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# ARCHIMEDEAN AND BLOCK-FINITE LATTICE EFFECT ALGEBRAS

**Abstract.** We show that every complete effect algebra is Archimedean. Moreover, a block-finite lattice effect algebra has the MacNeille completion which is a complete effect algebra iff it is Archimedean. We apply our results to orthomodular lattices.

## 1. Basic definitions

*Effect algebras* (introduced by Foulis D.J. and Bennett M.K. in [7], 1994) are important for modelling unsharp measurements in Hilbert space: The set of all effects is the set of all self-adjoint operators $T$ on a Hilbert space $H$ with $0 \leq T \leq 1$. In a general algebraic form an effect algebra is defined as follows:

DEFINITION 1.1. A structure $(E; \oplus, 0, 1)$ is called an *effect-algebra* if 0, 1 are two distinguished elements and $\oplus$ is a partially defined binary operation on $P$ which satisfies the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined,
(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined,
(Eiii) for every $a \in P$ there exists the unique $b \in P$ such that $a \oplus b = 1$ (we put $a' = b$),
(Eiv) if $1 \oplus a$ is defined then $a = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by $E$. In every effect algebra $E$ we can define the partial operation $\ominus$ and the partial order $\leq$ by putting

$$a \leq b \text{ and } b \ominus a = c \text{ iff } a \oplus c \text{ is defined and } a \oplus c = b\,.$$

Since $a \oplus c = a \oplus d$ implies $c = d$, $\ominus$ and $\leq$ are well defined. For more details we refer the reader to [7], [8], [13]–[17], [20]–[23] and the references given

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there. We review only a few properties without proof.

LEMMA 1.2. *Elements of an effect algebra* $(E;\oplus,0,1)$ *satisfy the properties:*

(i) $a\oplus b$ *is defined iff* $a\le b'$,

(ii) $a=(a\wedge b)\oplus(a\ominus(a\wedge b))$, *if* $a\wedge b$ *exists,*

(iii) *if* $a\oplus b$ *and* $a\vee b$ *exist then* $a\wedge b$ *exists, and* $a\oplus b=(a\wedge b)\oplus(a\vee b)$,

(iv) $a\oplus b\le a\oplus c$ *iff* $b\le c$ *and* $a\oplus c$ *is defined,*

(v) $a\ominus b=0$ *iff* $a=b$,

(vi) $a\le b\le c$ *implies that* $c\ominus b\le c\ominus a$ *and* $b\ominus a=(c\ominus a)\ominus(c\ominus b)$.

DEFINITION 1.3. $1\in Q\subseteq E$ is called a *sub-effect algebra* of an effect algebra $(E;\oplus,0,1)$ iff for all $a,b,c\in E$ such that $a\oplus b=c$, if at least two elements are in $Q$ then $a,b,c\in Q$.

In the sequel, for a poset $P$ and a subset $A\subseteq P$ we will denote by $\bigvee_P A$ and $\bigwedge_P A$ the supremum and infimum of $A$ in $P$ if exist.

## 2. Lattice effect algebras of mutually compatible elements and blocks

An effect algebra $(E;\oplus,0,1)$ is called a *lattice effect algebra* iff $(E;\le)$ is a lattice. If $(E;\le)$ is a complete lattice then $(E;\oplus,0,1)$ is called a *complete effect algebra.* Two elements $x$ and $y$ of a lattice effect algebra are *compatible* (we write $x\leftrightarrow y$) iff $(x\vee y)\ominus y=x\ominus(x\wedge y)$, (see [17]). We can easily see that $x\leftrightarrow y$ iff $y\oplus(x\ominus(x\wedge y))$ exists. A maximal subset of mutually compatible elements (i.e. every two are compatible) of a lattice effect algebra $E$ is called a *block* of $E$. In [23] was proved that blocks of $E$ are sub-effect algebras of E (i.e., with inherited operations $\oplus$, 0, and 1, blocks are affect algebras in their own right). Moreover blocks are MV-algebras (introduced by Chang [5]), therefore as recently Kôpka and Chovanec have shown, MV-algebras in some sense are equivalent to lattice effect algebras of mutually compatible elements [16].

