DEMONSTRATIO MATHEMATICA
Vol. XXXIII No 3 2000

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GENERAL PARTIAL DIFFERENTIAL SUBORDINATIONS FOR HOLOMORPHIC MAPPINGS IN COMPLEX BANACH SPACES

Abstract. In this paper, we will consider classes of subordinations involving partial derivatives of holomorphic mappings in complex Banach spaces.

1. Introduction

S. S. Miller and P. T. Mocanu [7], [8] considered the analytic functions defined on the unit disc, which satisfy some differential inequalities, and obtained several results including inclusion relations, inequalities and some sufficient conditions for univalence. P. Liczberski [6] and also G. Kohr and M. Kohr-Ile [2], [3] obtained some results concerning partial differential subordination for holomorphic mappings defined on the unit Euclidean ball and on the unit polydisc, respectively. G. Kohr and P. Liczberski [5] obtained some similar results for holomorphic mappings defined on the unit ball in $\mathbf{C}^n$ with an arbitrarily fixed norm.

It is natural to extend the above results to complex Banach spaces. In this paper, we will present some similar results for holomorphic mappings defined on the unit ball in an arbitrary complex Banach space.

2. Preliminaries

Let X be a complex Banach space with norm $\|\cdot\|$. The open ball $\{x \in X : \|x\| < \rho\}$ is denoted by B_ρ and the unit ball is abbreviated by $B_1 = B$. A holomorphic mapping f from a domain Ω in X into X is said to be biholomorphic if the inverse f^{-1} exists, is holomorphic on an open set $V \subset X$ and $f^{-1}(V) = \Omega$. Let X^* be the dual space of X. For each

1991 *Mathematics Subject Classification*: 32A30, 32H02, 30C45.

Key words and phrases: Banach space, differential subordinations, holomorphic mappings.

$z \in X \setminus \{0\}$, we define

$$\mathcal{T}(z) = \{z^* \in X^* : \|z^*\| = 1, z^*(z) = \|z\|\}.$$

By the Hahn-Banach theorem, $\mathcal{T}(z)$ is nonempty. The symbol $Df(z)$, as usual, means the Fréchet derivative of f at z.

Let U denote the unit disc in $\mathbf{C}$. The following Jack's lemma for holomorphic mappings into a complex Banach space [9] plays an important role in this paper (cf. [1], [4], [6]).

LEMMA 1. *Let $f : U \to X$ be a holomorphic mapping with $f(0) = 0$. If for a $\zeta_0 \in U \setminus \{0\}$, we have*

$$\|f(\zeta_0)\| = max\{\|f(\zeta)\| : |\zeta| \leq |\zeta_0|\} > 0,$$

then there exists a real number $m \geq 1$ such that

$$b^*(\zeta_0 Df(\zeta_0)) = mb^*(f(\zeta_0))$$

for all $b^* \in \mathcal{T}(b)$, where $b = f(\zeta_0)$ and

$$\|Df(\zeta_0)(\zeta_0)\| = s\|f(\zeta_0)\|,$$

where $s \geq m \geq 1$.

3. Main results

We shall give the well known definition of the subordination.

DEFINITION 1. Let $f, g : B \to X$ be holomorphic mappings on the unit ball B in X. We say that f is subordinate to g on B, written $f \prec g$ $(f(z) \prec g(z))$ if there exists a Schwarz mapping $q : B \to X$ (i.e. q is holomorphic in B, $q(0) = 0$, $\|q(z)\| < 1$, for all $z \in B$) such that $f(z) = g(q(z))$, $z \in B$.

REMARK 1. If f is subordinate to g in B, then $f(0) = g(0)$ and $f(B) \subset g(B)$. Moreover, if g is biholomorphic on B, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(B) \subset g(B)$.

Now, applying the result of Lemma 1, we obtain the following theorem (cf. [3], [5], [6]).

