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NONLINEAR SINGULAR DIFFERENCE INEQUALITIES SUITABLE FOR DISCRETIZATIONS OF PARABOLIC EQUATIONS

Abstract. In this paper we solve a nonlinear, singular difference inequality which is a discrete version of generalized integral inequalities of Henry-Gronwall type and their Bihari nonlinear version.

1. Introduction

Many problems in the theory of parabolic partial differential equations can be written as a Cauchy initial value problem

$$\frac{du}{dt} + Au = f(t,u),\ u \in X,\ \ u(0) = u_0, \tag{1}$$

where X is an appropriate Banach space and $A : X \to X$ is a linear sectorial operator. In the theory of such problems developed by D. Henry in his book [3] an important role is played by inequalities of the form

$$u(t) \leqq a(t) + \int_0^t (t-s)^{\beta-1} F(s)\omega(u(s))ds, \tag{2}$$

where $0 < \beta < 1$. The case $\beta = 1$, a, F, u continuous, nonnegative, ω linear is covered by the Gronwall lemma and the case $\beta = 1, \omega$ continuous, nonnegative, nonlinear is covered by the Bihari result published in [2] and its generalizations (see [4]). The case $0 < \beta < 1$ and ω linear is solved by D. Henry [3]. In the paper [6] a new method for solving the case $0 < \beta < 1$ and ω nonlinear is developed. This method is also applied in the paper [7] in the proof of a stability theorem for a class of initial value problems of type (1)

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and in the proofs of results on nonlinear singular integral inequalities in two and n independent variables published in [8].

The numerical analysis of the abstract Cauchy initial value problem (1) provides strong motivation for the study of a discrete analogue of the inequality (2). In connection with an error estimate for the discretization in time

$$(x_i - x_{i-1})\tau^{-1} + Ax_i = f(t_i, x_{i-1}),\ x_{n|n=0} = x_0 \tag{3}$$

($i \in N, \tau$ is a time step, $t_i = i\tau$; A is a sectorial operator in a Banach space $X, x_i \in X_\beta = D(A^\beta), 0 < \beta < 1$; see [3]) of the equation (1), the linear inequality

$$u_n \leqq a_n + L\sum_{k=1}^{n-1}(t_n - t_k)^{\beta-1}u_k\tau \tag{4}$$

($\{a_n\}_{n=1}^{\infty}, \{u_n\}_{n=1}^{\infty}$ are sequences of nonnegative real numbers, $L > 0, 0 < \beta < 1$) is solved by M. Slodička in the paper [12]. By an iteration argument, applied also by D. Henry [3] in the proof of his result on linear singular integral inequality, it is proven in [12] that if $0 < \tau < 1, 0 < \beta < 1, L > 0$, $t_n = n\tau$ and (4) is satisfied, then

$$u_n \leqq L[a_n + \sum_{k=1}^{n-1}(t_n - t_k)^{\beta-1}a_k\tau + \sum_{k=1}^{n-1}a_k\tau], \quad n \geqq 1. \tag{5}$$

In the paper [5] linear inequalities of type (4) are also solved.

In this paper we apply our method developed in [6] for solving nonlinear singular difference inequalities. In the case of the inequality (4) we obtain an exponential estimate for u_n.

2. Discrete inequalities

In the paper [6] we have defined a special class of nonlinear functions and proposed a new method of solving nonlinear integral inequalities with singular kernels and nonlinearity of that class. Let us recall the definition of this class of functions.

Definition 1. Let $q > 0$ and $0 < T \leqq \infty$. We say that a function $\omega : R^+ \to R$ ($R^+ = \langle 0, \infty)$) satisfies a condition (q), if

$$e^{-qt}[\omega(u)]^q \leqq R(t)\omega(e^{-qt}u^q) \quad \text{for all } u \in R^+,\ t \in \langle 0, T), \tag{q}$$

where $R(t)$ is a continuous, nonnegative function and $0 < T \leqq \infty$.

