DEMONSTRATIO MATHEMATICA
Vol. XXXIII No 3 2000

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AN ABSTRACT SECOND ORDER CAUCHY PROBLEM WITH NON-DENSELY DEFINED OPERATOR, I

Abstract. By using the theory of extrapolation space X_{-1} associated with an operator A which is non densely defined in Banach space$X$, the existence and uniqueness of solutions of linear second order differential initial value problem (1) is proved.

1. Introduction

Our main objective is to investigate the abstract semilinear second order initial value problem

$$\begin{cases} \dfrac{d^2u}{dt^2} = Au + f\left(t, u, \dfrac{du}{dt}\right), & t \in (0,T] \\ u(0) = u_0, \dfrac{du}{dt}(0) = u_1, & u_0, u_1 \in X, \end{cases} \tag{1}$$

where X is a Banach space, u is a mapping from R to X, f is a nonlinear mapping from $R \times X \times X$ into X. The problem (1) was considered by many authors and their results are presented in the great number of papers. Usually on the operator A in (1) is assumed that it is the infinitesimal generator of a strongly continuous cosine family of linear operators in X. It is known that this assumption follows among other that the operator $A : X \to X$ is densely defined in X.

In this paper we try to give a treatment of the problem of existence, uniqueness and smoothness of solutions of the linear problem corresponding to (1) when the operator A is non-densely defined. The nonlinear problem (1) will be the subject of the forthcoming paper.

Our main tools are the theory of strongly continuous cosine family of linear operators in Banach space, the certain weak continuous cosine family and some extrapolation spaces associated to a linear operator A.

2. Preliminaries

Let the operator A defined in Section 1 be the generator of strongly continuous cosine family $\{C(t); t \in R\}$ of bounded operators from X into itself. Recall that a one parameter family $\{C(t); t \in R\}$ bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if

(2) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s, t \in R$,
(3) $C(0) = I$,
(4) $R \ni t \to C(t)x$ is continuous for each $x \in X$.

We define the associated sine family $\{S(t); t \in R\}$ by

(5) $S(t)x = \int_0^t C(s)x ds, x \in X, t \in R.$

The infinitesimal generator of a strongly continuous cosine family $\{C(t); t \in R\}$ is the operator $A : X \to X$ defined by

$$Ax = \frac{d^2C(t)x}{dt^2} |_{t=0} \quad \text{for} \quad x \in D(A), \tag{6}$$

where

(7) $D(A) := \{x \in X : R \ni t \to C(t)x$ is twice continuously differentiable$\}$.

Let

$E := \{x \in X : R \ni t \to C(t)x$ is once continuously differentiable$\}$.

PROPOSITION 1 ([8; Prop. 2.1 and 2.2]). *Let $\{C(t); t \in R\}$ be a strongly continuous cosine family in X with infinitesimal generator A. The following are true:*

(1.1) $C(t) = C(-t)$ *for all* $t \in R$,
(1.2) $C(s), S(s), C(t)$ *and* $S(t)$ *comute for all* $s, t \in R$,
(1.3) *the mapping* $R \ni t \to S(t)x$ *is continuous for each fixed* $x \in X$,
(1.4) $S(t) = -S(-t)$ *for all* $t \in R$,
(1.5) $S(t+s) = S(s)C(t) + S(t)C(s)$ *for all* $s, t \in R$,
(1.6) *there exist constants* $M \geq 1$, $\omega \geq 0$ *such that*

$$\|C(t)\| \leq Me^{\omega|t|}, \|S(t)\| \leq M \left| \int_0^t e^{\omega|s|} ds \right| \textit{for} \quad t \in R,$$

(1.7) *for* $x \in X$ *and* $s, r \in R$ *we have*

$$\int_s^r S(t)x dt \in D(A) \quad \textit{and} \quad A\int_s^r S(t)x\, dt = [C(r) - C(s)]x,$$

(1.8) $\int_0^s \int_0^r C(t)C(\tau)x\, dt\, d\tau \in D(A)$ *for all* $s, r \in R$, $x \in X$, (1.9) *if* $x \in D(A)$ *then* $C(t)x \in D(A)$ *and* $d^2/dt^2C(t)x = AC(t)x = C(t)Ax$,

(1.10) *if* $x \in E$ *then* $\lim_{t\to 0} AS(t)x = 0$,

(1.11) *if* $x \in E$ *then* $S(t)x \in D(A)$ *and* $d/dtC(t)x = AS(t)x$,

(1.12) $D(A)$ *is dense in* X *and* A *is a closed operator in* X.