For subsets $U$ and $Q$ of a poset $(P;\le)$ we will write $U\le Q$ iff $u\le q$ for all $u\in U$ and $q\in Q$. We write $a\le Q$ instead of $\{a\}\le Q$ and $U\le b$ instead of $U\le\{b\}$. If $(E;\oplus,0,1)$ is an effect algebra and $U$ and $Q$ are subsets of $E$ such that $U\le Q$ then we put

$$Q\ominus U=\{q\ominus u\mid u\in U,q\in Q\}.$$

THEOREM 2.1. *Let* $(E;\oplus,0,1)$ *be a lattice effect algebra of mutually compatible elements. Let* $U,Q\subseteq E$ *such that* $U\le Q$ *then*

$$\bigwedge_E(Q\ominus U)=0 \textit{ implies } \{a\in E\mid a\le Q\}\le\{b\in E\mid U\le b\}.$$

Proof. Assume that $U \leq Q$ and $\bigwedge_E(Q \ominus U) = 0$. Let $a \leq Q$ and $U \leq b$. In view of the compatibility of elements of $E$ we have for all $u \in U$ and $q \in Q$ that

$$a \vee u = u \oplus (a \ominus (a \wedge u)) \leq a \vee q = q = u \oplus (q \ominus u)$$

which implies that $a \ominus (a \wedge b) \leq a \ominus (a \wedge u) \leq q \ominus u$. It follows that $a \ominus (a \wedge b) = 0$ and hence $a = a \wedge b$ and $a \leq b$.

It is well known that every poset has the MacNeille completion (i.e., a completion by cuts; see [2]). By J. Schmidt [25] the *MacNeille completion* $MC(P)$ of a poset $P$ is (up to isomorphism) any complete lattice into which $P$ can be supremum-densely and infimum-densely embedded (i.e., for every element $x \in MC(P)$ there exist $M, Q \subseteq P$ such that $x = \bigvee \varphi(M) = \bigwedge \varphi(Q)$, where $\varphi : P \to MC(P)$ is the embedding). We usually identify $P$ with $\varphi(P)$. In this sense $MC(P)$ preserves all infima and suprema existing in $P$.

From now on, by $(\widehat{P}, \leq)$ (or $\widehat{P}$ for brevity) we will denote the MacNeille completion of a poset $(P; \leq)$. It is easy to see that if $U, Q \subseteq P$ such that $U \leq Q$ then $\bigvee_{\widehat{P}} U = \bigwedge_{\widehat{P}} Q$ iff $\{a \in P \mid a \leq Q\} \leq \{b \in P \mid U \leq b\}$.

Since every effect algebra $(E; \oplus, 0, 1)$ is a poset (under the partial order defined by $a \leq b$ iff there is $c \in E$ such that $a \oplus c = b$) there exists the MacNeille completion $MC(E)$ of the poset $(E; \leq)$.

DEFINITION 2.2. ([23]) We say that the *MacNeille completion of an effect algebra* $(E; \oplus_E, 0_E, 1_E)$ is a complete effect algebra $(\widehat{E}; \oplus_{\widehat{E}}, 0_{\widehat{E}}, 1_{\widehat{E}})$ iff $\widehat{E} = MC(E)$ and the embedding $\varphi$ of the poset $(E; \leq)$ into a complete lattice $\widehat{E}$ is such that $\varphi(a) \oplus_{\widehat{E}} \varphi(b)$ exists for $a, b \in E$ iff $a \leq b'$, and then $\varphi(a) \oplus_{\widehat{E}} \varphi(b) = \varphi(a \oplus_E b)$.

LEMMA 2.3. *For elements of a lattice effect algebra* $(E; \oplus, 0, 1)$*:*

(i) *If* $x \leftrightarrow y$ *and* $x \leftrightarrow z$ *then* $x \leftrightarrow y \vee z$,

(ii) *If* $x \leftrightarrow a$ *for all* $a \in A \subseteq E$ *and* $\bigvee_E A$ *exists then* $x \leftrightarrow \bigvee_E A$.

Proof. (i) This is proved in [24], Theorem 2.1.

