_{i=1}^N \alpha_i u_i$, for some (uniquely determined) $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$. Write further α_i as $n_i + m_i$, where n_i stands for the integral part of number α_i and $m_i := \alpha_i - n_i$ ($i \in \{1, \ldots, N\}$). Then

$$x = \sum_{i=1}^N (n_i u_i + m_i u_i).$$

Define a map $F : X \to Y$ by the formula

$$F(x) := \sum_{i=1}^N (n_i f(u_i) + f(m_i u_i)).$$

Moreover, let $F_i(x)$ ($i \in \{1, \ldots, N\}$) stands for the i-th summand of the above sum. Fix $x \in B$. Since

$$x = \sum_{i=1}^N \alpha_i u_i,$$

and vectors u_i are pairwise orthogonal, we deduce that

$$\|x\|^2 = \|\alpha_1 u_1\|^2 + \ldots + \|\alpha_N u_N\|^2,$$

which implies that $\alpha_i u_i \in B$ for all $i \in \{1, \ldots, N\}$.

Observe that

$$F(x) - f(x) = \sum_{i=1}^N (n_i f(u_i) + f(m_i u_i)) - f\Big(\sum_{i=1}^N \alpha_i u_i\Big)$$

$$= \Big(\sum_{i=1}^N (n_i f(u_i) + f(m_i u_i)) - \sum_{i=1}^N f(\alpha_i u_i)\Big) + \Big(\sum_{i=1}^N f(\alpha_i u_i) - f\Big(\sum_{i=1}^N \alpha_i u_i\Big)\Big)$$

$$= \sum_{i=1}^N (n_i f(u_i) + f(m_i u_i) - f(n_i u_i + m_i u_i)) + \Big(\sum_{i=1}^N f(\alpha_i u_i) - f\Big(\sum_{i=1}^N \alpha_i u_i\Big)\Big).$$

An easy induction argument shows that

$$\sum_{i=1}^{N} f(\alpha_i u_i) - f\Big(\sum_{i=1}^{N} \alpha_i u_i\Big) \in (N-1)V.$$

Put

$$A_i := n_i f(u_i) + f(m_i u_i) - f(n_i u_i + m_i u_i), \qquad i \in \{1, \dots, N\}.$$

Observe that if $x \in B$ then

$$1 > \|x\|^2 = \sum_{i=1}^{N} \alpha_i^2 \|u_i\|^2 = \sum_{i=1}^{N} \frac{1}{4}\alpha_i^2,$$

whence $\sum_{i=1}^{N} \alpha_i^2 < 4$, and consequently $|\alpha_i| < 2$ for all $i \in \{1, \dots, N\}$. Moreover, for at least three $i \in \{1, \dots, N\}$, we have $|\alpha_i| > 1$. Let us distinguish four cases.

(a) $1 \le \alpha_i < 2$. Then $n_i = 1$ and, on account of Lemma 4, we state that

$$A_i = f(u_i) + f(m_i u_i) - f(u_i + m_i u_i) \in 3V.$$

(b) $0 \le \alpha_i < 1$. Then $n_i = 0$ and $A_i = 0$.

(c) $-1 \le \alpha_i < 0$. Then $n_i = -1$ and

$$A_i = -f(u_i) + f(m_i u_i) - f(-u_i + m_i u_i) \in 3V.$$

(d) $-2 < \alpha_i < -1$. In this case $n_i = -2$. Since $(-1 + m_i)u_i \in B$,

$$\begin{aligned} A_i &= -2f(u_i) + f(m_i u_i) - f(-2u_i + m_i u_i) \\ &= (-f(u_i) + f(m_i u_i) - f(-u_i + m_i u_i)) \\ &\quad + (f(-u_i + m_i u_i) - f(u_i) - f(-2u_i + m_i u_i)) \in 6V. \end{aligned}$$

Consequently,

$$F(x) - f(x) \in \begin{cases} 13V & \text{for } N = 2, \\ (4N+8)V & \text{for } N \ge 3. \end{cases} \tag{5}$$

We shall show now that for every $x, y \in X$ one has

$$F(x+y) - F(x) - F(y) \in 6N\, V.$$

For this purpose fix $x, y \in X$. Obviously x and y we can represented in the form

$$x = \sum_{i=1}^{N} \alpha_i u_i = \sum_{i=1}^{N} (n_i u_i + m_i u_i),$$

$$y = \sum_{i=1}^{N} \beta_i u_i = \sum_{i=1}^{N} (k_i u_i + l_i u_i)$$

with some (uniquely determined) real numbers α_i, β_i ($i \in \{1, \dots, N\}$); n_i, k_i stand here for the integral parts of α_i and β_i, respectively, and $m_i := \alpha_i - n_i$, $l_i := \beta_i - k_i$ ($i \in \{1, \dots, N\}$). Fix $i \in \{1, \dots, N\}$. Assume first that $m_i + l_i < 1$. Then

$$\begin{aligned} F_i(x+y) - F_i(x) - F_i(y) &= ((n_i + k_i)f(u_i) + f((m_i + l_i)u_i)) \\ &\quad - (n_i f(u_i) + f(m_i u_i)) - (k_i f(u_i) + f(l_i u_i)) \\ &= f((m_i + l_i)u_i) - f(m_i u_i) - f(l_i u_i) \in 3V. \end{aligned}$$