THEOREM 1. ([4; Th. 3.1]). *The operator* A *is the infinitesimal generator of a cosine family satisfying* (1.6) *if and only if resolvent* $R(\lambda^2, A)$ *exists for* $\lambda > \omega$ *is strongly infinitely differentiable there and such that the inequality*

$$(8) \qquad \|\frac{d^n}{d\lambda^n}[\lambda R(\lambda^2, A)]\| \le \frac{Mn!}{(\lambda-\omega)^{n+1}} \text{ for } \lambda > \omega, n \in N \text{ hold.}$$

3. The adjoint cosine family

Let $\{C(t); t \in R\}$ be a cosine family on a Banach space X. The adjoint cosine family $\{C^*(t); t \in R\}$ on the dual space X^* is the family of operators obtained from $\{C(t); t \in R\}$ by taking pointwicely in t the adjoint operator $C^*(t) := [C(t)]^*$. It is elementary to see that the family $\{C^*(t); t \in R\}$ satisfies the equations (2) and (3), i.e. $\{C^*(t); t \in R\}$ is a cosine family on X^*, which is weak*-continuous, but it need not to be strongly continuous on X^*.

Analogously as in the theory of adjoint semigroups we define the weak*-generator of a weak*-continuous cosine family on X^*.

Let $\{U(t); t \in R\}$ be a weak*-continuous cosine family on X^*. The weak*-generator of $U(t)$ is the linear operator B on X^* defined by $D(B) := \{x^* \in X^* : R \ni t \to U(t)x^*$ is twice weak*-continuously differentiable$\}$;

$$Bx^* := weak * -\frac{d^2U(t)x^*}{dt^2}\Big|_{t=0} \text{ for } x^* \in D(B).$$

We have the following

THEOREM 2. A^* *is the weak*-generator of the cosine family* $\{C^*(t); t \in R\}$, $D(A^*)$ *is a* $C^*(t)$ *- invariant subspace of* X^* *and for all* $x^* \in D(A^*)$ *we have* $A^*C^*(t)x^* = C^*(t)A^*x^*$.

The proof of this theorem is similar to the proof of an analogous theorem in the theory of semigroups and is omitted.

Let $\{C(t); t \in R\}$ be a strongly continuous cosine family on X. We define

$$X^{\odot} := \{x^* \in X^* : \lim_{t\to 0} \|C^*(t)x^* - x^*\| = 0\}.$$

THEOREM 3 ([7]). *We have:*

(i) $X^{\odot}$ *is a closed, weak*-dense,* $C^*(t)$ *-invariant linear subspace of* X^*;

(ii) $X^{\odot} = \overline{D(A^*)}$ *where the closure is in norm of* X^*,

(iii) $C^{\odot}(t) := C^*(t)|_{X^{\odot}}$ *for* $t \in R$ *is a strongly continuous cosine family on* $X^{\odot}$,

(iv) *the part of* A^* *in* $X^\odot$, *which is denoted by* $A^\odot$, *is the generator of family* $\{C^\odot(t); t \in R\}$.

(v) *for each* $t \in R$ $C^*(t)$ *is a closure of* $C^\odot(t)$ *in weak*-topology on* X^*.

REMARK 1. Similarly to (5) we define the adjoint sine family $\{\mathcal{S}^*(t); t \in R\}$ on the dual space X*, taking $\mathcal{S}^*(t) := [\mathcal{S}(t)]^*$ for each $t \in R$. From this by (5) we have

$$\mathcal{S}^*(t)x^* = \int_0^t C^*(s)x^*\,ds,\ x^* \in X^*,\ t \in R, \tag{9}$$

where the integral in (9) is the weak*-integral.