(ii) Let $u_\alpha = \bigvee \alpha$ for every finite subset $\alpha \subseteq A \subseteq E$. By (i) we have $x \leftrightarrow u_\alpha$ for every finite $\alpha \subseteq A$ and by [24] Lemma 4.2 we obtain that $x \leftrightarrow \bigvee_E A = \bigvee_E \{u_\alpha \mid \alpha \subseteq A,\ \alpha \text{ is finite}\}$.

It was proved in [23] that for an effect algebra $(E; \oplus, 0, 1)$ the partial operation $\oplus$ can be extended on $\widehat{E} = \mathrm{MC}(E)$ such that $(\widehat{E}; \oplus, 0, 1)$ is a MacNeille completion of $(E; \oplus, 0, 1)$ iff $E$ is *strongly D-continuous*, which means that if $U, Q \subseteq E$ are such that $U \leq Q$ then $\bigwedge_E(Q \ominus U) = 0$ iff $\{a \in E \mid a \leq Q\} \leq \{b \in E \mid U \leq b\}$.

THEOREM 2.4. *A lattice effect algebra* $(E; \oplus, 0, 1)$ *of mutually compatible elements has the MacNeille completion* $\widehat{E}$ *a complete effect algebra containing* $E$ *as a sub-effect algebra (up to embedding) iff for all* $U, Q \subseteq E$ *such that* $U \leq Q$,

$$\{a \in E \mid a \leq Q\} \leq \{b \in E \mid U \leq b\} \text{ implies } \bigwedge_E (Q \ominus U) = 0.$$

*In such case elements of* $\widehat{E}$ *are mutually compatible.*

Proof. It remains to prove that elements of $\widehat{E}$ are mutually compatible. Let $x, y \in \widehat{E}$. Then there exist $U_x$, $U_y \subseteq E$ such that $x = \bigvee_{\widehat{E}} U_x$ and $y = \bigvee_{\widehat{E}} U_y$. Since $u \leftrightarrow v$ for all $u \in U_x$ and $v \in U_y$ we obtain that $x \leftrightarrow y$ by Lemma 2.3.

## 3. Archimedean effect algebras

The well known fact is that an MV-algebra $M$ has the MacNeille completion which is a complete MV-algebra iff $M$ is Archimedean. Since MV-algebras are lattice effect algebras of mutually compatible elements ([17]), the notion to be Archimedean makes sense also for effect algebras.

DEFINITION 3.1. An effect algebra $(E; \oplus, 0, 1)$ is called *Archimedean* iff for no nonzero element $e \in E$, $e \oplus e \oplus \ldots \oplus e$ ($n$-times) exists for all natural numbers $n \in N$. Set $ne = e \oplus e \oplus \ldots \oplus e$ ($n$-times).

THEOREM 3.2. *For an effect algebra* $(E; \oplus, 0, 1)$ *the following conditions are equivalent:*

(i) *$E$ is Archimedean,*

(ii) *If* $U, Q \subseteq E$ *are such that* $U \leq Q$ *then* $\{a \in E \mid a \leq Q\} \leq \{b \in E \mid U \leq b\}$ *implies that* $\bigwedge_E (Q \ominus U) = 0$.

Proof. (i)⇒(ii): Assume that $U, Q \subseteq E$ are such that $U \leq Q$ and $a \leq b$ for all $a \leq Q$ and $U \leq b$. Let $e \in E$ be such that $e \leq Q \ominus U$. Then $e \leq q \ominus u$ for all $u \in U$ and $q \in Q$ which implies that $u \leq q \ominus e$ and $e \oplus u \leq q$. Thus $U \leq q \ominus e$ and $e \oplus u \leq Q$ which implies $e \oplus u \leq q \ominus e$ and hence $e \oplus e \oplus u \leq q$ for all $u \in U$ and $q \in Q$. It follows that $e \oplus e \oplus u \leq Q$ and since $U \leq q \ominus e$ we obtain $e \oplus e \oplus u \leq q \ominus e$ which implies that $e \oplus e \oplus e \oplus u \leq q$ for all $u \in U$ and $q \in Q$. By induction we obtain that $ne$ is defined for every natural number $n \in N$ and hence $e = 0$. Thus $\bigwedge_E (Q \ominus U) = 0$.