Let now $1 \le m_i + l_i < 2$. Then $(m_i - 1)u_i \in B$ and

$$\begin{aligned} &F_i(x+y) - F_i(x) - F_i(y) \\ &\quad = ((n_i + k_i + 1)f(u_i) + f((m_i + l_i - 1)u_i)) \\ &\qquad - (n_i f(u_i) + f(m_i u_i)) - (k_i f(u_i) + f(l_i u_i)) \\ &\quad = f(u_i) + f((m_i + l_i - 1)u_i) - f(m_i u_i) - f(l_i u_i) \\ &\quad = (f(u_i) + f((m_i - 1)u_i) - f(m_i u_i)) \\ &\qquad + (f((m_i + l_i - 1)u_i) - f((m_i - 1)u_i) - f(l_i u_i)) \in 6V. \end{aligned}$$

Hence

$$F(x+y) - F(x) - F(y) = \sum_{i=1}^{N} (F_i(x+y) - F_i(x) - F_i(y)) \in 6NV.$$

From J. Rätz's paper [5] we derive the existence of an additive function $a : X \to Y$ such that for all $x \in X$ we have

$$a(x) - F(x) \in 6N \operatorname{seq\,cl} V \quad \text{and} \quad a(x) = \lim_{n \to \infty} \frac{1}{2^n} F(2^n x).$$

This jointly with (5) gives (4), what ends the proof.

A thorough inspection of the proof of the above lemma allows to observe that the condition $x \perp y$ in (2) and the oddness of function f were used in the inner product space only for estimating the Cauchy difference for vectors that were linearly dependent. So, the above result can be reformulated in a slightly different form.

LEMMA 6. *Let $(X, |\!|\!| \cdot |\!|\!|)$ be a real normed space, $\dim X = N$, let $\widetilde{B}_r := \{x \in X : |\!|\!| x |\!|\!| < r\}$ for some positive constant r and let $f : \widetilde{B}_r \to Y$ fulfil the condition*

$$x, y, x+y \in \widetilde{B}_r \quad \textit{implies} \quad f(x+y) - f(x) - f(y) \in V.$$

Then there exist an additive function $a : X \to Y$ and a real constant $k_2 = k_2(N, |\!|\!| \cdot |\!|\!|)$ such that

$$a(x) - f(x) \in k_2 \operatorname{seq\,cl} V \quad \textit{for all } x \in \widetilde{B}_r.$$

P r o o f. Let $\|\cdot\|$ be any norm in X coming from an inner product. Then, as in the previous lemma, we get the existence of an additive mapping $a : X \to Y$ such that for all $x \in B_r := \{x \in X : \|x\| < r\}$ one has

$$a(x) - f(x) \in k' \operatorname{seq\,cl} V, \tag{6}$$

where

$$k' = \begin{cases} (5N-1) & \text{for } N < 3, \\ (4N+2) & \text{for } N \geq 3. \end{cases}$$

Since X is finite dimensional, the norms $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ are equivalent; there exist then positive constants α and β such that

$$\alpha |\!|\!|x|\!|\!| \leq \|x\| \leq \beta |\!|\!|x|\!|\!| \tag{7}$$

for all $x \in X$. Without loss of generality we may assume that the balls $\widetilde{B}_r$ and B_r are unit balls ($r = 1$) and put $\widetilde{B} := \widetilde{B}_1$ and $B_\alpha := \alpha B_1$. From (7) we have $B_\alpha \subset \widetilde{B}$. We continue as Z. Kominek in [3]. There exists a $p \in \mathbb{N}$ such that $\widetilde{B} \subset 2^p B_\alpha$. If $x \in \widetilde{B}$ then $\frac{1}{2^p}x \in B_\alpha$. Take now an arbitrary $x \in \widetilde{B}$. Then also $\frac{1}{2^l}x \in \widetilde{B}$ for $l \in \{1, \ldots, p\}$ and

$$f\left(\frac{1}{2^{l-1}}x\right) - 2f\left(\frac{1}{2^l}x\right) \in V, \quad l \in \{1, \ldots, p\}.$$

It is easy to check that

$$f(x) - 2^p f\left(\frac{1}{2^p}x\right) \in (2^p - 1)V. \tag{8}$$

Finally, from (6) and (8), for an arbitrary $x \in \widetilde{B}$, we have

$$\begin{aligned} a(x) - f(x) &= 2^p\left(a\left(\frac{1}{2^p}x\right) - f\left(\frac{1}{2^p}x\right)\right) + \left(2^p f\left(\frac{1}{2^p}x\right) - f(x)\right) \\ &\in 2^p k' \operatorname{seq\,cl} V + (2^p - 1)V \subset (2^p(k'+1) - 1) \operatorname{seq\,cl} V, \end{aligned}$$

and we get the assertion of the lemma with $k_2 = 2^p(k'+1) - 1$, where $p \in \mathbb{N}$ depends on $|\!|\!|\cdot|\!|\!|$ only.

Next results concern even mappings.