Remark. If $\omega(u) = u^m$, $m > 0$ then

$$e^{-qt}[\omega(u)^q] = e^{(m-1)qt}\omega(e^{-qt}u^q) \tag{1}$$

for any $q > 1$, i. e. the condition (q) is satisfied with $R(t) = e^{(m-1)qt}$.

For $\omega(u) = u + au^m$, where $0 \leqq a \leqq 1$, $m \geqq 1$ the function ω satisfies the condition (q) with $q > 1$ and $R(t) = 2^{q-1}e^{qmt}$ (see [6]).

Let us recall a discrete analogue of Gronwall–Bihari theorem.

LEMMA 1 ([4, Theorem 3.3]). *Let $\omega : R^+ \to R$ be a continuous, nondecreasing function with $\omega(y) > 0$ for $y > 0$, $c > 0$ and $\{y_n\}_{n=0}^{\infty}$ be a sequence of nonnegative numbers satisfying the inequality*

$$y_n \leqq c + \sum_{k=0}^{n-1} b_k \omega(y_k), \quad n \geqq 0, \tag{2}$$

where $0 \leqq y_0 \leqq c$ and $\{b_n\}_{n=0}^{\infty}$ is a sequence of nonnegative numbers. Then

$$y_n \leqq \Omega^{-1}[\Omega(c) + \sum_{k=0}^{n-1} b_k], \quad 1 \leqq n \leqq N_0, \tag{6}$$

where

$$N_0 = \sup\{i \,|\, \Omega(c) + \sum_{k=0}^{i-1} b_k \in \Omega(R^+)\}, \Omega(v) = \int_{v_0}^{v} \frac{du}{\omega(u)}, \quad v \geqq v_0 > 0,$$

v_0 is a constant.

If $\Omega(\infty) \neq \infty$ then we also assume that

$$\sum_{k=0}^{i-1} b_k < \Omega(\infty) \quad \text{for } i = 1, 2, \ldots .$$

REMARK. The inequality (6) is one of discrete analogues of the well-known Gronwall - Bihari inequality. Inequlities of such kind can be found e. g. in the monograph [1] by R. P. Agarwall and in the papers [9]–[11] by B. G. Pachpatte.

COROLLARY.

1. If $\omega(u) = u$ then (6) yields the inequality

$$y_n \leqq c \exp \sum_{k=0}^{n-1} b_k. \tag{7}$$

2. If $\omega(u) = u^m$, where $m > 1$, then

$$y_n \leqq c[1 - (m-1)c^{m-1} \sum_{k=0}^{n-1} b_k]^{-\frac{1}{m-1}}, \quad 0 \leqq n \leqq N_0, \tag{8}$$

where

$$N_0 = \sup\{i \,|\, (m-1)c^{m-1} \sum_{k=0}^{n-1} b_k < 1\}.$$

Applying the method developed in the paper [6] for solving nonlinear integral inequalities with weakly singular kernels and using Lemma 1 we shall prove the following theorem.

THEOREM 2. *Let* $c > 0, d > 0, \{a_n\}_{n=0}^{\infty}, \{F_n\}_{n=0}^{\infty}$ *be sequences of nonnegative numbers,* $\{t_n\}_{n=0}^{\infty}$ *be an increasing sequence of positive numbers,* $\omega : R^+ \to R$ *be a continuous, nondecreasing function,* $\omega(y) > 0$ *for* $y > 0$ *and* $\{u_n\}_{n=0}^{\infty}$ *be a sequence of nonnegative numbers with*

$$u_n \leqq a_n + \sum_{k=0}^{n-1}(t_n - t_k)^{\beta-1}\tau_k F_k \omega(u_k), \quad n \geqq 0, \tag{9}$$

where $\beta > 0, \tau_k = t_{k+1} - t_k$ *with* $\tau = \sup_{k\geqq 0} \tau_k < \infty$. *Then the following assertions hold:*

