DEMONSTRATIO MATHEMATICA
Vol. XXXIII No 3 2000

Adam Piskorek, Arkadiusz Szymaniec

LOCAL IN TIME EXISTENCE OF SOLUTION TO THE DIRICHLET INITIAL-BOUNDARY VALUE PROBLEM IN NONLINEAR HYPOELASTICITY

Dedicated to the memory of Witold Pogorzelski

0. Introduction

The basic equations of motion governing a standard linear constitutive model in hypoelasticity (cf. Piskorek, (1994) equations (0.1) and (1.1), pp. 549-550) are given by

$$\begin{cases} \partial_t \rho = -v\partial_x \rho - \rho \partial_x v, \\ \partial_t v = -v\partial_x v + \dfrac{1}{\rho}\partial_x \sigma, \\ \partial_t \sigma = -v\partial_x \sigma + \kappa \partial_x v, \end{cases} \tag{0.1}$$

where ρ denotes the density of the hypoelastic medium, v- the velocity field of its motion, σ-the stress tensor field, and κ is the physical constant.

In this paper we study the existence, uniquenes and regularity for solutions of the Dirichlet initial-boundary value problem to this system of partial differential equations (p.d.e.).

The arrangement of the paper is as follows. In Section 1 we formulate the initial-boundary value problem to the quasilinear system (0.1) of p.d.e. in one-dimensional case and examine this system. This system is strictly hyperbolic and diagonalizable. In Section 2 we prove existence of C^r — smooth solution $r \geq 1$ to the related linearized system of p.d.e. using classical iterative method due to Courant (1961) and (1962). The Section 3 is devoted to the construction of solution (local in time) of the initial-boundary value problem to the quasilinear system (0.1) of p.d.e. as the limit of sequences of solutions of linearized initial value problems (the contraction mapping principle). In this manner we prove local in time well posedness of the initial-boundary value problem in the space* of C^r — smooth and bounded functions with bounded derivatives up to order r in $\bar{R}_+ \times \bar{R}_+$ which we denote by $C_b^r(\bar{R}_+ \times \bar{R}_+)$, where $\bar{R}_+ = \{t \in R \ : \ t \geq 0\}$.

1. Formulation of the problem

We consider the quasilinear system of p.d.e. in the first quadrant of (t, x) — space, i.e. in $\bar{R}_+ \times \bar{R}_+$ for three unknown functions ρ, v, σ

$$\begin{cases} \partial_t \rho = -v\partial_x \rho - \rho \partial_x v, \\ \partial_t v = -v \partial_x v + \dfrac{1}{\rho}\partial_x \sigma, \\ \partial_t \sigma = -v\partial_x \sigma + \kappa \partial_x v. \end{cases} \tag{1.1}$$

The quasilinear system (1.1) of p.d.e. describes the motion of the one-dimensional linear hypoelastic body, which occupies half axis $x > 0$ and for which ρ — denotes the mass density, v-the velocity and σ-the stress. Of course the mass density is positive functions, more precisely

$$\inf_{(t,x)\in \bar{R}_+ \times \bar{R}_+} \rho(t,x) > 0. \tag{1.2}$$

We denote by **

$$V = (V^1, V^2, V^3)^* := (\rho, v, \sigma)^*, \tag{1.3}$$

$$A(V) = \begin{pmatrix} -V^2 & -V^1 & 0 \\ 0 & -V^2 & \frac{1}{V^1} \\ 0 & \kappa & -V^2 \end{pmatrix} := \begin{pmatrix} -v & -\rho & 0 \\ 0 & -v & \frac{1}{\rho} \\ 0 & \kappa & -v \end{pmatrix} \tag{1.4}$$

and rewrite the system (1.1) of p.d.e. in the form

$$\partial_t V = A(V)\partial_x V. \tag{1.5}$$

The characteristic matrix $A(V)$ of this system of p.d.e. has three distinct real eigenvalues

$$\lambda_1 = -\left(v + \sqrt{\frac{\kappa}{\rho}}\right) \quad \lambda_2 = -v, \quad \lambda_3 = -\left(v - \sqrt{\frac{\kappa}{\rho}}\right) \tag{1.6}$$