LEMMA 7. *Let $f : B_r \to Y$ be an even function satisfying* (2). *Then for all $x, y \in B_r$ such that $x + y, x - y \in B_r$ one has*

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) \in 34V. \tag{9}$$

Proof. Fix $x, y \in B_r$ such that $\|x\| = \|y\|$. Then $\frac{x+y}{2} \perp \frac{x-y}{2}$ and

$$f(x) - f\left(\frac{x+y}{2}\right) - f\left(\frac{x-y}{2}\right) \in V,$$
$$f(y) - f\left(\frac{x+y}{2}\right) - f\left(\frac{y-x}{2}\right) \in V.$$

Hence for all $x, y \in B_r$ such that $\|x\| = \|y\|$

(10) $$f(x) - f(y) \in 2V.$$

Since $\dim X \geq 2$, for an arbitrary $x \in B_r$ there exists a vector $y \in B_r$ such that $x \perp y$ and $\|x\| = \|y\|$. Using (2), (10) and the eveness of f we get

$$f(x) - f\left(\frac{x+y}{2}\right) - f\left(\frac{x-y}{2}\right) \in V,$$
$$f\left(\frac{x+y}{2}\right) - f\left(\frac{x}{2}\right) - f\left(\frac{y}{2}\right) \in V,$$
$$f\left(\frac{x-y}{2}\right) - f\left(\frac{x}{2}\right) - f\left(-\frac{y}{2}\right) \in V,$$
$$2f\left(\frac{y}{2}\right) - 2f\left(\frac{x}{2}\right) \in 4V,$$

whence

(11) $$f(x) - 4f\left(\frac{x}{2}\right) \in 7V \quad \text{for all } x \in B_r.$$

Fix now an $x \in B_r$ and a real number $\lambda > 0$ such that λx, $(\lambda + 1)x$, $(\lambda - 1)x \in B_r$. Then there exists a vector $y \in B_r$ such that $x \perp y$ and $x + y \perp \lambda x - y$. It is easy to check that also $x + y, \lambda x - y, 2y \in B_r$. From of (2), (11) and eveness of function f, we obtain

$$\begin{aligned}
&f(x + \lambda x) + f(x - \lambda x) - 2f(x) - 2f(\lambda x) \\
&\quad = (f(x + y + \lambda x - y) - f(x + y) - f(\lambda x - y)) \\
&\qquad + 2(f(x + y) - f(x) - f(y)) + 2(f(\lambda x - y) - f(\lambda x) - f(-y)) \\
&\qquad + (-f(x + y - \lambda x + y) + f(x - \lambda x) + f(2y)) \\
&\qquad + (f(x + y - \lambda x + y) - f(x + y) - f(-\lambda x + y)) \\
&\qquad + (4f(y) - f(2y)) \in V + 2V + 2V + V + V + 7V = 14V.
\end{aligned}$$

Therefore

(12) $$f(x + \lambda x) + f(x - \lambda x) - 2f(x) - 2f(\lambda x) \in 14V$$

for all $x \in B_r$ and $\lambda \in \mathbb{R}^+$ such that $\lambda x, (\lambda+1)x, (\lambda-1)x \in B_r$. Observe further that for an arbitrary $x \in X$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha x, \beta x, (\alpha+\beta)x$, $(\alpha-\beta)x \in B_r$ we have

$$f(\alpha x+\beta x)+f(\alpha x-\beta x)-2f(\alpha x)-2f(\beta x) \in 14V. \tag{13}$$

In fact, when $\alpha = 0$ or $\beta = 0$, then condition (13) obviously holds. When $\frac{\beta}{\alpha} > 0$, then we apply (12) for $\lambda := \frac{\beta}{\alpha}$. If $\frac{\beta}{\alpha} < 0$ then (12) applied for $\lambda := -\frac{\beta}{\alpha}$ and the eveness of f give the required relationship.

Fix arbitrary $x, y \in B_r$ such that $x+y, x-y \in B_r$. If x and y are linearly dependent, then from (13) it follows that condition (9) holds. Assume that x and y are linearly independent. Let u and v be vectors from the subspace $\operatorname{lin}\{x,y\}$ such that $u, v \in B_r$ and $u \perp v$. Therefore $x = \alpha u+\beta v$, $y = \gamma u+\delta v$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Using conditions (2) and (13) we get

$$\begin{aligned}
&f(x+y)+f(x-y)-2f(x)-2f(y)\\
&\quad = f((\alpha+\gamma)u+(\beta+\delta)v)+f((\alpha-\gamma)u+(\beta-\delta)v)\\
&\qquad -2f(\alpha u+\beta v)-2f(\gamma u+\delta v)\\
&\quad = (f((\alpha+\gamma)u+(\beta+\delta)v)-f(\alpha u+\gamma u)-f(\beta v+\delta v))\\
&\qquad +(f((\alpha-\gamma)u+(\beta-\delta)v)-f(\alpha u-\gamma u)-f(\beta v-\delta v))\\
&\qquad +(f(\alpha u+\gamma u)+f(\alpha u-\gamma u)-2f(\alpha u)-2f(\gamma u))\\
&\qquad +(f(\beta v+\delta v)+f(\beta v-\delta v)-2f(\beta v)-2f(\delta v))\\
&\qquad +2(f(\alpha u)+f(\beta v)-f(\alpha u+\beta v))+2(f(\gamma u)+f(\delta v)-f(\gamma u+\delta v))\\
&\quad \in V+V+14V+14V+2V+2V = 34V,
\end{aligned}$$

which ends the proof.