(i) *Suppose* $\frac{1}{2} < \beta < 1, a_n^2 e^{-2\tau t_n} \leqq c$ *for all* $n \geqq 0$ *and* ω *satisfies the condition* (q) *with* $q = 2$. *Then*

$$u_n \leqq e^{\tau t_n}\{\Omega^{-1}[\Omega(2c) + \tau^{2(1-\beta)}B\sum_{k=0}^{n-1} F_k^2 R(\tau t_k)]\}^{\frac{1}{2}}, \quad 1 \leqq n \leqq N_0, \tag{10}$$

where Ω *is as in Lemma 1,*

$$N_0 = \sup\{i |\, \Omega(2c) + \tau^{2(1-\beta)}B\sum_{k=0}^{i-1} F_k^2 R(\tau t_k) \in \Omega(R^+)\},$$

$B = \frac{1}{4^{\beta-1}}\Gamma(2\beta - 1)$, Γ *is the Eulerian Gamma function. If* $\Omega(\infty) \neq \infty$ *then we assume*

$$\tau^{2(1-\beta)}B\sum_{k=0}^{n-1} F_k^2 R(\tau t_k) < \Omega(\infty), \quad n = 1, 2, \dots$$

(ii) *Let* $0 < \beta = \frac{1}{1+z}$, $z \geqq 1$, ω *satisfies the condition* (q) *with* $q = z + 2, p = \frac{z+2}{z+1}$, *i. e.* $\frac{1}{p} + \frac{1}{q} = 1$, $a_n^q e^{-q\tau t_n} \leqq d$ *for all* $n \geqq 0$. *Then*

$$u_n \leqq e^{\tau t_n}\{\Omega^{-1}[\Omega(2^{q-1}d) + \tau^{\kappa}G\sum_{k=0}^{n-1} F^q R(\tau t_k)\}^{\frac{1}{q}}, \quad 1 \leqq n \leqq n_0, \tag{11}$$

where

$$n_0 = \sup\{i |\, \Omega(2^{q-1}d) + \tau^{\kappa}G\sum_{k=0}^{i-1} F_k^q R(\tau t_k) \in \Omega(R^+)\},$$

$G = 2^{q-1}[\frac{\Gamma(1-\alpha p)}{p^{1-\alpha p}}]^{\frac{1}{p}}$, $\alpha = 1 - \beta = \frac{z}{1+z}$, $\kappa = 1 - (1 - \alpha p)^{\frac{q}{p}} = 1 - \frac{1}{(1+z)(2+z)^2} > 0$.

If $\Omega(\infty) \neq \infty$, *then we assume*

$$\tau^{\kappa} G \sum_{k=0}^{n-1} F_k^q < \Omega(\infty), \quad n = 1, 2, \ldots$$

Proof. First we shall prove the assertion (i). Using the Cauchy–Schwarz inequality and the condition (q) we obtain from (9)

$$\begin{aligned} u_n &\leqq a_n + \tau^{\frac{1}{2}} \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k^{\frac{1}{2}} e^{\tau t_k} e^{-\tau t_k} F_k \omega(u_k) \\ &\leqq a_n + \tau^{\frac{1}{2}} \Big[\sum_{k=0}^{n-1} (t_n - t_k)^{2\beta-2} \tau_k e^{2\tau t_k} \Big]^{\frac{1}{2}} \Big[\sum_{k=0}^{n-1} F_k^2 e^{-2\tau t_k} \omega(u_k)^2 \Big]^{\frac{1}{2}} \\ &\leqq a_n + \tau^{\frac{1}{2}} \Big[\sum_{k=0}^{n-1} (t_n - t_k)^{2\beta-2} \tau_k e^{2\tau t_k} \Big]^{\frac{1}{2}} \Big[\sum_{k=0}^{n-1} F_k^2 R(\tau t_k) \omega(e^{-2\tau t_k} u_k^2) \Big]^{\frac{1}{2}}. \end{aligned}$$