(ii)⇒(i): Let $e \in E$ be such that $ne$ exists for every natural number $n \in N$. Let $Q = \{1 \ominus ne \mid n \in N\}$ and $U = \{a \in E \mid a \leq Q\}$. Then $\bigwedge_E (Q \ominus U) = 0$. Moreover, for all $a \in U$ and $n, m \in N$ we have $a \leq 1 \ominus (m + n)e = (1 \ominus ne) \ominus me$. It follows that $me \leq (1 \ominus ne) \ominus a$, which implies $me \leq Q \ominus U$. We obtain that $me = 0$ and hence $e = 0$. This proves that $E$ is Archimedean.

Combining Theorem 3.2 and the necessary and sufficient condition for effect algebras to have MacNeille completion — the strongly D-continuity — we obtain the following assertion:

THEOREM 3.3. *Every complete effect algebra is Archimedean.*

For lattice effect algebras we obtain the following consequence of Theorems 2.4 and 3.2:

THEOREM 3.4. *For a lattice effect algebra $(E;\oplus,0,1)$ of mutually compatible elements, the following conditions are equivalent:*

(i) *$E$ is Archimedean,*

(ii) *The MacNeille completion $\widehat{E}$ of $E$ is a complete effect algebra containing $E$ as a sub-effect algebra. In such case elements of $\widehat{E}$ are mutually compatible.*

Now, using mentioned above result by Kôpka-Chovanec [17] we obtain the following well known fact for MV-algebras:

COROLLARY 3.5. *For an MV-algebra $M$ the following conditions are equivalent:*

(i) *$M$ is Archimedean,*

(ii) *the MacNeille completion of $M$ is a complete MV-algebra.*

## 4. Block-finite effect algebras

DEFINITION 4.1. A lattice effect algebra $(E;\oplus,0,1)$ is called *block-finite* iff there is a finite set $\{M_1,M_2,\dots,M_n\}$ of blocks of $E$ such that $E=\bigcup\limits_{k=1}^{n} M_k$.

LEMMA 4.2. *Let $(E;\oplus,0,1)$ be a block-finite lattice effect algebra and $E=\bigcup\limits_{k=1}^{n} M_k$, where $M_k$ are blocks of $E$. Then for every net $(x_\alpha)_{\alpha\in\mathcal{E}}$ of elements of $E$ ($\mathcal{E}$ is a directed set) there exist $k_0\in\{1,2,\dots,n\}$ and a cofinal subset $\mathcal{E}_1\subseteq\mathcal{E}$ such that $x_\alpha\in M_{k_0}$ for all $\alpha\in\mathcal{E}_1$.*

Proof. Assume that for every $k\in\{1,2,\dots,n\}$ there is $\alpha_k\in\mathcal{E}$ such that $x_\alpha\notin M_k$ for all $\alpha\geq\alpha_k$. Then for $\gamma\in\mathcal{E}$ such that $\gamma\geq\alpha_k$ for every $k\in\{1,2,\dots,n\}$ and all $\alpha\geq\gamma$ we have $x_\alpha\notin M_k$ for every $k\in\{1,2,\dots,n\}$, a contradiction. It follows that there exists $k_0\in\{1,2,\dots,n\}$ such that for every $\beta\in\mathcal{E}$ there is $\alpha\in\mathcal{E}$, $\alpha\geq\beta$ and $x_\alpha\in M_{k_0}$. Hence $\mathcal{E}_1'=\{\alpha\in\mathcal{E}\mid x_\alpha\in M_{k_0}\}$ is cofinal in $\mathcal{E}$.