THEOREM 1. *Let $f : B \to X$ be a holomorphic mapping and let $g : B_\rho \to X$ be a biholomorphic mapping, where $\rho > 1$. Suppose that $f(0) = g(0)$ and f is not subordinate to g on B. Then there exist numbers $r \in (0, 1)$ and $s \geq 1$, and points $a \in B$ and $w \in \partial B$ such that*

$$f(a) = g(w), \tag{1}$$

$$f(\zeta a) \subset g(\bar{B}), \quad \zeta \in \bar{U}, \tag{2}$$

$$s\|(Dg(w))^{-1}\|^{-1} \leq \|Df(a)(a)\| \leq s\|Dg(w)\|. \tag{3}$$

Proof. Since f is not subordinate to g, there exists a $z \in B$ such that $f(z) \notin g(B)$. Since $f(0) \in g(B)$, there exists a $\zeta_0 \in \bar{U} \setminus \{0\}$ such that $f(\zeta_0 z) \in g(\bar{B}) \setminus g(B)$ and that $f(\zeta z) \in g(B)$ for all ζ with $|\zeta| < |\zeta_0|$. Let $a = \zeta_0 z \in B$ and let $w = g^{-1}(f(a)) \in \partial B$. Then we obtain the equality (1) and the inclusion (2). The mapping

$$h(\zeta) = (g^{-1} \circ f)(\zeta a)$$

is holomorphic on $\bar{U}$ and satisfies the conditions:

$$h(0) = 0, \quad 1 = \|h(1)\| = max\{\|h(\zeta)\| : \zeta \in \bar{U}\}.$$

Thus h satisfies the assumptions of Lemma 1, so there exists a real number $s \geq 1$ such that $\|Dh(1)(1)\| = s$. Since $Dh(1) = (Dg(w))^{-1}Df(a)(a)$, we obtain the inequalities (3).

Now we will consider the differential subordinations. First, we will define the concept of the admissible class for mappings in X.

DEFINITION 2. Let $\Omega \subset X$ be a subset and let $g : B_\rho \to X$ be a biholomorphic mapping, where $\rho > 1$. By $\Psi(\Omega, g)$, we denote the class of all mappings $\psi : X \times X \times B \to X$ which satisfy the conditions

$$\psi(g(0), 0, 0) \in \Omega$$

and

$$\psi(u, v, z) \notin \Omega \tag{4}$$

for all $z \in B$ and all $(u, v) \in Q(g)$, where

$$Q(g) = \bigcup_{s \geq 1, \|w\|=1} Q_s(g, w)$$

with

$$Q_s(g, w) \\ = \{(u, v) \in X \times X : u = g(w),\ s\|(Dg(w))^{-1}\|^{-1} \leq \|v\| \leq s\|Dg(w)\|\}.$$

The class $\Psi(\Omega, g)$ is called the "admissible class".

EXAMPLE 1. Let $g = M \cdot id_X$ that is $g(z) = Mz$, $z \in X$, where M is a positive number. Then g is biholomorphic on X and by simple computations we deduce in this case that

$$Q_s(g, w) = Q_s(M \cdot id_X, w) = \{(u, v) \in X \times X : u = Mw,\ \|v\| = sM\}.$$

Hence

$$Q(M \cdot id_X) = \{(u, v) \in X \times X : \|u\| = M,\ \|v\| \geq M\}.$$

In this case $\Psi(\Omega, g) = \Psi(\Omega, M \cdot id_X)$ consists of those mappings $\psi : X \times X \times B \to X$, $\psi(0,0,0) \in \Omega$, which satisfy the condition $\psi(u,v,z) \notin \Omega$ for all (u,v) with $\|u\| = M$, $\|v\| \geq M$ and $z \in B$.