---

* This space endowed with a norm

$$\|u\|_{C^r} = \sup\{ \sum_{l+k \leq r} |(\partial_t^l \partial_x^k u)(t,x)| : (t,x) \in \bar{R}_+ \times \bar{R}_+\}, \quad r \geq 1$$

is a Banach space. $C_b^0(\bar{R}_+ \times \bar{R}_+)$ denotes the space continuous and bounded function in $\bar{R}_+ \times \bar{R}_+$ where:

$$\partial_t = \frac{\partial}{\partial t}, \qquad \partial_x = \frac{\partial}{\partial x}.$$

** ( )* denotes transpose of a ( )

with property

$$\lambda_3 - \lambda_2 = \lambda_2 - \lambda_1 = \sqrt{\frac{\kappa}{\rho}} > 0$$

it means that the quasilinear system of p.d.e. (1.1) or (1.5) is strictly hyperbolic and diagonalizable (cf. Mizohata (1977), Chap. 6, §3, p. 324).

In the quadrant $\bar{R}_+ \times \bar{R}_+$ we seek a solution of (1.1) or (1.5) which satisfies the initial condition

$$V(0,x) = V_0(x) := (\rho_0, v_0, \sigma_0)^*(x) \qquad x \in R_+ \tag{1.7}$$

together with boundary condition

$$(BV)(t,0) = V_b(t) = (\rho_b, v_b, \sigma_b)(t) \qquad t \in R_+ \tag{1.8}$$

and we assume that $\rho_0, v_0, \sigma_0, \rho_b, v_b, \sigma_b \in C_b^r(\bar{R}_+)$ $r \geq 1$

The conditions (1.7), (1.8) should establish a well posed problem, here we assume the non-characteristic condition

$$\det A(V) = v\left(\frac{\kappa}{\rho} - v^2\right) \neq 0 \tag{1.9}$$

on the boundary $x = 0$, $t > 0$, i.e.

$$0 \neq v \neq \pm\sqrt{\frac{\kappa}{\rho}} \qquad x = 0, t > 0 \tag{1.10}$$

and the eigenvalues (1.6) of $A(V)$ split into two groups $\{\lambda_j\}_{j=1,\ldots,l}$, and $\{\lambda_j\}_{j=l+1,\ldots,3}$ satisfying

$$\lambda_1 \leq \ldots \leq \lambda_l < 0 < \lambda_{l+1} \leq \ldots \leq \lambda_3. \tag{1.11}$$

By this assumption $l(= 1,2,3)$ is the number of characteristics throughout the origin 0 which point upward in to the first quadrant of the (t,x)-plane. Then the first characteristics through a point $t > 0$ near to 0 also point into the first quadrant. The characteristic C_l through 0 separates the quadrant adjacent to 0 in two regions such that all l characteristics drawn from a point (t,x) in the region left from C_l towards decreasing t intersect the positive t-axis, prowided we restrict ourselfs to x sufficiently small region adjacent to 0.

Thus, the boundary condition (1.8) admit a following form

$$BV = V_- - MV_+ \tag{1.12}$$

where V_- denotes the projection of V onto the negative l-dimensionale eigenspace of $A(V)$, and V_+ — the projection of V onto the positive $(3-r)$ — dimensional eigenspace of $A(V)$, M is given matrix $l \times (3-l)$ with $\|M\| < 1$. Without loss of generality (cf. Courant, (1962), p. 472, (8)–(10)) we can assume that $M = 0$.

REMARK 1.1. Along the characteristics through 0 the solution will have discontinuities unless "consistency conditions" at 0 for the data are stipulated for $t, x = 0$. First order consistency, i.e., continuity of V depends on the conditions

$$g^j(0) = V_0^j(0) \qquad j = 1, \dots, l.$$

Similar conditions are obtained by differentiation for consistency derivatives to the r — order.

In order to obtain existence results for the related initial value problem (1.1)–(1.2) or (1.8)–(1.9) in next section we consider the related initial value problem to the linearized system of p.d.e..

2. Solution of linearized system of p.d.e.

We consider the related linearized system of p.d.e.

$$\partial_t U = A(W)\partial_x U \tag{2.1}$$

where U is an unknown vector function with the components U^1, U^2, U^3, i.e. $U = (U^1, U^2, U^3)^*$. Here

$$W = (W^1, W^2, W^3)^* = (\pi, \omega, \tau)^* \tag{2.2}$$

is given $C_b^r\left(\bar{R}_+ \times \bar{R}_+\right)$ vector function, which satisfies the condition (2.3)

$$\inf_{t,x} W^1(t,x) \equiv \inf_{t,x} \pi(t,x) > 0 \tag{2.3}$$

and matrix $A(W)$ has (cf. formula (1.4)) following form

$$A(W) = \begin{pmatrix} -W^2 & -W^1 & 0 \\ 0 & -W^2 & \frac{1}{W^1} \\ 0 & \kappa & -W^2 \end{pmatrix} := \begin{pmatrix} -\omega & -\pi & 0 \\ 0 & -\omega & \frac{1}{\pi} \\ 0 & \kappa & -\omega \end{pmatrix}. \tag{2.4}$$