Starting from strongly continuous cosine family $\{C^\odot(t); t \in R\}$ on $X^\odot$, the duality construction can be repeated. We define $C^{\odot *}(t)$ to be adjoint of $C^\odot(t)$ and write $X^{\odot\odot}$ for $(X^\odot)^\odot; C^{\odot\odot}(t)$ and $A^{\odot\odot}$ are defined analogously. It is known that the mapping

$$j : X \to X^{\odot *} \tag{10}$$

defined by

$$< jx, x^\odot > = < x^\odot, x > \text{ for } x \in X, x^\odot \in X^\odot \tag{11}$$

is an imbending isomorphism and jX is a closed subspace of $X^{\odot\odot}$. If $jX = X^{\odot\odot}$ then X is said to be $\odot$-reflexive with respect to $C(t)$. Moreover, $C^{\odot\odot}(t)$ is an extension of $jC(t)$ and $A^{\odot\odot}$ is an extension of jA and $jD(A) = D(A^{\odot\odot}) \cap jX$.

Similary to the C_0-semigroup we may to charakterize of the space $X^\odot$ in terms of the resolvent for the generator A of the cosine family $\{C(t); t \in R\}$.

PROPOSITION 2 ([6;Prop.1.4.4]). $x^* \in X^\odot$ *if and only if*

$$\lim_{\lambda \to \infty} \|\lambda R(\lambda, A^*)x^* - x^*\| = 0 \tag{12}$$

4. Ekstrapolation spaces

Here we introduce the concept of the extrapolation spaces X^{-1} and X_{-1}. The extrapolation space X^{-1} was introduce by Da Prato and Grisvard [2] and X_{-1} by R. Nagel [5]. This section is based on [6;Ch.3].

Let A be a closed linear operator on the Banach space X with non-empty resolvent set $\rho(A)$. We do not assume that A is densely defined. We define (see [6]).

$$X^{-1} := (X \times X)/G_A, \tag{12}$$

where G_A denote the graph of the operator A. Note that G_A - is a closed linear subspace of $X \times X$ since A is closed. Let us define

$$i : X \ni x \to ix := (0, x) \in X^{-1}. \tag{13}$$

The function (13) maps the space X onto the linear subspace iX of X^{-1}. This allows us to identify X with iX. We also define a linear operator A^{-1} on X^{-1} by

$$D(A^{-1}) := iX, \tag{14}$$

$$A^{-1}(0,x) := (-x,0) \quad \text{for} \quad x \in X. \tag{15}$$

Note that if $x \in D(A)$ then $(-x,0) = (0,Ax) = A^{-1}(0,x)$.

The operator A^{-1} should not to be confused with the inverse of A if this inverse exists. If we identify iX with X, we may regard A^{-1} as a bounded linear operator $X \to X^{-1}$. In fact if $x \in D(A)$ then $A^{-1}x := A^{-1}(0,x) = Ax$ so A^{-1} is an extension of A. In the space X^{-1} it may be defined an equivalent norm by formula

$$|(x,y)|_\mu := \|AR(\mu,A)x - R(\mu,A)y\| \tag{16}$$

for each $\mu \in \rho(A)$ and $(x,y) \in X^{-1}$.

THEOREM 4 ([6;Th.3.1.6]). *The space X is dense in X^{-1} if and only if A is densely defined, i.e. $\overline{D(A)} = X$.*

If the operator A is closed with nonempty resolvent set, we define the space X_{-1} as the closure of X in the norm of X^{-1}. From this and Theorem 4 follows that if A is densely defined, then $X_{-1} = X^{-1}$.

Let us denote by A_{-1} the part of A^{-1} in X_{-1} and by A_0 the part of A in $X_0 := \overline{D(A)}$. Clearly, A_{-1} is an extension of A.

