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NOTES ON BCK-ALGEBRAS WITH CONDITION (S)

Abstract. Some further properties related to BCK-algebras with the condition (S) are obtained. The main results are as follows: (i) If a commutative BCK-algebra X is a lattice with respect to the BCK-ordering $\leq$, then X need not be with the condition (S); (ii) A positive implicative BCK-algebra X with the condition (S) may not be a lattice with respect to $\leq$, moreover, if $(X;\leq)$ is a lattice, it must be a distributive lattice; (iii) Each involutory BCK-algebra is with the condition (S).

All of the notions in this paper can be found in [7] (or in [5] and [4]). We denote a BCK-algebra by $(X;*,0)$, or X in short, and the BCK-ordering on X by $\leq$ which is defined by $x \leq y \iff x * y = 0$. A BCK-algebra X is called to be with *the condition* (S) if, for any $a, b \in X$, the set $A(a,b) = \{x \in X \mid x * a \leq b\}$ has the greatest element, denoted by $a \circ b$. BCK-algebras are closely related to lattices. For example, it is known that $(X;\leq)$ is a partially ordered set (see, [5], page 4); S. Tanaka [8] proved that if X is commutative, $(X;\leq)$ is a lower semilattice; K. Iséki and S. Tanaka [5] showed that if X is bounded and commutative, $(X;\leq)$ is a lattice; T. Traczyk [9] further gave the following significant result: if X is a commutative BCK-algebra and if $(X;\leq)$ is a lattice, then the lattice is distributive. K. Iséki [2] also proved the following result:

PROPOSITION 1. *If X is a positive implicative BCK-algebra with the condition (S), then $(X;\leq)$ is an upper semilattice where the least upper bound $a \vee b$ of each pair of elements a and b in X is equal to $a \circ b$.*

As a preliminary we give the following three results (see, [5], [3] and [10], respectively).

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PROPOSITION 2. *Suppose that X is a BCK-algebra. Then for all $x, y, z \in X$,*

(1) $(x * y) * z = (x * z) * y$;

(2) *If X is with the condition (S), then* $(x * y) * z = x * (y \circ z)$;

(3) *If there exists a binary operation $\cdot$ on X such that* $(x*y)*z = x*(y \cdot z)$, *then X is with the condition (S).*

The motivation of the present paper is based on the following three open problems which were arised from Y.H. Lin [6].

PROBLEM 1. If X is a commutative BCK-algebra and $(X; \leq)$ a lattice, does X satisfy the condition (S)?

PROBLEM 2. Does each nontrivial ideal of a positive implicative BCK-algebra with the condition (S) form an initial section?

PROBLEM 3. If X is a positive implicative BCK-algebra with the condition (S), does $(X; \leq)$ form a lattice?

It is a pity that the answers to these problems are negative. Besides, we will prove the following results.

(1) If X is a positive implicative BCK-algebra with the condition (S) and if $(X; \leq)$ is a lattice, then the lattice must be distributive.
(2) Every involutory BCK-algebra is with the condition (S).

We start our discussion with giving the following counter examples. Let us first recall that a BCK-algebra X is said *commutative* if $x * (x * y) = y * (y * x)$ for all $x, y \in X$.

EXAMPLE 1. Let X be the interval $\{x \in R \mid 0 \leq x < 2\}$ of real numbers. We define the operation $*$ on X by putting $x * y = \max\{0, x - y\}$ where $-$ is the subtraction as usual. Then X is a commutative BCK-algebra. Because the BCK-ordering $\leq$ on X is the same as the natural ordering of real numbers, $(X; \leq)$ is a totally ordered set, thus a lattice. Next, for any $x \in X$, since $0 \leq x < 2$, we have $\max\{0, x - 1\} \leq 1$, namely, $x * 1 \leq 1$. Hence the set $A(1, 1) = \{x \in X \mid x * 1 \leq 1\}$ is just X itself. Note that X is a left closed and right open interval, $A(1, 1)$ has not any greatest element. Therefore X is not with the condition (S).