To the system (2.1) of p.d.e. we add initial data

$$U(0,x) = V_0(x) = (\rho_0, v_0, \sigma_0)^*(x) \tag{2.5}$$

and boundary data

$$V_b(t) = (\rho_b, v_b, \sigma_b)(t). \tag{2.6}$$

The linear system (2.1) of p.d.e. is strictly hyperbolic, since its characteristic matrix $A(W)$ has three different real eigenvalues

$$\Lambda_1 = -\left(\omega + \sqrt{\frac{\kappa}{\pi}}\right), \quad \Lambda_2 = -\omega, \quad \Lambda_3 = -\left(\omega - \sqrt{\frac{\kappa}{\pi}}\right) \tag{2.7}$$

with (cf. (1.6)) property $\Lambda_3 - \Lambda_2 = \Lambda_2 - \Lambda_1 = \sqrt{\frac{\kappa}{\pi}} > 0$, and we assume (cf.(1.9)) $\det \mathrm{A}(W) \neq 0$.

System (2.1) of p.d.e. with initial-boundary condition (2.5)–(2.6) is the initial-boundary value problem for which it is desired to prove the existence

C^r-smooth solution. In order to be able to do this by applying the iterative method due to Courant (see Courant (1962), Chapt V, § 6, pp. 461–474) the system (21) of p.d.e. must be converted to a first order diagonal system of p.d.e.

With the aid (cf. Mizohata, (1977), Chapt 6, § 3, pp. 324–326; § 9, pp. 356–362) of the eigencovectors of the characteristic matrix $A(W)$, which have following form

$$\begin{aligned} l_1 &= \left(0, \sqrt{\frac{\kappa}{\pi}}, -1\right) \\ l_2 &= (-\kappa, 0, -\pi) \\ l_3 &= \left(0, -\sqrt{\frac{\kappa}{\pi}}, -1\right) \end{aligned} \tag{2.8}$$

we construct the matrix

$$N(W) = \begin{pmatrix} 0 & \sqrt{\frac{\kappa}{\pi}} & -1 \\ -\kappa & 0 & -\pi \\ 0 & -\sqrt{\frac{\kappa}{\pi}} & -1 \end{pmatrix} \tag{2.9}$$

with property

$$N(W)A(W), N^{-1}(W) = \begin{pmatrix} -\left(\omega + \sqrt{\frac{\kappa}{\pi}}\right) & 0 & 0 \\ 0 & -\omega & 0 \\ 0 & 0 & -\left(\omega + \sqrt{\frac{\kappa}{\pi}}\right) \end{pmatrix}. \tag{2.10}$$

Setting (see Mizohata (1977), Chapt 6, § 3, p. 325)

$$u = N(W)U \tag{2.11}$$

in the system (2.1) of p.d.e., after simple calculation, we obtain the diagonal form of this system with new unknown vector function $u = (u^1, u^2, u^3)^*$

$$\begin{aligned} \partial_t u^1 = &- \left(\omega + \sqrt{\frac{\kappa}{\pi}}\right)(t,x)\partial_x u^1 + \frac{1}{4\pi(t,x)} \\ &\times \left[\partial_t \pi + \left(\omega + \sqrt{\frac{\kappa}{\pi}}\right)\partial_x \pi\right](t,x)(u^1 + u^3), \\ \partial_t u^2 = &- \omega(t,x)\partial_x u^2 + \frac{1}{2}[\partial_t \pi + \omega \partial_x \pi](t,x)(u^1 + u^3), \\ \partial_t u^3 = &- \left(\omega - \sqrt{\frac{\kappa}{\pi}}\right)(t,x)\partial_x u^3 \\ &- \frac{1}{4\pi(t,x)}\left[\partial_t \pi + \left(\omega - \sqrt{\frac{\kappa}{\pi}}\right)\partial_x \pi\right](t,x)(u^1 + u^3) \end{aligned} \tag{2.12}$$

and with new initial and boundary data (2.13)