We have the following

THEOREM 5 ([6;Prop.3.1.9]). *If A is a closed and $\lambda \in \rho(A)$, then*

(i) *$D(A_{-1}) = X_0$ and $\lambda - A_{-1} : X_0 \to X_{-1}$,*

(ii) *A is the part of A_{-1} in X; if $\lambda \in \rho(A)$, then $\lambda \in \rho(A_{-1})$ and $R(\lambda,A) = R(\lambda,A_{-1})|_X$.*

Now we may prove the following theorem which is analogous to the Theorem 3.1.10 in [6].

THEOREM 6. *Let A be linear closed operator on X which resolvent $R(\lambda^2,A)$ exists for $\lambda > \omega$ and which satisfies the inequality (8). Then:*

(i) *A_0 generates a cosine family $\{C_0(t); t \in R\}$ on X_0 and $R(\lambda^2,A_0) = R(\lambda^2,A)|_{X_0}$,*

(ii) *X_0 is X_{-1} dense in X and $(X_0)_{-1}$ is isomorphic to X_{-1},*

(iii) *under the identification $(X_0)_{-1} = X_{-1}$ we have $(A_0)_{-1} = A_{-1}$.*

Proof. (see [6; proof of Th.3.1.10]). To prove (i) we remark that the operator A_0 is densely defined in X_0. This follows from the fact that $\lambda R(\lambda,A)R(\mu,A)x \in D(A_0)$ for each $\lambda, \mu \in \rho(A)$ and $x \in X$ and

$$\lim_{\lambda\to\infty} \lambda R(\lambda,A)R(\mu,A)x = R(\mu,A)x,$$

that $D(A_0)$ is dense in the dense subspace $R(\mu, A)X$ of X_0. The assertion concerning the resolvents is obvious. From this, by Theorem 1, follows the assertion (i).

(ii) At first we remark that by (16) and by definition of the space X_{-1} it follows that an equivalent norm on X_{-1} may be defined by

$$\|x\|_{-1} = \|R(\mu, A)x\|, \quad \mu \in \rho(A) \quad \text{and } x \in X. \tag{17}$$

Let $x \in X$ be arbitrary but fixed. Since $R(\mu, A)x \in X_0$ and $D(A_0)$ is dense in X_0 then there is a sequence $\{x_n\} \subset X_0$ such that $R(\mu, A)x_n \to R(\mu, A)x$.

Hence $\|R(\mu, A)(x_n - x)\| = \|x_n - x\|_{-1} \to 0$.

This proves the density of X_0 in X_{-1} in norm $\|\cdot\|_{-1}$.

The inclusion map $i_0 : X_0 \to X$ may be extended to an isomorphism $(X_0)_{-1} \to X_{-1}$.

(iii) By Theorem 5 if $\lambda \in \rho(A)$ then $\lambda \in \rho(A_{-1})$ and $R(\lambda, A_{-1})|_X = R(\lambda, A)$ for $\lambda \in \rho(A)$. An application of the above to A_0 shows that

$$R(\lambda, (A_0)_{-1})|_{X_0} = R(\lambda, A_0) = R(\lambda, A)|_{X_0} = R(\lambda, A_{-1})|_{X_0}.$$

But X_0 is dense in X_{-1} and so it follows that $R(\lambda, (A_0)_{-1}) = R(\lambda, A_{-1})$. Therefore $(A_0)_{-1} = A_{-1}$.

THEOREM 7 ([6;Th.3.1.11]). *Under the assumptions of Theorem 6, the cosine family $\{C_0(t); t \in R\}$ generated by A_0 on X_0 extends to the cosine family $\{C_{-1}(t); t \in R\}$ on X_{-1} whose generator is the operator A_{-1}.*

5. The Favard class defined by cosine family

In this section we define and study some properties of the so-called Favard class. The definition and the properties of this class in entirely similar to Favard class defined by semigroups (see for exemple [6; sec.3.2]).

Let $\{C(t); t \in R\}$ be a cosine family on X.