A BCK-algebra X is called *positive implicative* if $(x*z)*(y*z) = (x*y)*z$, or equivalently, $(x * y) * y = x * y$ for any $x, y, z \in X$. X is called *implicative* if $x * (y * x) = x$ for all $x, y \in X$. It is known very well that every implicative BCK-algebra is positive implicative. *The initial section of an element a in a BCK-algebra X* is the set $\{x \in X \mid x \leq a\}$, denoted by $A(a)$. It is obvious that if X has the greatest element, say 1, i.e., X is *bounded with the unit 1*, then $X = A(1)$. *The Iséki's extension $X \cup \{\alpha\}$ by adding an element α to a*

BCK-algebra $(X; *, 0)$ means that $(X \cup \{\alpha\}; *, 0)$ is a bounded BCK-algebra with the greatest element α, where the operation $*$ on $X \cup \{\alpha\}$ is given by

$$x * y = \begin{cases} x * y, & \text{if } x, y \in X, \\ \alpha, & \text{if } x = \alpha \text{ and } y \in X, \\ 0, & \text{if } y = \alpha. \end{cases}$$

It is easily seen that (i) X is a maximal ideal of $X \cup \{\alpha\}$; (ii) X is with the condition (S) if and only if $X \cup \{\alpha\}$ is with the condition (S); (iii) X is positive implicative if and only if $X \cup \{\alpha\}$ is positive implicative.

EXAMPLE 2. Let N be the set of all natural numbers and X the collection of the whole finite subset of N (containing the empty set $\varnothing$). Denote $*$ for the set difference $-$ on X. Then $(X; *, \varnothing)$ is an implicative BCK-algebra with the condition (S) where $x \circ y = x \cup y$, the union of x and y. It is clear that X is not bounded, i.e., there does not exist an element a in X such that X is equal to the initial section $A(a)$. We now make an Iséki's extension $X \cup \{\alpha\}$ of X, then $X \cup \{\alpha\}$ is a positive implicative BCK-algebra with the condition (S) and X is a nontrivial ideal of $X \cup \{\alpha\}$. However, as we have seen, the ideal X is not an initial section.

EXAMPLE 3. Let X be as in Example 2 and α, β, ω be three distinct elements such that $X \cap \{\alpha, \beta, \omega\} = \varnothing$. Denote $X' = X \cup \{\alpha, \beta, \omega\}$. Make two Iséki's extensions $X \cup \{\alpha\}$ and $X \cup \{\beta\}$. Then $(X \cup \{\alpha\}; *, 0)$ and $(X \cup \{\beta\}; *, 0)$ are positive implicative BCK-algebras with the condition (S). If we extensively define that

$$\alpha * \beta = \alpha, \;\; \beta * \alpha = \beta \quad \text{and} \quad \alpha * \alpha = \beta * \beta = 0,$$

then $(X \cup \{\alpha, \beta\}; *, 0)$ is a positive implicative BCK-algebra. Further, put

$$\omega * \alpha = \beta, \;\; \omega * \beta = \alpha, \;\; \omega * x = \omega \text{ for } x \in X \text{ and } x * \omega = 0 \text{ for } x \in X'.$$

It is no difficulty to verify that $(X'; *, 0)$ is a bounded and positive implicative BCK-algebra with the condition (S), where the greatest element of X' is ω and the operation $\circ$ is as follows: for any $x, y \in X$ and $\lambda \in \{\alpha, \beta, \omega\}$,

$$x \circ y = x \cup y, \; x \circ \lambda = \lambda \circ \lambda = \lambda \text{ and } \alpha \circ \beta = \alpha \circ \omega = \beta \circ \omega = \omega.$$

However, $(X'; \leq)$ is not a lattice, in fact, the set of all lower bounds of α and β is just the X, but as we have seen from Example 2, X has not any greatest element.