$$(2.13)\begin{cases} u^1(0)=u_0^1(x):=v_0(x)\sqrt{\dfrac{\kappa}{\rho_0(x)}}-\sigma_0(x), \;\; u_b^1(t)=v_b(t)\sqrt{\dfrac{\kappa}{\rho_b(t)}}-\sigma_b(t),\\ u^2(0)=u_0^2(x):=-\kappa\rho_0(x)-\rho_0(x)\sigma_0(x), \;\; u_b^2(t)=-\kappa\rho_b(t)-\rho_b(t)\sigma_b(t),\\ u^3(0)=u_0^3(x):=-v_0(x)\sqrt{\dfrac{\kappa}{\rho_0(x)}}-\sigma_0(x), \; u_b^3(t)=-v_b(t)\sqrt{\dfrac{\kappa}{\rho_b(t)}}-\sigma_b(t). \end{cases}$$

REMARK 2.1. Our assumptions insure the equivalence of the initial-boundary value problem for U and u.

In order to use iterative method due to Courant (cf. Courant (1962), Chapt. V, § 6, pp. 461–474) we consider in the quadrant (t,x), $t \geq 0$, $x \geq 0$ the characteristic curves C_1, C_2, C_3 of the system (2.12) of p.d.e., which pass through a given point (t,x). They are, of course, the solutions of the initial value problems to the ordinary differential equations

$$(2.14)\qquad \frac{dy}{ds} = -\Lambda_j(s,y), \qquad j=1,2,3, \quad y(t)=x,$$

where Λ_j are given by (2.7). Under our assumptions (2.2) they exist uniquely and have the form

$$(2.15)\qquad y = y_j(s;t,x), \qquad j=1,2,3$$

here the functions y_j possess continuous derivatives of order up to r with respect to s, t and x.

Now we consider in (t,x) quadrant a closed domain G so that the characteristic curves C_1, C_2, C_3 followed from a point (t,x) in G backwards towards decreasing values s, meet a given sections I_1 of the positive x-axis in points $(0, y_1(0;t,x))$, $(0, y_2(0;t,x))$, $(0, y_3(0;t,x))$ respectively, and I_2 of the positive t-axis in point $(0, y_1^{-1}(0;t,x), 0)$, $(0, y_2^{-1}(0;t,x), 0)$, $(0, y_3^{-1}(0;t,x), 0)$ respectively, so that $I_1 \cup I_2$ contains the domain of dependence for all points of G. In order to construed the C_b^r — smooth solution of the initial-boundary problem (2.12)–(213), eo ipso of the initial-boundary problem (2.1)–(2.6), we integrate (2.12), taking into account (2.13), (2.14) and (2.15), along the characteristic curves C_1, C_2, C_3 from $\max(0, y_1^{-1}(0;t,x))$, $\max(0, y_2^{-1}(0;t,x))$, $\max(0, y_3^{-1}(0;t,x))$ to t, and we obtain the following related system of integral equations

$$(2.16)\qquad u^1(t,x) = f^1(t,x) + \int\limits_{\max(0,y_1^{-1}(0;t,x))}^{t} \left[\partial_s \ln \sqrt[4]{\pi(s, y_1(s;t,x))}\right] (u^1+u^3)(s; y_1(s;t,x))\, ds,$$

(2.16)cont.
$$u^2(t,x) = f^2(t,x) + \int\limits_{\max(0,y_2^{-1}(0;t,x))}^{t} \left[\frac{1}{2}\partial_s \pi(s, y_1(s;t,x))\right] (u^1+u^3)(s; y_2(s;t,x))\, ds,$$
$$u^3(t,x) = f^3(t,x) - \int\limits_{\max(0,y_3^{-1}(0;t,x))}^{t} \left[\partial_s \ln \sqrt[4]{\pi(s, y_1(s;t,x))}\right] (u^1+u^3)(s; y_3(s;t,x))\, ds$$

where

$$f^j(t,x) = \begin{cases} u_b^j(y^{-1}(0;t,x)) & \text{for } y_j^{-1}(0;t,x) > 0 \\ u_0^j(y^{-1}(0;t,x)) & \text{for } y_j^{-1}(0;t,x) = 0 \end{cases} \quad j = 1,2,3.$$