Define its Favard class by

$$Fav(C(t)) := \{x \in X : \limsup_{t \to 0} \frac{2}{t^2}\|C(t)x - x\| < \infty\}. \tag{18}$$

From (18) it follows that $D(A) \subset Fav(C(t))$.

We have the following

THEOREM 8 ([6;Th.3.2.1]). *If $x^* \in X^*$, then the following assertions are equivalent:*

(i) $x^* \in D(A^*)$,
(ii) $\limsup_{t \to 0} \frac{2}{t^2}\|C^*(t)x^* - x^*\| < \infty$,
(iii) $\liminf_{t \to 0} \frac{2}{t^2}\|C^*(t)x^* - x^*\| < \infty$.

COROLLARY 1. $Fav(C^{\odot}(t)) = D(A^{\odot})$.

Proof. If $x^{\odot} \in Fav(C^{\odot}(t))$, then, by Theorem 8, $x^{\odot} \in D(A^*)$. Conversely, if $x^* \in D(A^*)$, then $x^* \in X^{\odot}$ and we have

$$\{x^* \in X^{\odot} : \limsup_{t\to 0} \frac{2}{t^2}\|C^*(t)x^{\odot} - x^{\odot}\| < \infty\}$$
$$= \{x^{\odot} \in X^{\odot} : \limsup_{t\to\infty} \frac{2}{t^2}\|C^{\odot}(t)x^{\odot} - x^{\odot}\| < \infty\} = Fav(C^{\odot}(t)).$$

From this we get

THEOREM 9 ([6;Th.3.2.3]). *If we identify the space X with its image jX in $X^{\odot *}$, then*

$$Fav(C(t)) = D(A^{\odot *}) \cap X. \tag{19}$$

COROLLARY 2. *If X is $\odot$ -reflexive, then $Fav(C(t)) = D(A^{\odot *})$ and if X is reflexive, then*

$$Fav(C(t)) = D(A). \tag{20}$$

Let A be a closed linear operator on a Banach space X with non-empty resolvent set $\rho(A)$, satisfying (8). Restriction $A_0 := A|_{X_0}$ where $X_0 := \overline{D(A)}$, generates the cosine family $\{C_0(t);\ t \in R\}$. We denote by

$$X_0^{\odot} = (X_0)^{\odot} := \{x^* \in X^* : \lim_{t\to 0} \|C_0^*(t)x^* - x^*\| = 0\} \tag{21}$$

where $C_0^*(t)$ denotes the adjoint of $C_0(t)$.

The $\{C_0^*(t); t \in R\}$ is the cosine family on $X_0^{\odot}$ generated by $A_0^{\odot}$ on $X_0^{\odot}$. Moreover $C_0^{\odot}(t) = C_0^*(t)|_{X_0^{\odot}}$.

In the sequel it will be important to have a representation of the Favard class of the cosine family $\{C_{-1}(t); t \in R\}$ on X_{-1}.

In this purpose we define

$$X^{\odot\times} := \{x^{\odot *} \in X^{\odot *} : R(\lambda, A^{\odot *})x^{\odot *} \in jX\} \tag{22}$$

(cf.[6;sec.3.2]).

The subspace $X^{\odot\times}$ is a closed $C^{\odot *}(t)$ invariant of $X^{\odot *}$.

If the operator A satisfies the inequality (8) and is not densely defined we define

$$X_0^{\odot\times} := (X_0)^{\odot\times}. \tag{23}$$

We have the following

THEOREM 10 ([6, Th.3.2.6;4.3.6 and 4.3.7]). *If the operator A satisfies the inequality* (8), *then*

(i) *$Fav(C_{-1}(t))$ is isomorphic to $X_0^{\odot\times}$,*
(ii) *$X \subset X_0^{\odot\times}$ and this inclusion is continuous.*

PROPOSITION 3 ([6;Prop.4.3.1]). *If A is the generator of the cosine family $\{C(t); t \in R\}$ on the space X, then*

$$X = \{x^{\odot\times} \in X^{\odot\times} : \lim_{t\to 0} \|C^{\odot\times}(t)x^{\odot\times} - x^{\odot\times}\| = 0\}. \tag{24}$$