So far we have negatively answered the Y.H. Lin's problems.

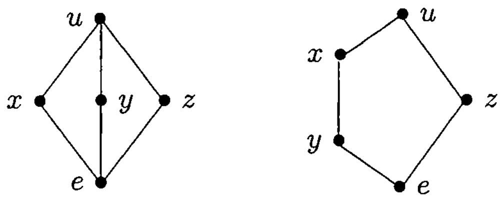
It is known (see, e.g., [4]) that a partially ordered set $(X; \leq)$ with the least element 0 may induce a positive implicative BCK-algebra $(X; *, 0)$ in which the binary operation $*$ on X is given by

$$x * y = \begin{cases} 0, & \text{if } x \leq y, \\ x, & \text{otherwise.} \end{cases}$$

From this reason we see if a positive implicative BCK-algebra forms a lattice with respect to $\leq$, the lattice is generally not distributive. Nevertheless, we still have the next interesting result.

THEOREM 3. *Let X be a positive implicative BCK-algebra with the condition (S). If $(X; \leq)$ is a lattice, it must be distributive.*

Proof. From the theory of lattices, a lattice is distributive if and only if it contains neither a rhombus sublattice nor a pentagon sublattice (see, e.g., [1]). Now, if our assertion is not true, the lattice $(X; \leq)$ contains either a rhombus sublattice or a pentagon sublattice whose Hasse diagrams are respectively assumed as follows:



For the first diagram, we have $x \vee y = u$ and $x \vee z = u$, which mean from Proposition 1 that $x \circ y = u$ and $x \circ z = u$, then by Proposition 2(2),

$$(u * x) * y = u * (x \circ y) = u * u = 0,$$

namely, $u * x \leq y$. Likewise, $u * x \leq z$. So $u * x \leq y \wedge z$. Noticing $y \wedge z = e$, it follows $u * x \leq e$. Also, since $e \leq x$, by Proposition 1, $x \circ e = x \vee e = x$. Now, Proposition 2(2) gives

$$u * x = u * (x \circ e) = (u * x) * e \leq e * e = 0.$$

Therefore $u \leq x$, a contradiction with $u > x$.

For the second diagram, we have $y \vee z = u$. Then Proposition 1 implies that $y \circ z = u$. Applying Proposition 2(2) and the fact that $x \leq u$, we derive

$$(x * y) * z = x * (y \circ z) = x * u = 0,$$

that is, $x * y \leq z$. Also, obviously $x * y \leq x$. Then $x * y \leq x \wedge z = e$, and so $(x * y) * e = 0$. Using Proposition 2(2) again, it follows $x * (y \circ e) = 0$. Next, because $e \leq y$, by Proposition 1, $y \circ e = y \vee e = y$. Hence $x * y = x * (y \circ e) = 0$, proving $x \leq y$, which is impossible since $x > y$. The proof is complete.

Finally, let us consider the relation between involutory BCK-algebras and BCK-algebras with the condition (S). A BCK-algebra X is said *involutory* if X is bounded and $NNx = x$ where $Nx = 1 * x$ and 1 denotes the greatest element of X.

THEOREM 4. *Every involutory BCK-algebra X is with the condition (S).*

Proof. We assume 1 is the greatest element of X and $x, y, z \in X$. Because X is involutory, by Proposition 2(1), we have

$$(I) \qquad Nx * Ny = (1 * x) * (1 * y) = (1 * (1 * y)) * x = NNy * x = y * x.$$

We now define a binary operation $\circ$ on X as follows:

$$x \circ y = N(Nx * y).$$

Using the involutory property of X and (I) as well as Proposition 2(1), we obtain

$$\begin{aligned} x * (y \circ z) &= NNx * N(Ny * z) = (Ny * z) * Nx \\ &= (Ny * Nx) * z = (x * y) * z. \end{aligned}$$

Hence Proposition 2(3) tells us that X is with the condition (S).

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