Let A be a linear integral operator defined by the right-side of the system (2.16) with the domain D_A which consists all C^r — smooth vector functions v in G i.e.:

$$(Av)(t,x) = f(t,x) + \int_0^t \theta(s,t,x)(\partial_s a(t,s,x))v(s; y(s;t,x))\, ds, \tag{2.17}$$

where $f(t,x) = (f^1(t,x),\ f^2(t,x),\ f^3(t,x))$ and $a(t,s,x)$ denotes (3×3) — matrix of the form

$$a(t,s,x) = \begin{bmatrix} \ln \sqrt[4]{\pi(s,y_1(s,t,x))} & 0 & \ln \sqrt[4]{\pi(s,y_1(s,t,x))} \\ \frac{1}{2}\pi(s,y_2(s;t,x)) & 0 & \frac{1}{2}\pi(s,y_2(s;t,x)) \\ -\ln \sqrt[4]{\pi(s,y_3(s,t,x))} & 0 & -\ln \sqrt[4]{\pi(s,y_3(s,t,x))} \end{bmatrix} \tag{2.18}$$

$v(s,y(s;t,x)) = (v^1(s,y_1(s;t,x)),\ v^2(s,y_2(s;t,x)),\ v^3(s,y_3(s;t,x))$ and $\theta(t,s,x)$ denotes

$$\theta(t,s,x) = \begin{bmatrix} H(s-\max(0,y_1^{-1}(0;t,x))) & 0 & 0 \\ 0 & H(s-\max(0,y_2^{-1}(0;t,x))) & 0 \\ 0 & 0 & H(s-\max(0,y_3^{-1}(0;t,x))) \end{bmatrix}$$
$$H(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0. \end{cases}$$

It is easily seen that the unique solution of the initial-boundary value problem (2.12)–(2.13) is a unique fixed point of the operator A. For suitable narrow strip $G_T = G \cap \{(s,y) : 0 \le s \le T,\ y \in R_+\}$ the desired fixed point will be constructed by iterations as the uniform limit $n \to \infty$ of

$$V_{n+1} = Av_n, \qquad n = 1,2,\ldots \tag{2.19}$$

starting with an arbitrary element $v_1 = f(t,x)$ of D_A. For simplicity we assume that $r = 1$ and in this case we prove the existence results. We set

$$\|v_1\|_{C^0} = R \tag{2.20}$$

and consider in D_A the subset Z defined by

$$Z = \{v \in D_A : \|v_1\|_{C^0} = R_0, \|v\|_1 \leq R_1\} \tag{2.21}$$

where (cf. footnote *), p. 504):

$$\|v_1\|_{C^0} = \sup\{|v(t,x)| : (t,x) \in G_T\}, \tag{2.22}$$

$$|v|_1 = \sup\left\{\left|\frac{\partial v}{\partial t}(t,x)\right| + \left|\frac{\partial v}{\partial x}(t,x)\right| : (t,x) \in G_T\right\}, \tag{2.23}$$

R_0 and R_1 are large enough so that: $R < R_0$ and $R + R_0 < R_1$.

In order to examine the regularity of Av for each $v \in Z$ we remark first that from (214),(215) and regularity of Λ_j, $j = 1, 2, 3$ (cf. (2.2), (2.7) follow (cf. Antontzev, Kazhikhov, Monakhov, (1983), Chap. II, §ż 1, pp. 46–47) after a long but easy calculations the formulae

$$\frac{\partial y}{\partial t}(s;\, t, x) = \Lambda_j(t,x)\exp\Big(-\int_t^s(\partial_x\Lambda_j)(s_1, y_j(s_1; t, x))\text{ṣ}_1\Big), \tag{2.24}$$

$$\frac{\partial y}{\partial t}(s;\, t, x) = \exp\Big(-\int_t^s(\partial_x\Lambda_j)(s_1, y_j(s_1; t, x))\, ds_1\Big). \tag{2.25}$$

With the aid of above formulae we can differentiate with respect to variables t and x a function Av for each $v \in Z$. The first order derivatives $\frac{\partial(Av)}{\partial t}(t,x)$, $\frac{\partial(Av)}{\partial x}(t,x)$ exist and are continuous. In fact, the first term in right-side (2.16) we differentiate using the chain rule and formulate (2.24), (2.25), and with respect to the second one we must combine the chain rule and the integration by parts Finally (cf. (2.17), (2.18), (2.24), (22.5)) we can write