6. The linear problem corresponding to (1)

In this section we study the following linear Cauchy problem corresponding to (1)

$$\begin{cases} \frac{d^2u}{dt^2} = Au + f & \text{for } t \in (0,T], \\ u(0) = u_0, \frac{du}{dt}(0) = u_1. \end{cases} \tag{25}$$

We prove the following

LEMMA 1. *If:*

1° $A : X \to X$ is a linear operator satisfying the inequality (8),

2° $f : [0,T] \to X$ is continuous,

then $s \to C_{-1}(t-s)f(s)$ is Bochner integrable in X_{-1} and the mapping

$$[0,T] \ni t \to v(t) := \int_0^t C_{-1}(t-s)f(s)\,ds \tag{26}$$

is a norm-continuous X_0-valued function such that

$$\|v(t)\|_{X_0} \leq Mt\|f\|_{C([0,T],X_0^{\odot\times})},$$

where $X_0 := \overline{D(A)}$ and $M := sup\{\|C_0(t)\| : t \in [0,T]\}$.

Proof. Since the operator A satisfies the inequality (8), then the operator $A_0 := A|_{X_0}$ is the generator of the cosine family $\{C_0(t); t \in R\}$ on the Banach space X_0. By the asumption 2° and the continuity inclusion $X \subset X_{-1}$ it follows that $f : [0,T] \to X_{-1}$ is continuous and so $s \to C_{-1}(t-s)f(s)$ is Bochner integrable in X_{-1}. On the other hand, since the inclusion $X \subset X_0^{\odot\times}$ is continuous (see Theorem 10), we can regard f as a continuous map $f : [0,T] \to X_0^{\odot\times}$.

Because $C_{-1}(t) |_{X_0^{\odot\times}} = C_0^{\odot\times}(t)$, in order to show that $v(t) \in X_0$, by Proposition 3 it is enough to check that

$$\lim_{r\to 0} \|C_0^{\odot\times}(r)v(t) - v(t)\|_{X_0^{\odot\times}} = 0. \tag{27}$$

We have (cf.[1])

$$C_0^{\odot\times}(r)\int_0^t C_0^{\odot\times}(t-s)f(s)\,ds - \int_0^t C_0^{\odot\times}(t-s)f(s)\,ds$$

$$= \int_0^t C_0^{\odot*}(r)C_0^{\odot\times}(t-s)f(s)\,ds - \int_0^t C_0^{\odot\times}(t-s)f(s)\,ds$$

$$= \frac{1}{2}\int_0^t [C_0^{\odot\times}(t+r-s) + C_0^{\odot\times}(t-r-s)] f(s)\,ds - \int_0^t C_0^{\odot\times}(t-s) f(s)\,ds$$

$$= \frac{1}{2}\Big[\int_0^t C_0^{\odot\times}(t+r-s) f(s)\,ds - \int_0^t C_0^{\odot\times}(t-s) f(s)\,ds\Big]$$

$$+ \frac{1}{2}\Big[\int_0^t C_0^{\odot\times}(t-r-s) f(s)\,ds - \int_0^t C_0^{\odot\times}(t-s) f(s)\,ds\Big].$$

After some rearrangements we obtain

$$(28)\quad 2[C_0^{\odot\times}(r)v(t) - v(t)]$$
$$= \int_0^t C_0^{\odot\times}(\tau)[f(t-\tau+r) - f(t-\tau)]\,d\tau$$
$$+ \int_0^t C_0^{\odot\times}\tau)[f(t-\tau-r) - f(t-\tau)]\,d\tau$$
$$+ \int_t^{t+r} C_0^{\odot\times}(\tau) f(t+r-\tau)\,d\tau - \int_{t-r}^{t} C_0^{\odot\times}(\tau) f(t-r-\tau)\,d\tau$$
$$- \int_0^r C_0^{\odot\times}(\tau) f(t+r-\tau)\,d\tau + \int_{-r}^{0} C_0^{\odot\times}(\tau) f(t-r-\tau) d\tau.$$

The norms of the first two terms of the right hand of (28) are less than

$$Mt \sup\{\|f(t+r) - f(t)\|_{X_0^{\odot\times}} : t \in [0,T]\}$$

and the norms of the last four terms are less than

$$Mr \sup\{\|f(t)\|_{X_0^{\odot\times}} : t \in [0,T]\}.$$

The latter inequalities imply (27).