EXAMPLE 2. Let

$$\ell_2 = \{z = (z_1, z_2, \ldots) : \|z\|_2^2 = \sum_{n=1}^{\infty} |z_n|^2 < \infty\}.$$

Let us consider the biholomorphic mapping on ℓ_2 defined by

$$g(z) = (z_1 + az_2^2, z_2, \ldots),$$

where a is a positive real number. Then

$$\|Dg(w)\|_2 = \|(Dg(w))^{-1}\|_2 = a|w_2| + \sqrt{a^2|w_2|^2 + 1}$$

for every w with $\|w\|_2 = 1$. Therefore the family $\Psi(\Omega, g)$ consists of those mappings $\psi : \ell_2 \times \ell_2 \times B \to \ell^2$, $\psi(0,0,0) \in \Omega$ and $\psi(u,v,z) \notin \Omega$ for all $z \in B$, and $u, v \in \ell_2$, such that

$$|u_1 - au_2^2|^2 + \sum_{n=2}^{\infty} |u_n|^2 = 1, \quad \sqrt{a^2|u_2|^2 + 1} - a|u_2| \leq \|v\|_2.$$

EXAMPLE 3. Let

$$\ell_\infty = \{z = (z_1, z_2, \ldots) : \|z\|_\infty = \sup\{|z_n| : n \geq 1\} < \infty\}.$$

Let g be as in Example 2. Then $\|Dg(w)\|_\infty = \|(Dg(w))^{-1}\|_\infty = 1 + 2a|w_2|$ for every w with $\|w\|_\infty = 1$. Therefore the family $\Psi(\Omega, g)$ consists of those mappings $\psi : \ell_\infty \times \ell_\infty \times B \to \ell_\infty$, $\psi(0,0,0) \in \Omega$ and $\psi(u,v,z) \notin \Omega$ for all $z \in B$, and $u, v \in \ell_\infty$, such that

$$\max(|u_1 - au_2^2|, \sup\{|u_n| : n \geq 2\}) = 1, \quad 1/(1 + 2a|u_2|) \leq \|v\|_\infty.$$

Now, we have the following theorem.

THEOREM 2. *Let $f : B \to X$ be a holomorphic mapping and let $g : B_\rho \to X$ be a biholomorphic mapping with $g(0) = f(0)$, where $\rho > 1$. If there exists a mapping $\psi \in \Psi(\Omega, g)$ such that for all $z \in B$*

$$\psi(f(z), Df(z)(z), z) \in \Omega, \tag{5}$$

then $f \prec g$ on B.

Proof. If the subordination $f \prec g$ does not hold on B, then in view of Theorem 1, there exist points $a \in B$, $w \in \partial B$ and a real number $s \geq 1$ such that (1) and (3) hold. Let $u = f(a)$ and $v = Df(a)(a)$, then $(u,v) \in Q_s(g,w) \subset Q(g)$ and according to (4), we deduce $\psi(u,v,a) \notin \Omega$. However, this contradicts with (5). Hence $f \prec g$.

Furthermore, we suppose that Ω is a subset of X such that there exists a holomorphic mapping h from B onto Ω. Then using the result of Theorem 2 and the notation $\Psi(h, g)$ for the class $\Psi(h(B), g)$, we obtain the following corollary.

COROLLARY 1. *Let $g : B_\rho \to X$ be a biholomorphic mapping, where $\rho > 1$ and let $f, h : B \to X$ be holomorphic mappings with $f(0) = g(0)$. If there exists a holomorphic mapping $\psi \in \Psi(h, g)$ such that the subordination*

$$\psi(f(z), Df(z)(z), z) \prec h(z) \tag{6}$$

holds on B, then $f \prec g$ on B.

DEFINITION 3. We say that g is a dominant for the subordination (6) if (6) implies the subordination $f \prec g$ on B. Also, we say that g is the best dominant of (6) if g is a dominant of (6) and g is subordinate to all other dominants of (6) in B.

Now we will give a sufficient condition such that g will be the best dominant of (6).