$$\begin{aligned}(2.26)\quad &\frac{\partial(Av)}{\partial t}(t,x)\\ &= \partial_t f(t,x) + (\partial_s a(t,s,x))v(s, y(s;t,x))\big|_{s=t}\\ &\quad - (\partial_s a(t,s,x))v(s,y(s;t,x))\big|_{s=\max(0,y^{-1}(0;t,x))}\frac{\partial y^{-1}}{\partial t}(0;t,x)H(y^{-1}(0;t,x))\\ &\quad + \int_0^t \theta(s,t,x)(\partial_x a(t,s,x))(\partial_x v)(s,y(s;t,x))\frac{\partial y}{\partial t}(s;t,x)\,ds\end{aligned}$$

$$- \int_0^t \theta(s,t,x)(\partial_t a(t,s,x)) \left[(\partial_t v)(s,y(s;t,x)) + (\partial_t v)(s,y(s;t,x)) \frac{\partial y}{\partial t}(s;t,x) \right] ds$$

$$+ (\partial_t a(t,t,x))v(t,x) - (\partial_t a(t,s,x))v(s,y(s;t,x))\big|_{s=\max(0,y^{-1}(0;t,x))},$$

$$\text{(2.27)} \qquad \frac{\partial(Av)}{\partial t}(t,x) = \partial_t f(t,x)$$

$$- (\partial_s a(t,s,x))v(s,y(s;t,x))\big|_{s=\max(0,y^{-1}(0;t,x))} \cdot \frac{\partial y^{-1}}{\partial t}(0;t,x) H(y^{-1}(0;t,x))$$

$$+ \int_0^t \theta(s,t,x)(\partial_s a(t,s,x))(\partial_x v)(s,y(s;t,x)) \frac{\partial y}{\partial t}(s;t,x)\, ds$$

$$- \int_0^t \theta(s,t,x)(\partial_x a(t,s,x)) \cdot \left[(\partial_t v)(s,y(s;t,x)) + (\partial_x v)(s,y(s;t,x)) \frac{\partial y}{\partial t}(s;t,x) \right] ds$$

$$+ (\partial_x a(t,t,x))v(t,x) - (\partial_x a(t,s,x))v(s,y(s;t,x))\big|_{s=\max(0,y^{-1}(0;t,x))}.$$

By virtue of formulae (2.17), (2.18), (2.24)–(2.27) we obtain the estimates of $\|Av\|_{C^0}$ and $|Av|_1$ in terms of norm $\|v\|_{C^0}$ and seminorm $|v|_1$, then, if we pick R_0 and R_1 large enough, for T chosen sufficiently small it is easy to show that the integral operator leaves Z invariant, i.e.

$$\text{(2.28)} \qquad AZ \subset Z.$$

To prove uniform convergence or iterations $v_{n+1} = Av_n$, $n = 1, 2, \dots$ for $n \to \infty$, it is sufficient to show the integral operator A is a contraction in $C_b^0(G_T)$. Since the kernel $\partial_s a(t,s,x)$ is continuous and bounded (cf. (2.18)),

$$\text{(2.29)} \qquad \|Av - A\hat{v}\|_{C^0} \le T\|a\|_{C^1}\|v - \hat{v}\|_{C^0}.$$

Therefore, it follows that A is a contraction for T small enough, hence the iterations v_n converge uniformly to a continuous vector function u in G_T. This limit vector function u has obviously the initial-boundary value f, is a fixed point of the operator A and solves the system of integral equation (2.16).

However the existence and continuity of the derivatives $\frac{\partial v_n}{\partial t}$, $\frac{\partial v_n}{\partial x}$ of the iterations v_n, $n = 1, 2, \dots$ (cf. (2.26), (2.27)) do not imply existence and continuity of the derivatives $\frac{\partial v}{\partial t}$, $\frac{\partial v}{\partial x}$ of the limit function u. That these derivatives exist and are continuous will now be proved by showing that the derivatives $\frac{\partial v_n}{\partial t}$, $\frac{\partial v_n}{\partial x}$ converge uniformly as the iterations v_n for $n \to \infty$. Then according to classical calculus, the limit functions u have these limit functions as the first order derivatives $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$ respectively.