Since $\|C_{-1}(t)\| = \|C_0^{\odot\times}(t)\| = \|C_0(t)\| \le M$, the estimate of $v(t)$ is trivial.

To show the contimity of the mapping (26) we notice that for $t, t+h \in [0,T]$

$$v(t+h) - v(t) = \int_0^{t+h} C_{-1}(t+h-s) f(s)\,ds - \int_0^t C_{-1}(t-s) f(s)\,ds$$
$$= \int_0^t C_{-1}(\tau)[f(t+h-\tau) - f(t-\tau)]\,d\tau + \int_t^{t+h} C_{-1}(\tau) f(t+h-\tau)\,d\tau.$$

From the above it follows that

$$\|v(t+h)-v(t)\| \\ \leq M\{T\sup\|f(t+h)-f(t)\|:t\in[0,T]+|h|\sup\|f(t)\|:t\in[0,T]\}$$

and this implies the norm-continuity of the mapping (26).

THEOREM 11. *If:*

1° $f:[0,T]\to X$ *is of class* C^1,

2° $u_0\in D(A_0)$ *and* $u_1\in E_0$,

then the problem (25) *has exactly one solution* u *of class* C^2 *in* $[0,T]$, *given by formula*

$$u(t)=C_0(t)u_0+\mathcal{S}_0(t)u_1+\int_0^t\mathcal{S}_{-1}(t-s)f(s)ds, t\in[0,T], \tag{29}$$

where

$$E_0:=\{x\in X_0:C_0(t)x \textit{ is once continuously differentiable in } t\}.$$

Proof. The basic idea of this proof comes from [6;Ch.4]. In [6;Ch.4] author considers the Cauchy problem of the first order. For given problem (25) in which the operator A is non densely defined we first consider the following problem

$$\begin{cases}\dfrac{d^2u}{dt^2}=A_{-1}u+f, & t\in(0,T],\\ u(0)=u_0,\ \frac{du}{dt}(0)=u_1\end{cases} \tag{30}$$

in the space $X_0^{\odot\times}$. Solutions of the problem (30), which lie in X_0, are likely to be also solutions to the problem (25)(cf[6.Ch.4]).

Since A_{-1} is a generator of cosine family $\{C_{-1}(t);t\in R\}$ on X_{-1} and the function $f:[0,T]\to X$ which is of class C^1 is also C^1 class as $f;[0,T]\to X_{-1}$, then some standard arguments of cosine theory (see[8]) show that the function $u:[0,T]\to X_{-1}$ given by formula

$$u(t)=C_{-1}(t)u_0+\mathcal{S}_{-1}(t)u_1+\int_0^t\mathcal{S}_{-1}(t-s)f(s)ds, \quad t\in[0,T] \tag{31}$$

is twice continuously differentiable in [0,T] and satisfies (30).

We shall show that u given by (31) is a classical solution of (25) i.e. u satisfies the conditions:

(i) $u(t)\in D(A)$ for $t\in[0,T]$,

(ii) $u:[0,T]\to X$ is of class C^2,

(iii) u satisfies the problem (25).

It is obvious that if $u_0 \in D(A_0)$ and $u_1 \in E_0$ then

$$w(t) := C_{-1}(t)u_0 + \mathcal{S}_{-1}(t)u_1 = C_0(t)u_0 + \mathcal{S}_0(t)u_1 \tag{32}$$

and so $w : [0,T] \to X_0$ is of class $C^2, w(t) \in D(A)$ for $t \in [0,T]$ and w satisfies the homogeneous problem (25).