LEMMA 8. *Let $(X, |\!|\!| \cdot |\!|\!|)$ be a real normed space,* $\dim X = N$, *let* $\widetilde{B}_r := \{x \in X : |\!|\!|x|\!|\!| < r\}$ *for some positive constant r and let $\varphi : \widetilde{B}_r \times \widetilde{B}_r \to Y$ be a symmetric function such that*

$$\varphi(x_1+x_2,y)-\varphi(x_1,y)-\varphi(x_2,y) \in V \textit{ whenever } x_1, x_2, x_1+x_2, y \in \widetilde{B}_r.$$

Then there exist a symmetric and biadditive mapping $\psi : \widetilde{B}_r \times \widetilde{B}_r \to Y$ and a constant $k_3 = k_3(N, |\!|\!| \cdot |\!|\!|)$ such that

$$\psi(x,y)-\varphi(x,y) \in k_3 \operatorname{seq\,cl} V \quad \textit{for all } x, y \in \widetilde{B}_r.$$

P r o o f. Like in the proof of Lemma 6, assume first additionally, that $\|\cdot\|$ is a norm in X coming from an inner product. Fix a $y \in B_r := \{x \in X : \|x\| < r\}$ and define a mapping $\varphi_y : B_r \to Y$ as follows

$$\varphi_y(x) := \varphi(x,y) \quad \text{for all } x \in B_r.$$

From the assumption we get

$$\varphi_y(x_1 + x_2) - \varphi_y(x_1) - \varphi_y(x_2) \in V$$

for all vectors $x_1, x_2 \in B_r$ such that $x_1 + x_2 \in B_r$. Writing, as previously, an arbitrary $x \in X$ in the form $x = \sum_{i=1}^{N}(n_i u_i + m_i u_i)$, define $\Phi_y : X \to Y$ by the formula

$$\Phi_y(x) := \sum_{i=1}^{N}(n_i \varphi_y(u_i) + \varphi_y(\mu_i)),$$

where $\mu_i = m_i u_i$, $i \in \{1, \ldots, N\}$. Similar arguments as in the proof of Lemma 5 (cf. also the first part of the proof of Lemma 6) show that there exists an additive function $G_y : X \to Y$ such that

$$G_y(x) - \varphi_y(x) \in \begin{cases} (5N-1)\operatorname{seq\,cl} V & \text{for } N < 3, \\ (4N+2)\operatorname{seq\,cl} V & \text{for } N \geq 3 \end{cases}$$

for all $x \in B_r$.

Let a mapping $G : X \times B_r \to Y$ be defined by the formula

$$G(x, y) := G_y(x) \quad \text{for all } x \in X,\ y \in B_r.$$

Then for all $x, y \in B_r$ we have

$$G(x, y) - \varphi(x, y) \in \begin{cases} (5N-1)\operatorname{seq\,cl} V & \text{for } N < 3, \\ (4N+2)\operatorname{seq\,cl} V & \text{for } N \geq 3. \end{cases} \tag{14}$$

In view of the additivity of G_y, the function G is additive with respect to the first variable.

Now, we shall show that for every $x, y, z \in B_r$ such that $y + z \in B_r$ we have

$$G(x, y + z) - G(x, y) - G(x, z) \in 2N \operatorname{seq\,cl} V.$$

Fix an $x \in B_r$. Using previous notations, for every $k \in \mathbb{N}$, we represent the vector $2^k x$ in the form

$$2^k x = \sum_{i=1}^{N}(n_{i,k} u_i + \mu_{i,k}).$$

Then

$$\left| \frac{1}{2}|n_{i,k}| - \|\mu_{i,k}\| \right| \leq \|n_{i,k} u_i + \mu_{i,k}\| \leq \|2^k x\| < 2^k,$$

whence

$$\frac{1}{2}|n_{i,k}| - \|\mu_{i,k}\| < 2^k,$$

$$\frac{1}{2}|n_{i,k}| < 2^k + \|\mu_{i,k}\| \leq 2^k + \frac{1}{2},$$

implying that

$$|n_{i,k}| \le 2^{k+1} + 1, \quad i \in \{1, \dots, N\},\ k \in \mathbb{N}.$$

Using the above estimation and the fact that $G_y(x) = \lim_{k\to\infty} \frac{1}{2^k}\Phi_y(2^k x)$ (cf. J. Rätz [5]), we may write

$$\begin{aligned}
&G(x, y+z) - G(x,y) - G(x,z) = G_{y+z}(x) - G_y(x) - G_z(x)\\
&= \lim_{k\to\infty} \frac{1}{2^k}(\Phi_{y+z}(2^k x) - \Phi_y(2^k x) - \Phi_z(2^k x))\\
&= \lim_{k\to\infty} \frac{1}{2^k}\Big(\sum_{i=1}^{N}(n_{i,k}\varphi_{y+z}(u_i) + \varphi_{y+z}(\mu_{i,k})) - \sum_{i=1}^{N}(n_{i,k}\varphi_y(u_i) + \varphi_y(\mu_{i,k}))\\
&\quad - \sum_{i=1}^{N}(n_{i,k}\varphi_z(u_i) + \varphi_z(\mu_{i,k}))\Big)\\
&= \lim_{k\to\infty} \frac{1}{2^k}\sum_{i=1}^{N}(n_{i,k}(\varphi_{y+z}(u_i) - \varphi_y(u_i) - \varphi_z(u_i))\\
&\quad + (\varphi_{y+z}(\mu_{i,k}) - \varphi_y(\mu_{i,k}) - \varphi_z(\mu_{i,k})))\\
&\in \bigcap_{k\in\mathbb{N}} \left(\frac{1}{2^k}(2^{k+1}+1)N\,V + \frac{1}{2^k}NV\right) = 2N \operatorname{seq} \operatorname{cl} V.
\end{aligned}$$