Let us estimate the first sum

$$\begin{aligned} \sum_{k=0}^{n-1} (t_n - t_k)^{2\beta-2} \tau_k e^{2\tau t_k} &\leqq \int_0^{t_n} (t_n - s)^{2\beta-2} e^{2\tau s}\, ds = e^{2\tau t_n} \int_0^{t_n} \eta^{2\beta-2} e^{-2\tau\eta}\, d\eta \\ &= \frac{e^{2\tau t_n}}{(2\tau)^{2\beta-1}} \int_0^{2\tau t_n} \sigma^{2\beta-2} e^{-\sigma}\, d\sigma \leqq \frac{e^{2\tau t_n}}{(2\tau)^{2\beta-1}} \Gamma(2\beta-1) \end{aligned}$$

and thus

$$u_n \leqq a_n + \Big[\frac{e^{2\tau t_n}}{2^{2\beta-1}} \tau^{2(1-\beta)} \Gamma(2\beta - 1) \Big]^{\frac{1}{2}} \Big[\sum_{k=0}^{n-1} F_k^2 R(\tau t_k) \omega(e^{-2\tau t_k} u_k^2) \Big]^{\frac{1}{2}}. \tag{12}$$

Since the elementary inequality $(a+b)^2 \leqq 2(a^2 + b^2)$ is satisfied for any $a \geqq 0, b \geqq 0$ we obtain from (12):

$$u_n^2 \leqq 2[a_n^2 + \frac{e^{2\tau t_n}}{2^{2\beta-1}} \tau^{2(1-\beta)} \Gamma(2\beta - 1) \sum_{k=0}^{n-1} F_k^2 R(\tau t_k) \omega(e^{-2\tau t_k} u_k^2)], \quad n \geqq 0,$$

and this yields

$$v_n \leqq 2c + \tau^{2(1-\beta)} \frac{\Gamma(2\beta - 1)}{4^{\beta-1}} \sum_{k=0}^{n-1} F_k^2 R(\tau t_k) \omega(v_k), \quad n \geqq 0, \tag{13}$$

where

$$v_n = u_n^2 e^{-2\tau t_n}, \quad n \geqq 0. \tag{14}$$

From Lemma 1 we obtain the inequality

$$v_n \leqq \Omega^{-1}\left[\Omega(2c) + \tau^{2(1-\beta)}\frac{\Gamma(2\beta-1)}{4^{\beta-1}}\sum_{k=0}^{n-1} F_k^2 R(\tau t_k)\right], \quad n \geqq 1 \tag{15}$$

and from (14) we obtain the inequality (10).

Now we shall prove the assertion (ii). Let p, q be as in theorem. Applying Hölder inequality and using the condition (q) we obtain

$$\begin{aligned} u_n &\leqq a_n + \tau^{\frac{1}{q}}\sum_{k=0}^{n-1}(t_n - t_k)^{-\alpha}\tau_k^{\frac{1}{p}} e^{\tau t_k} e^{-\tau t_k} F_k \omega(u_k) \\ &\leqq a_n + \tau^{\frac{1}{q}}\left[\sum_{k=0}^{n-1}(t_n - t_k)^{-\alpha p}\tau_k e^{p\tau t_k}\right]^{\frac{1}{p}}\left[\sum_{k=0}^{n-1} F_k^q e^{-q\tau t_k}\omega(u_k)^q\right]^{\frac{1}{q}} \\ &\leqq a_n + \tau^{\frac{1}{q}}\left[\sum_{k=0}^{n-1}(t_n - t_k)^{-\alpha p}\tau_k e^{p\tau t_k}\right]^{\frac{1}{p}}\left[\sum_{k=0}^{n-1} F_k^q R(\tau t_k)\omega(e^{-q\tau t_k}u_k^q)\right]^{\frac{1}{q}}. \end{aligned}$$