To prove the uniform convergence of derivatives $\frac{\partial v_n}{\partial t}$, $\frac{\partial v_n}{\partial x}$ we use the same arguments as above (the contraction property of A with respect to iterations v_n, $n = 1, 2, \ldots$). Then it is sufficient to show that the integral operator A has the contraction property with respect to the first order derivatives $\frac{\partial v}{\partial t}$, $\frac{\partial v}{\partial x}$. From (2.17), (2.18), (2.24)–(2.27) following estimates

$$(2.30) \qquad |Av - A\hat{v}|_1 \leq 3\|a\|_{C^1}\|v - \hat{v}\|_{C^0} + CT\|a\|_{C^1}|v - \hat{v}|_1.$$

With sufficiently small T such that

$$(2.31) \qquad CT\|a\|_{C^1} < 1$$

we assure the contraction property of A with respect to the first order derivatives $\frac{\partial v}{\partial t}$, $\frac{\partial v}{\partial x}$. This property allows us to state the uniform convergence of $\frac{\partial v_n}{\partial t}$, $\frac{\partial v_n}{\partial x}$ in a suitable narrow strip G_T if in this strip the iterations v_n converge uniformly (cf. (2.29)). Thus the limit function u has continuous first order derivatives in G_T and therefore indeed is the solution of the linearized system (2.12) of p.d.e. satisfying the initial-boundary condition (2.13) under assumptions (2.2), (2.3), (1.2) for $r = 1$.

We have thus showed.

THEOREM 2.1. *Under assumptions* (1.2), (1.8), (2.2), (2.3) *for* $r = 1$ *initial boundary value problem* (2.1), (2.5), (2.6) *possesses a unique* C^1*-smooth solution* G_T *for sufficiently small* T.

REMARK 2.2. The preceding reasoning extends without change to continuous derivatives up to the order $r > 1$.

REMARK 2.3. The C^r-smooth solution u of the initial-boundary value problem (2.1), (2.5), (2.6) $r \geq 1$ can be extended into the wholes domain G and in larger domain too as long as the assumptions of continuity and boundenes of the coefficients and the initial data remain satisfied (cf. (2.29), (2.30)).

REMARK 2.4. The C^r-smooth solution U, $r \geq 1$, of the initial-boundary value problem (2.1), (2.5), (2.6) depends continuously on the initial and bound data in C^r topology (cf. (2.16)-(2.18), (2.22)-(2.27)).

3. Solution of quasilinear system of p.d.e.

We turn now to the initial-boundary value problem to the quasilinear system (1.1) of p.d.e. with the initial and bound conditions (1.7), (1.8) and briefly describe how this problem can be solved on the basis of existence and uniqueness results of Section 2 by a slightly different iteration scheme. The results are exactly the same as those for linearized system (2.1) of p.d.e. (see. Theorem 2.1).

First we consider (cf. (2.2)) C^r-smooth vector function

$$W = (W^1, W^2, W^3)^* = (\pi, \omega, \tau)^*,$$

for simplicity we assume $r = 1$, with the fixed initial and boundary values

$$(3.1) \qquad \begin{aligned} W(0) &= V_0 = (\rho_0, v_0, \sigma_0)^* \\ W_b &= V_b = (\rho_b, v_b, \sigma_b)^* \end{aligned}$$

and satisfying inequalities

$$(3.2) \qquad \|W\|_{C^0} \le M_0, \qquad |W|_s \le M_1$$

with fixed M_0 and M_1. The matrix $A(W)$ given by (2.4) has three distinct real eigenvalues Λ_j, $j = 1, 2, 3$ and matrix $N(W)$ of independent eigencovector of $A(W)$ given by formulae (2.7), (2.8), (2.9) having continuous first order derivatives with respect to the variables t and x in a fixed domain G_T to be described presently provided we substitute for W any C^1-smooth vector function satysfying conditions (3.1), (3.2). The solutions of the initial-boundary value problems to ordinary differential equations (2.14) are called the characteristic C_j, $j = 1, 2, 3$ of the W-fields. For C^1-smooth vector function W satisfying (3.1), (3.2) their slopes are uniformly bounded, then we specify that a closed domain G and in it the strip G_T consist of point such that the characteristics C_j, $j = 1, 2, 3$ of the W-fields backwards towards $t = 0$ remain in G and intersect the section I of the x-axis or t-axis.

The set of all C^1-smooth function W satisfying (3.1), (3.2) in G_T we denote by S.

Now we use the classical iteration method due to Curant (1961). After substituting C^1-smooth vector function $W \in S$ in matrix A instead of V in the system (1.5) of p.d.e., the initial-boundary value problem (1.5), (1.7), (1.8) becomes linear of the same form (cf. (2.1), (2.5), (2.6)) as we treated in Section 2.