From Lemma 6 (more precisely, from the part of the proof concerning the inner product space) we state that there exists a function $\Psi : B_r \times X \to Y$ additive with respect to the second variable and such that for all $x, y \in B_r$ one has

$$\Psi(x,y) - G(x,y) \in \begin{cases} 2N(5N-1) \operatorname{seq} \operatorname{cl} V & \text{for } N < 3,\\ 2N(4N+2) \operatorname{seq} \operatorname{cl} V & \text{for } N \ge 3.\end{cases} \tag{15}$$

From the form of Ψ (defined as the limit of a suitable Cauchy sequence, cf. J. Rätz [5]) it follows that it is additive with respect to the first variable as well. Moreover, from (14) and (15), we get that if $x, y \in B_r$ then

$$\Psi(x,y) - \varphi(x,y) \in \begin{cases} (2N+1)(5N-1) \operatorname{seq} \operatorname{cl} V & \text{for } N < 3,\\ (2N+1)(4N+2) \operatorname{seq} \operatorname{cl} V & \text{for } N \ge 3.\end{cases} \tag{16}$$

Define a mapping $\psi : B_r \times B_r \to Y$ by the formula

$$\psi(x,y) := \frac{\Psi(x,y) + \Psi(y,x)}{2} \quad \text{for all } x, y \in B_r.$$

Obviously ψ is symmetric. Moreover, using the symmetry of φ and (16), and equalities

$$\psi(x,y)-\varphi(x,y)=\frac{\Psi(x,y)+\Psi(y,x)}{2}-\varphi(x,y)$$
$$=\frac{\Psi(x,y)-\varphi(x,y)}{2}+\frac{\Psi(y,x)-\varphi(y,x)}{2}$$

for all $x,y\in B_r$, we have

$$\psi(x,y)-\varphi(x,y)\in\begin{cases}(2N+1)(5N-1)\operatorname{seq}\operatorname{cl}V & \text{for } N<3,\\ (2N+1)(4N+2)\operatorname{seq}\operatorname{cl}V & \text{for } N\geq 3.\end{cases}\tag{17}$$

In the finite dimensional space X the norms $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ are equivalent. Now we proceed in the same way as in the proof of Lemma 6. This completes the proof.

LEMMA 9. *Let $(X,|\!|\!|\cdot|\!|\!|)$ be a real normed space, $\dim X=N$, let $\widetilde{B}_r:=\{x\in X: |\!|\!|x|\!|\!|<r\}$ for some positive constant r and let $f:\widetilde{B}_r\to Y$ satisfy the condition*

$$x,y,x+y,x-y\in\widetilde{B}_r \quad \textit{implies } f(x+y)+f(x-y)-2f(x)-2f(y)\in V.$$

Then there exist a quadratic function $q:X\to Y$ and a constant $k_4=k_4(N,|\!|\!|\cdot|\!|\!|)$ such that

$$q(x)-f(x)\in k_4\operatorname{seq}\operatorname{cl}V \quad \textit{for all } x\in\widetilde{B}_r.\tag{18}$$

Proof. Functions f_o, $f_e: B\to Y$, given by the formulas

$$f_o(x)=\frac{f(x)-f(-x)}{2},\qquad f_e(x)=\frac{f(x)+f(-x)}{2},\qquad x\in\widetilde{B}_r,$$

are the odd and even parts of f, respectively. For all $x,y\in\widetilde{B}_r$ we have

$$f_o(x+y)+f_o(x-y)-2f_o(x)-2f_o(y)\in V$$

and

$$f_e(x+y)+f_e(x-y)-2f_e(x)-2f_e(y)\in V.$$

Since f_o is odd we also have

$$f_o(x-y)+f_o(x+y)-2f_o(x)+2f_o(y)\in V.$$

Hence

$$4f_o(y)\in 2V \quad \text{for all } y\in\widetilde{B}_r,$$

and so

$$f_o(y)\in\frac{1}{2}V \quad \text{for all } y\in\widetilde{B}_r.\tag{19}$$

It is easy to check that

$$f_e(0)\in\frac{1}{2}V\tag{20}$$

and

$$f_e(2x) - 4f_e(x) \in \frac{3}{2}V \quad \text{for all } x \in \frac{1}{2}\widetilde{B}_r. \tag{21}$$

Let $\widetilde{B}_{r/2} := \frac{1}{2}\widetilde{B}_r$. Define $\varphi : \widetilde{B}_{r/2} \times \widetilde{B}_{r/2} \to Y$ by the formula

$$\varphi(x,y) := \frac{1}{4}[f_e(x+y) - f_e(x-y)] \quad \text{for all } x,y \in \widetilde{B}_{r/2}.$$

Obviously, φ is also symmetric. Moreover, for all $x_1, x_2, y \in \widetilde{B}_{r/2}$ such that $x_1 + x_2 \in \widetilde{B}_{r/2}$, we have

$$\begin{aligned}
&4(\varphi(x_1+x_2,y) - \varphi(x_1,y) - \varphi(x_2,y)) \\
&\quad = f_e(x_1+x_2+y) - f_e(x_1+x_2-y) - f_e(x_1+y) + f_e(x_1-y) \\
&\qquad - f_e(x_2+y) + f_e(x_2-y) \\
&\quad = (f_e(x_1+x_2+y) + f_e(x_1-x_2-y) - 2f_e(x_1) - 2f_e(x_2+y)) \\
&\qquad + (f_e(x_2+y) + f_e(x_2-y) - 2f_e(x_2) - 2f_e(y)) \\
&\qquad + (-f_e(x_1+x_2-y) - f_e(x_1-x_2-y) + 2f_e(x_1-y) + 2f_e(x_2)) \\
&\qquad + (-f_e(x_1-y) - f_e(x_1+y) + 2f_e(x_1) + 2f_e(y)) \in 4V,
\end{aligned}$$

whence

$$\varphi(x_1+x_2,y) - \varphi(x_1,y) - \varphi(x_2,y) \in V \quad \text{whenever } x_1, x_2, x_1+x_2, y \in \widetilde{B}_{r/2}.$$