The following estimates hold

$$\begin{aligned} \sum_{k=0}^{n-1}(t_n - t_k)^{-\alpha p}\tau_k e^{p\tau t_k} &\leqq \int_0^{t_n}(t_n - s)^{-\alpha p} e^{p\tau s}\, ds = e^{p\tau t_n}\int_0^{t_n}\eta^{-\alpha p}e^{-p\tau\eta}\, d\eta \\ &= \frac{e^{p\tau t_n}}{(p\tau)^{1-\alpha p}}\int_0^{p\tau t_n}\sigma^{-\alpha p}e^{-\sigma}\, d\sigma \leqq \frac{e^{p\tau t_n}}{(p\tau)^{1-\alpha p}}\Gamma(1-\alpha p). \end{aligned}$$

Obviously $1 - \alpha p = \frac{1}{(1+z)^2} > 0$, i. e. $\Gamma(1-\alpha p) \in R$. Therefore we obtain

$$\begin{aligned} u_n &\leqq a_n \\ &+ \tau^{\frac{1}{q}}\left[\frac{e^{p\tau t_n}}{(p\tau)^{1-\alpha p}}\Gamma(1-\alpha p)\right]^{\frac{1}{p}}\left[\sum_{k=0}^{n-1} F_k^q R(\tau t_k)\omega(e^{-q\tau t_k}u_k^q)\right]^{\frac{1}{q}}, \quad n \geqq 0. \end{aligned} \tag{16}$$

Since $(a+b)^q \leqq 2^{q-1}(a^q + b^q)$ for any $a \geqq 0, b \geqq 0$ (see [6]) we obtain from (16):

$$u_n^q \leqq 2^{q-1}\left[a_n^q + e^{q\tau t_n}K^{\frac{q}{p}}\tau^{1+(\alpha p-1)(q/p)}\sum_{k=0}^{n-1} F_k^q R(\tau t_k)\omega(e^{-qt_k}u_k^q)\right], \tag{17}$$

where $K = \frac{\Gamma(1-\alpha p)}{p^{1-\alpha p}}$. This inequality yields

$$v_n \leqq 2^{q-1}d + \tau^{\kappa}G\sum_{k=0}^{n-1} F_k^q R(\tau t_k)\omega(v_k), \tag{18}$$

where G, d, κ are as in the assertion (ii) and

$$v_n = u_n^q e^{-q\tau t_n}. \tag{19}$$

From Lemma 1 we obtain

$$v_n \leqq \Omega^{-1}\Big[\Omega(2^{q-1}d) + \tau^\kappa G \sum_{k=0}^{n-1} F_k^q R(\tau t_k)\Big], \quad 1 \leqq n \leqq n_0$$

and from (19) we have the inequality (11).

As a consequence of Theorem 2 we obtain a discrete version of [6, Theorem 2].

THEOREM 3. *Let $c > 0, d > 0, \{a_n\}_{n=0}^\infty, \{F_n\}_{n=0}^\infty, \{t_n\}_{n=0}^\infty, \{\tau_n\}_{n=0}^\infty$ and τ be as in Theorem 2 and $\{u_n\}_{n=0}^\infty$ be an increasing sequence of positive numbers and $\{u_n\}_{n=0}^\infty$ be a sequence of nonnegative numbers with*

$$u_n \leqq a_n + \sum_{k=0}^{n-1}(t_n - t_k)^{\beta-1}\tau_k^{\frac{1}{\tau}} F_k u_k, \quad n \geqq 1, \tag{20}$$

where $\beta > 0$. Then the following assertions hold:

(i) *If $\frac{1}{2} < \beta < 1, a_n^2 e^{-2\tau t_n} \leqq c$ for all $n \geqq 0$, then*

$$u_n \leqq \sqrt{2c}\exp\left(\tau t_n + \frac{1}{2}\tau^{2(1-\beta)}B\sum_{k=0}^{n-1}F_k^2\right), \quad n \geqq 1, \tag{21}$$

where $B = \frac{1}{4^{\beta-1}}\Gamma(2\beta-1)$.