The solution U of this linearized problem is called $U = F(W)$, is F denotes the map, carries $W \in S$ into the solution U of initial-boundary value problem (2.1), (2.5), (2.6):

$$(3.3) \qquad W \to F(W) = U.$$

The solution V of the initial-boundary value problem (1.5), (1.7), (1.8) is a fixed point of the map F and we can obtain it as the uniform limit for $n \to \infty$ of iterations

$$V_{n+1} = F(V_n) \text{ with } V_1 = \begin{cases} u_b(y^{-1}(0; t, x)) & \text{for } y^{-1}(0; t, x) > 0 \\ u_0(y^{-1}(0; t, x)) & \text{for } y^{-1}(0; t, x) = 0. \end{cases}$$

It is easy to show that fixed $M_0 > \|V_1\|_{C^0}$ we can choose a sufficiently large and a sufficiently small T such that any vector function $U = F(W)$ belonging to S.

The map F has the contraction property. In fact, it follows directly, as in Section 2, by considering for the difference $z = U - \hat{U}$ of the two functions $U = F(W)$ and $\hat{U} = F(\hat{W})$ with the difference $\xi = W - \hat{W}$, the system of p.d.e.

$$\partial_t z = A(W)\partial_x z + B(W, \hat{W}, \partial_x \hat{U})\xi \tag{3.4}$$

where (cf. (2.1), (2.4))

$$B(W, \hat{W}, \partial_x \hat{U}) = \begin{bmatrix} \partial_x \hat{U}^2 & \partial_x \hat{U}^2 & 0 \\ \frac{\partial_x \hat{U}^3}{W^1 \hat{W}^1} & \partial_x \hat{U}^2 & 0 \\ 0 & \partial_x \hat{U}^2 & 0 \end{bmatrix} \tag{3.5}$$

with the initial condition

$$z(0) = 0 \tag{3.6}$$

and we obtain as in Section 2 an inequality of the form

$$\|z\|_{C^0} \le CT\|\xi\|_{C^0}. \tag{3.7}$$

With sufficiently small T, such that $CT < 1$, we assure the contraction property for F. Therefore the iterations V_n converge uniformly to a limit function V in G_T which is a C^1-smooth solution of the initial-boundary value problem to the quasi-linear system (1.5) with the initial-boundary conditions (1.7), (1.8) eo ipso of the initial-boundary value problem (1.1), (1.7), (1.8).

Thus we proved:

THEOREM 3.1. *Under assumption* (1.2), (1.4) *for* $r = 1$ *the initial-boundary value problem* (1.1), (1.7), (1.8) *possesses a unique* C^1*-smooth solution in* C_T *for sufficiently small* T.

REMARK 3.1. The preceding reasoning extends without change to continuous derivatives up to the order $r > 1$.

REMARK 3.2. The C^1-smooth solution $V = (\rho, v, \sigma)^*$, $r \geq 1$, of the initial-boundary value problem (1.1), (1.7), (1.8) depends continuously on the initial and boundary conditions (1.7), (1.8).

References

[1] S. N. Antontzev, A. V. Kazhihov, V. N. Manakhov, *Boundary Value Problems for Mechanics of Nonhomogeneous Fluids*, Nauka, Novosybirsk 1983, (in Russian).

[2] R. Collrant, *Cauchy's problem for hyperbolic quasi-linear system for first order partial differential equations in two independent variables*, Comm. Pure and Appl. Math. 14 (1961), 257–265.
[3] R. Courant, (Russian transl. (1964)), *Partial Differential Equations*, Interscience Publ., New York, 1962.
[4] S. Mizohata, (Engl. transl. (1973), Russian transl. (1977)), *The Theory of Partial Differential Equation*, Cambridge University Press, Cambridge, (1981).
[5] A. Piskorek, *Local in time existence of solutions of the some linear constitutive model in hypoelasticity equation*, Demonstratio Math. 28 (1994), No 2, pp. 449–559.
[6] M. E. Taylor, *Pseudodifferential Operators*, Princeton Univ. Press (1981).
[7] M. E. Taylor, *Pseudodifferential Operators and Nonlinear PDE*, Birkhäuser Progress in Mathematics, Vol. 100, Boston, Basel, Berlin 1991.

Adam Piskorek
UNIVERSITY OF WARSAW
FACULTY OF MATHEMATICS, INFORMATICS AND MECHANICS
Banacha 2
02-097 WARSAW, POLAND;

Arkadiusz Szymaniec
INSTITUTE OF MATHEMATICS AND OPERATION RESEARCH
MILITARY ACADEMY OF TECHNOLOGY
Kaliskiego 2
00-908 WARSAW, POLAND
E-mail: a.szymaniec@imbo.wat.waw.pl

Received July 26, 1999.