From Lemma 8 we obtain the existence of a symmetric and biadditive function $\psi : \widetilde{B}_{r/2} \times \widetilde{B}_{r/2} \to Y$ such that

$$\psi(x,y) - \varphi(x,y) \in k_3 \operatorname{seq} \operatorname{cl} V \quad \text{for all } x,y \in \widetilde{B}_{r/2}. \tag{22}$$

Using (19), (20) and (21) we may write

$$\begin{aligned}
4(\varphi(x,x) - f(x)) &= (f_e(2x) - f_e(0)) - 4(f_o(x) + f_e(x)) \\
&= (f_e(2x) - 4f_e(x)) - f_e(0) - 4f_o(x) \\
&\in \frac{3}{2}V + \frac{1}{2}V + 4\cdot\frac{1}{2}V = 4V
\end{aligned}$$

for all $x \in \widetilde{B}_{r/2}$, so that

$$\varphi(x,x) - f(x) \in V \quad \text{for all } x \in \widetilde{B}_{r/2}. \tag{23}$$

Let $h(x) := \psi(x,x)$ for $x \in \widetilde{B}_{r/2}$. Obviously h is quadratic on the ball $\widetilde{B}_{r/2}$. There exists (cf. Lemma 1) a quadratic mapping $q : X \to Y$ such that $q|_{\widetilde{B}_{r/2}} = h$.

Fix $x \in \widetilde{B}_r$. If $x \in \widetilde{B}_{r/2}$ then, on account of (22) and (23), we have

$$\begin{aligned} q(x) - f(x) &= \psi(x,x) - f(x) = (\psi(x,x) - \varphi(x,x)) + (\varphi(x,x) - f(x)) \\ &\in k_3 \operatorname{seq}\operatorname{cl} V + V \subset (k_3+1)\operatorname{seq}\operatorname{cl} V. \end{aligned}$$

If $x \in \widetilde{B}_r \setminus \widetilde{B}_{r/2}$ then $\frac{1}{2}x \in \widetilde{B}_{r/2}$ and from the previous case we obtain

$$\begin{aligned} q(x) - f(x) &= 4\left(q\left(\frac{1}{2}x\right) - f\left(\frac{1}{2}x\right)\right) + \left(4f\left(\frac{1}{2}x\right) - f(x)\right) \\ &\in 4(k_3+1)\operatorname{seq}\operatorname{cl} V + \frac{3}{2}V \subset \left(4k_3 + \frac{11}{2}\right)\operatorname{seq}\operatorname{cl} V, \end{aligned}$$

which gives the assertion of the lemma with $k_4 = \left(4k_3 + \frac{11}{2}\right)$.

REMARK 1. If in Lemma 9 we assume additionally that f is even and X is an inner product space, then

$$k_4 = \begin{cases} 4(2N+1)(5N-1) + \frac{7}{2} & \text{for } N < 3, \\ 4(2N+1)(4N+2) + \frac{7}{2} & \text{for } N \geq 3. \end{cases} \tag{24}$$

LEMMA 10. *Let $f : B_r \to Y$ be an even mapping satisfying* (2). *Then there exist an additive function $b : \mathbb{R}_0^+ \to Y$ and a constant $k_5 = k_5(N)$ such that*

$$b(\|x\|^2) - f(x) \in k_5 \operatorname{seq}\operatorname{cl} V \quad \textit{for all } x \in B_r.$$

Proof. A consequence of Lemma 7, Lemma 9, Remark 1 and Lemma 3. The existence of the constant k_5 results from (9), (18) and (24).

3. Main result

The main result of the paper reads as follows.

THEOREM 1. *Let $(X, (\cdot|\cdot))$ be a real inner product space,* $\dim X = N$ $(N \geq 2)$, *Y be a real sequentially complete linear topological space and V let be a nonempty bounded convex and symmetric with respect to zero subset of Y. Let, further, B_r $(r > 0)$ denote an open ball in X centered at the origin and with radius r. If a function $f : B_r \to Y$ fulfils the condition* (2)

$$(x, y, x+y \in B_r,\ x \perp y) \quad \textit{implies} \quad f(x+y) - f(x) - f(y) \in V,$$

then there exist additive functions $a : X \to Y$, $b : \mathbb{R}_0^+ \to Y$ and a constant $k = k(N)$ such that

$$a(x) + b(\|x\|^2) - f(x) \in k \ \operatorname{seq}\operatorname{cl} V \quad \textit{for all} \quad x \in B_r.$$

Proof. Let functions f_o, $f_e : B_r \to Y$ denote the odd and even part of function f, respectively. Then, if f fulfils the condition (2), so do the functions f_o and f_e. From Lemma 5 we infer that there exist an additive function $a : X \to Y$ and a constant k_1 such that

$$a(x) - f_o(x) \in k_1 \operatorname{seq}\operatorname{cl} V \quad \text{for all } x \in B_r,$$

and from Lemma 10 we get the existence of an additive function $b : \mathbb{R}_0^+ \to Y$ and a constant k_5 such that

$$b(\|x\|^2) - f_e(x) \in k_5 \operatorname{seq\,cl} V \quad \text{for all } x \in B_r.$$

Consequently,

$$a(x) + b(\|x\|^2) - f(x) \in (k_1 + k_5) \operatorname{seq\,cl} V \quad \text{for all } x \in B_r,$$

which gives the assertion of the lemma with $k = k_1 + k_5$.