Let

$$v(t) := \int_0^t \mathcal{S}_{-1}(t-s)f(s)\,ds, \quad t \in [0,T]. \tag{33}$$

By Lemma 1 it follows that $\mathcal{S}_{-1}(t)|_{X_0^{\odot\times}} \in X_0$ for $t \in R$ and so $v(t) \in X_0$ for each $t \in [0,T]$.

On the other hand

$$\mathcal{S}_{-1}(t-s)f(s) = \int_s^t C_{-1}(\tau - s)f(s)\,d\tau, \quad 0 < s < t,$$

and from this we get

$$v(t) = \int_0^t\int_s^t C_{-1}(\tau - s)f(s)\,d\tau\,ds.$$

Changing the order of integration we have

$$v(t) = \int_0^t \left[\int_0^s C_{-1}(s-\tau)f(\tau)\,d\tau\right] ds. \tag{34}$$

Since by Lemma 1 the mapping

$$[0,T] \ni s \to \int_0^s C_{-1}(s-\tau)f(\tau)\,d\tau$$

is norm-continuous X_0-valued function, then the function v defined by (33) is of class C^1 with respect to the norm of X_0 and

$$\frac{dv}{dt} = \int_0^t C_{-1}(t-s)f(s)ds \quad \text{for} \quad t \in [0,T]. \tag{35}$$

By the assumption 1° we have

$$f(s) = f(0) + \int_0^s f'(\tau)\,d\tau \quad \text{for} \quad s \in [0,T].$$

This implies that

$$\frac{dv}{dt} = \int_0^t C_{-1}(t-s)f(0)\,ds + \int_0^t\int_0^s C_{-1}(t-s)f'(\tau)\,d\tau\,ds.$$

Changing the order integration in the last integral we obtain

$$\frac{dv}{dt} = \mathcal{S}_{-1}(t)f(0) + \int_0^t \mathcal{S}_{-1}(t-s)f'(s)\,ds. \tag{36}$$

Because the f' is a continuous function in $[0,T]$, analogously as in above, from (36) we conclude that the function v is of class C^2 in $[0,T]$ and

$$\frac{d^2v}{dt^2} = C_{-1}(t)f(0) + \int_0^t C_{-1}(t-s)f'(s)\,ds, \quad t \in [0,T]. \tag{37}$$

On the other hand, since $v(t) \in X_0 = D(A_{-1})$ for each $t \in [0,T]$, we get

$$A_{-1}v(t) = A_{-1}\Big[\int_0^t \mathcal{S}_{-1}(t-s)f(0)\,ds + \int_0^t\int_0^s \mathcal{S}_{-1}(t-s)f'(\tau)\,d\tau\,ds\Big].$$

Changing the order of integration in the laast integral and using Proposition 1, p.(1.7) we obtain

$$A_{-1}v(t) = [C_{-1}(t) - I]f(0) + \int_0^t [C_{-1}(t-s) - I]f'(s)\,ds$$

$$= C_{-1}(t)f(0) + \int_0^t C_{-1}(t-s)f'(s)ds - f(t).$$

From this and (37) follows that

$$A_{-1}v(t) = \frac{d^2v}{dt^2}(t) - f(t) \in X \quad \text{for each} \quad t \in [0,T].$$

Since A is the part of A_{-1} in X, then it follows that $v(t) \in D(A)$ for $t \in [0,T]$.

As we have seen above (cf.(31) and (32))

$$u(t) = w(t) + v(t), \quad t \in [0,T],$$

where $w(t) \in D(A)$ and w is of class C^2 in [0,T]. This implies that u is a classical solution of (25).

To show the uniqueness it is sufficient to notice that if $f : [0,T] \to X_{-1}$ is continuous, $u : [0,T] \to X_{-1}$ is twice continuously differentiable in $(0,T], u(t) \in D(A_{-1})$ for $t \in [0,T]$ and u satisfies (30) then u is given by (31)(cf.[8;Prop.2.4]). This completes the proof of Theorem 11.

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Received December 15, 1999.