(ii) *If $0<\beta = \frac{1}{1+z}, z \geqq 1, q = z+2, p = \frac{z+2}{z+1}$, i.e. $\frac{1}{p}+\frac{1}{q} = 1$, $a_n^q e^{-q\tau t_n} \leqq d$ for all $n \geqq 0$, then*

$$u_n \leqq (2^{q-1}d)^{\frac{1}{q}} \exp\left(\tau t_n + \frac{1}{q}\tau^\kappa G\sum_{k=0}^{n-1}F_k^q\right), \quad n \geqq 1, \tag{22}$$

where G, κ are as in Theorem 2.

REMARK. Obviously, if the sequence $\{a_n\}_{n=0}^\infty$ of nonnegative numbers from (4) satisfies the additional condition $a_n^2 e^{-\tau t_n} \leqq c$ for all $n \geqq 0$ and $\frac{1}{2} \leqq \beta < 1$, then instead of the Slodička's estimate (5) we obtain the exponential one

$$u_n \leqq \sqrt{2c}\exp(\tau t_n + \frac{1}{2}\tau^\kappa BnL^q), \quad n \geqq 0,$$

where B is as in the assertion (i) of Theorem 2. If $\beta = \frac{1}{1+z}$, $z \geqq 1, q, d, \kappa, G$ are as in the assertion (ii) of Theorem 2 and $a_n^q e^{-q\tau t_n} \leqq d$ for $n \geqq 0$, then

$$u_n \leqq (2^{q-1}d)^{\frac{1}{q}} \exp\left(\tau t_n + \frac{1}{q}\tau^\kappa GnL^q\right), \quad n \geqq 0.$$

As a consequence of Lemma 1 and Theorem 2 we obtain the following theorem.

THEOREM 4. *Let* $c > 0, d > 0, \{a_n\}_{n=0}^{\infty}, \{F_n\}_{n=0}^{\infty}, \{\tau_n\}_{n=0}^{\infty}, \tau$ *be as in Theorem* 2, $m > 1$ *and* $\{u_n\}_{n=0}^{\infty}$ *be a sequence of nonnegative numbers with*

$$u_n \leqq a_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k F_k u_k^m, \quad n \geqq 1, \tag{23}$$

where $\beta > 0$. *Then the following assertions hold:*

(i) *If* $\frac{1}{2} < \beta < 1, a_n^2 e^{-2\tau t_n} \leqq c$ *for all* $n \geqq 0$, *then*

$$u_n \leqq e^{\tau t_n} \sqrt{2c} \left[1 - (m-1)(2c)^{m-1} \tau^{2(1-\beta)} B \sum_{k=0}^{n-1} F_k^2 e^{2(m-1)\tau t_k} \right]^{-\frac{1}{2(m-1)}}, \quad 1 \leqq n \leqq N_0, \tag{24}$$

where B, κ *are as in the assertion* (i) *of Theorem* 2,

$$N_0 = \sup \left\{ i \,|\, (m-1)(2c)^{m-1} \tau^{2(1-\beta)} \sum_{k=0}^{i-1} F_k^2 e^{2(m-1)\tau t_k} < 1 \right\}.$$

(ii) *If* $0 < \beta = \frac{1}{1+z}$, $z \geqq 1$, $q = z + 2, p = \frac{z+2}{z+1}$, $a_n^q e^{-q\tau t_n} \leqq d$, *then*

$$u_n \leqq e^{\tau t_n} (2^{q-1} d)^{\frac{1}{q}} \times \left[1 - (m-1)(2^{q-1} d)^{m-1} \tau^{\kappa} G \sum_{k=0}^{n-1} F_k^q e^{2(m-1)\tau t_k} \right]^{-\frac{1}{q(m-1)}}, \tag{25}$$

where G *is as in the assertion* (ii) *of Theorem* 2, $1 \leqq n \leqq n_0$,

$$n_0 = \sup \left\{ i \,|\, (m-1)(2^{q-1} d)^{m-1} \tau^{\kappa} G \sum_{k=0}^{i-1} F_k^q e^{2(m-1)\tau t_k} < 1 \right\}.$$

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