REMARK 2. It is easy to show that, in general, g in the assertion of Theorem 1 is not uniquely determined.

4. Applications

Besides the Cauchy functional equation we can also study the stability problem for other functional equations. Now we will give three results, concerning the stability of the Jensen, Pexider and exponential functional equations on balls, as an application of the theorem just established (cf. Z. Kominek [3], K. Nikodem [4], R. Ger [1]).

THEOREM 2. *Under the assumptions of Theorem* 1, *if a function $f : B_r \to Y$ fulfils the condition*

$$(25) \qquad (x, y \in B_r,\ x \perp y) \quad \textit{implies} \quad f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \in V,$$

then there exist a function $g : B_r \to Y$ fulfilling for orthogonal vectors the Jensen functional equation on the ball B_r:

$$(x, y \in B_r,\ x \perp y) \quad \textit{implies} \quad g\left(\frac{x+y}{2}\right) = \frac{g(x)+g(y)}{2},$$

and a constant $k = k(N)$ such that

$$g(x) - f(x) \in 4k \operatorname{seq\,cl} V \quad \textit{for all} \quad x \in B_r.$$

Proof. Define $f_1 : B_r \to Y$ by the formula

$$f_1 := f - f(0).$$

From (25) we have

$$(26) \qquad (x, y \in B_r,\ \perp y) \implies f_1\left(\frac{x+y}{2}\right) - \frac{f_1(x)+f_1(y)}{2} \in V,$$

and $f_1(0) = 0$. Moreover, since for an arbitrary $x \in X$, we have $x \perp 0$ and $0 \perp x$, so

$$(27) \qquad f_1\left(\frac{x}{2}\right) - \frac{f_1(x)}{2} \in V, \quad x \in B_r.$$

Take $x, y \in B_r$ such that $x + y \in B_r$ and $x \perp y$. From (27)

$$f_1\left(\frac{x+y}{2}\right) - \frac{f_1(x+y)}{2} \in V,$$

which together with (26) and symmetry of V gives

$$f_1(x+y) - f_1(x) - f_1(y) \in 4V.$$

Now, using Theorem 1, we obtain the existence of additive functions $a : X \to Y$, $b : \mathbb{R}_0^+ \to Y$ and a constant $k = k(N)$ such that

$$a(x) + b(\|x\|^2) - f_1(x) \in 4k \operatorname{seq} \operatorname{cl} V, \qquad x \in B_r.$$

Let $g(x) := a(x) + b(\|x\|^2) + f(0)$, $x \in X$. Such g satisfies both conditions from the assertion of the theorem.

THEOREM 3. *Under the assumptions of Theorem 1, if functions $f, g, h : B_r \to Y$ fulfil the condition*

(28) $\quad (x, y, x+y \in B_r,\ x \perp y)$ *implies* $f(x+y) - g(x) - h(y) \in V,$

then there exist functions $f_1, g_1, h_1 : B_r \to Y$ fulfilling for orthogonal vectors the Pexider functional equation on the ball B_r:

$$(x, y, x+y \in B_r,\ x \perp y) \quad \textit{implies} \quad f_1(x+y) = g_1(x) + h_1(y),$$

and a constant $k = k(N)$ such that for all $x \in B_r$ one has

$$f_1(x) - f(x) \in 3k \operatorname{seq} \operatorname{cl} V,$$
$$g_1(x) - g(x) \in 4k \operatorname{seq} \operatorname{cl} V,$$
$$h_1(x) - h(x) \in 4k \operatorname{seq} \operatorname{cl} V.$$

Proof. Since $x \perp 0$ and $0 \perp x$ for all $x \in X$, from (28) we have

$$f(x) - g(x) - h(0) \in V, \qquad x \in B_r,$$

and

$$f(x) - g(0) - h(x) \in V, \qquad x \in B_r.$$

Define functions f_0, g_0, $h_0 : B_r \to Y$ by the formulas

$$f_0 := f - g(0) - h(0),$$
$$g_0 := g - g(0),$$
$$h_0 := h - h(0).$$

It is easy to see that

$$f_0(x) - g_0(x) \in V, \qquad x \in B_r,$$

and

$$f_0(x) - h_0(x) \in V, \qquad x \in B_r.$$

We show that the following condition is satisfied

$$(x, y, x+y \in B_r,\ x \perp y) \implies f_0(x+y) - f_0(x) - f_0(y) \in 3V.$$

Indeed, take $x, y \in B_r$ such that $x+y \in B_r$ and $x \perp y$. We have

$$\begin{aligned} f_0(x+y) - f_0(x) - f_0(y) &= f(x+y) - f(x) - f(y) + g(0) + h(0) \\ &= (f(x+y) - g(x) - h(y)) - (f(x) - g(x) - h(0)) \\ &\quad - (f(y) - g(0) - h(y)) \in 3V. \end{aligned}$$