THEOREM 3. *Let $f, h : B \to X$ be holomorphic mappings and let $g : B_\rho \to X$ be biholomorphic mappings with $g(0) = f(0)$, where $\rho > 1$. Suppose that there exists a holomorphic mapping $\psi \in \Psi(h, g)$ such that g fulfils the differential equation*

$$\psi(g(z), Dg(z)(z), z) = h(z), \quad z \in B. \tag{7}$$

If the subordination

$$\psi(f(z), Df(z)(z), z) \prec h(z)$$

holds on B, then $f \prec g$ and g is the best dominant.

Proof. By using the result of Corollary 1, we deduce that f is subordinate to g in B and because g satisfies the differential equation (7), g will be subordinate to all other dominants of (6), so g will be the best dominant of (6).

4. Applications

In this section, we will apply the results in the previous sections to some particular cases.

First, we will prove two theorems.

THEOREM 4. *Let M and N be positive numbers, and let $x, y : B \to \mathbf{C}$ be functions which satisfy the inequality*

$$|x(z) + my(z)| \geq NM^{-1}, \qquad z \in B$$

for all real numbers $m \geq 1$. Let $f : B \to X$ be a holomorphic mapping with $f(0) = 0$ such that

$$\|x(z)f(z) + y(z)Df(z)(z)\| < N, \quad z \in B.$$

Then $\|f(z)\| < M$ in B.

Proof. Suppose that there exists a $z \in B$ such that $\|f(z)\| \geq M$. Then there exists a point $a \in B$ such that

$$M = \|f(a)\| = max\{\|f(\zeta a)\| : \zeta \in \bar{U}\}.$$

Then $a \neq 0$ and $b = f(a) \neq 0$. Then the mapping F defined by

$$F(\zeta) = f(\zeta a\|a\|^{-1})$$

satisfies the assumption of Lemma 1 at $\zeta_0 = \|a\|$. So, from Lemma 1, there exists a real number m such that $m \geq 1$ and $b^*(Df(a)(a)) = m\|f(a)\| = mM$ for all $b^* \in \mathcal{T}(b)$. Then we have

$$\begin{aligned} \|x(a)f(a) + y(a)Df(a)(a)\| &\geq |b^*[x(a)f(a) + y(a)Df(a)(a)]| \\ &= |Mx(a) + mMy(a)| = M|x(a) + my(a)| \geq N, \end{aligned}$$

which contradicts with the hypothesis. Hence $\|f(z)\| < M$ in B.

THEOREM 5. *Let $f : B \to X$ be a holomorphic mapping and let $g : B_\rho \to X$ be a biholomorphic mapping with $g(0) = f(0)$, where $\rho > 1$. Suppose that*

$$0 < M = inf\{\|(Dg(w))^{-1}\|^{-1} : \|w\| = 1\}$$

and $\|Df(z)(z)\| < M$ in B. Then $f \prec g$.

Proof. If f is not subordinate to g in B, then from Theorem 1, we deduce that there exist points $a \in B,\ w \in \partial B$ such that

$$\|Df(a)(a)\| \geq \|(Dg(w))^{-1}\|^{-1} \geq M.$$

This contradicts with the hypothesis. So $f \prec g$.

If we put $g = M \cdot id_X$, we obtain the following corollaries from Theorems 2 and 5.

COROLLARY 2. *Let $f : B \to X$ be a holomorphic mapping with $f(0) = 0$. Suppose that there exists a mapping $\psi \in \Psi(\Omega, M \cdot id_X)$ such that for each $z \in B$*

$$\psi(f(z), Df(z)(z), z) \in \Omega.$$

Then $\|f(z)\| < M$ in B.

COROLLARY 3. *Let $f : B \to X$ be a holomorphic mapping with $f(0) = 0$. Suppose that $\|Df(z)(z)\| < M$ in B. Then $\|f(z)\| < M$ in B.*

Acknowledgements. The first author is partially supported by Grant-in-Aid for Scientific Research (C) no.11640194 from Japan Society for the Promotion of Science, 1999.

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Received October 25, 1999.