Applying Theorem 1 we get that there exist additive functions $a : X \to Y$, $b : \mathbb{R}_0^+ \to Y$ and a constant $k = k(N)$ such that

$$a(x) + b(\|x\|^2) - f_0(x) \in 3k \operatorname{seq\,cl} V, \qquad x \in B_r.$$

Define mappings $f_1, g_1, h_1 : X \to Y$ as follows

$$\begin{aligned} f_1 &:= a(x) + b(\|x\|^2) + g(0) + h(0), \\ g_1 &:= a(x) + b(\|x\|^2) + g(0), \\ h_1 &:= a(x) + b(\|x\|^2) + h(0). \end{aligned}$$

Such functions satisfy all conditions in the assertion of Theorem 3.

THEOREM 4. *Let $(X, (\cdot|\cdot))$ be a real inner product space, $\dim X = N$ ($N \geq 2$) and let B_r ($r > 0$) denote an open ball in X centered at the origin and with radius r. Given an $\varepsilon \in (0,1)$ and a mapping $f : B_r \to \mathbb{C} \setminus \{0\}$ such that*

$$(29) \qquad (x, y, x+y \in B_r,\ x \perp y) \quad \textit{implies} \quad \left| \frac{f(x+y)}{f(x)f(y)} - 1 \right| \leq \varepsilon,$$

there exist an orthogonally exponential mapping $g : B_r \to \mathbb{R} \setminus \{0\}$:

$$(x, y, x+y \in B_r,\ x \perp y) \quad \textit{implies} \quad g(x+y) = g(x)g(y)$$

and a constant $k = k(N)$ such that

$$\left| \frac{f(x)}{g(x)} - 1 \right| \leq \delta \quad \textit{and} \quad \left| \frac{g(x)}{f(x)} - 1 \right| \leq \delta$$

for all $x \in B_r$, where $\delta = \left(\frac{1}{1-\varepsilon}\right)^k + 1$.

Proof. Define $\varphi : B_r \to \mathbb{R}_0^+$ as $\varphi := |f|$. Then for all $x, y \in B_r$ such that $x + y \in B_r$ and $x \perp y$, from (29), we have

$$1 - \varepsilon \leq \frac{\varphi(x+y)}{\varphi(x)\varphi(y)} \leq 1 + \varepsilon.$$

Hence we get

$$(x, y, x+y \in B_r,\ x \perp y) \implies |\ln\varphi(x+y) - \ln\varphi(x) - \ln\varphi(y)| \le \ln\frac{1}{1-\varepsilon}.$$

Applying now Theorem 1 for $Y := \mathbb{R}$, $V := \left\{x \in \mathbb{R} : |x| \le \ln\frac{1}{1-\varepsilon}\right\}$ and function $\ln\circ\varphi$ we obtain the existence of additive functions $a : X \to \mathbb{R}$, $b : \mathbb{R}_0^+ \to \mathbb{R}$ and a constant $k = k(N)$ such that

$$|\ln\varphi(x) - a(x) - b(\|x\|^2)| \le k\ln\frac{1}{1-\varepsilon}, \qquad x \in B_r.$$

Define $g : X \to \mathbb{R}$ by

$$g(x) := \exp(a(x) + b(\|x\|^2)), \qquad x \in X.$$

Then

$$\left|\ln\frac{\varphi(x)}{g(x)}\right| \le k\ln\frac{1}{1-\varepsilon}, \qquad x \in B_r,$$

whence

$$(1-\varepsilon)^k \le \frac{\varphi(x)}{g(x)} \le \left(\frac{1}{1-\varepsilon}\right)^k, \qquad x \in B_r.$$

As a consequence we have

$$\left|\frac{f(x)}{g(x)} - 1\right| \le \left|\frac{f(x)}{g(x)}\right| + 1 \le \left(\frac{1}{1-\varepsilon}\right)^k + 1,$$

for all $x \in B_r$. Similarly we get the second inequality. This ends the proof of the theorem.

References

[1] R. Ger, *Superstability is not natural*, Rocznik Naukowo-Dydaktyczny WSP w Krakowie, Prace Mat. 159 (1993), 109–123.

[2] R. Ger, J. Sikorska, *Stability of the orthogonal additivity*, Bull. Polish Acad. Sci. Math. 43, No.2 (1995), 143–151.

[3] Z. Kominek, *On a local stability of the Jensen functional equation*, Demonstratio Math. 22, No.2 (1989), 499–507.

[4] K. Nikodem, *The stability of the Pexider equation*, Ann. Math. Sil. 5 (1991), 91–93.

[5] J. Rätz, *On approximately additive mappings*, General Inequalities 2, Internat. Ser. Numer. Math. 47, Birkhäuser Verlag, Basel, 1980, 233–251.

[6] J. Rätz, *On orthogonally additive mappings*, Aequationes Math. 28 (1985), 35–49.

[7] F. Skof, *Proprietá locali e approssimazione di operatori*, Rend. Semin. Mat. Fis. Milano 53 (1983), 113–129.

[8] F. Skof, *Sull'approssimazione delle applicazioni localmente δ-additive*, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. 117 (1983), 377–389.
[9] F. Skof, S. Terracini, *Sulla stabilità dell'equazione funzionale quadratica su un dominio ristretto*, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. 121 (1987), 153–167.

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