THEOREM 4.3. *Let a complete effect algebra $(\widehat{E};\oplus,0,1)$ be a MacNeille completion of a block-finite lattice effect algebra $(E;\oplus,0,1)$. Then $\widehat{E}=\bigcup\limits_{k=1}^{n}\widehat{M_k}$,* *where $\widehat{M_k}$ are blocks of $\widehat{E}$ such that there are blocks $M_k$ of $E$, $M_k \subseteq \widehat{M_k}$ and $E = \bigcup\limits_{k=1}^{n} M_k$.*

Proof. Let $E = \bigcup\limits_{k=1}^{n} M_k$. Since $M_k \subseteq \widehat{E}$ is a set of mutually compatible elements there is a block $\widehat{M_k} \subseteq \widehat{E}$ such that $M_k \subseteq \widehat{M_k}$, for $k = 1, 2, \dots, n$. Let $x \in \widehat{E}$. Then there exists $\mathcal{U} \subseteq E$ such that $x = \bigvee \mathcal{U}$. Assume that the set $\mathcal{E} = \{\alpha \subseteq \mathcal{U} \mid \alpha \text{ is finite }\}$ is directed by set inclusion. We put $u_\alpha = \bigvee \alpha$ for all $\alpha \in \mathcal{E}$. Then $\bigvee\{u_\alpha \mid \alpha \in \mathcal{E}\} = x$. By Lemma 4.2 there exist $k_0 \in \{1, 2, \dots, n\}$ and a cofinal subset $\mathcal{E}_1 \subseteq \mathcal{E}$ such that $u_\alpha \in M_{k_0}$ for all $\alpha \in \mathcal{E}_1$. Evidently, $x = \bigvee\{u_\alpha \mid \alpha \in \mathcal{E}_1\}$. Since by [24] Theorem 4.3 $\widehat{M_{k_0}}$ is $\tau_o$-closed we obtain $x \in \widehat{M_{k_0}}$. We conclude that $\widehat{E} = \bigcup\limits_{k=1}^{n} \widehat{M_k}$.

THEOREM 4.4. *Let $(E; \oplus, 0, 1)$ be a block-finite effect algebra and let $(\widehat{E}; \leq)$ be the MacNeille completion of the poset $(E; \leq)$. Let for every block $M$ of $E$, $\widehat{M} = \{x \in \widehat{E} \mid x = \bigvee_{\widehat{E}} U$ for some $U \subseteq M\}$. Then*

(i) *$\widehat{M}$ is a complete sublattice of $\widehat{E}$,*

(ii) *$\widehat{M}$ is the MacNeille completion of the block $M$.*

*Moreover, if $E = \bigcup_{k=1}^{n} M_k$, where $M_k$ are blocks of $E$ then $\widehat{E} = \bigcup_{k=1}^{n} \widehat{M_k}$, where $\widehat{M_k}$ is the MacNeille completion of $M_k$ for $k = 1, 2, \dots, n$.*

Proof. (i) Evidently for every block $M$ of $E$, $M \subseteq \widehat{M}$ and for all $x \in \widehat{M}$ we have $x = \bigvee_{\widehat{E}}\{u \in M \mid u \leq x\} = \bigvee_{\widehat{M}}\{u \in M \mid u \leq x\}$.

Assume that $D \subseteq \widehat{M}$. Then

(a) $\bigvee_{\widehat{E}} D = \bigvee_{\widehat{E}}\{\bigvee_{\widehat{E}} U_x^M \mid x \in D,\ U_x^M = \{u \in M \mid u \leq x\}\} = \bigvee_{\widehat{E}}(\bigcup\{U_x^M \mid x \in D\}) \in \widehat{M}$. It follows that $\bigvee_{\widehat{E}} D = \bigvee_{\widehat{M}} D$.

(b) In view of the part (a) of the proof we have $\bigvee_{\widehat{E}}\{a \in \widehat{M} \mid a \leq D\} = \bigvee_{\widehat{M}}\{a \in \widehat{M} \mid a \leq D\} = \bigwedge_{\widehat{M}} D$. We obtain $\bigwedge_{\widehat{M}} D = \bigvee_{\widehat{E}}\{a \in \widehat{M} \mid a \leq D\} \leq \bigvee_{\widehat{E}}\{a \in \widehat{E} \mid a \leq D\} = \bigwedge_{\widehat{E}} D$. Since $\bigwedge_{\widehat{M}} D$ is a lower bound of $D$ in $E$ we have also $\bigwedge_{\widehat{E}} D \leq \bigwedge_{\widehat{M}} D$. Thus $\bigwedge_{\widehat{E}} D = \bigwedge_{\widehat{M}} D$.

(ii) Let $x \in \widehat{M}$. Let us denote $U_x^M = \{u \in M \mid u \leq x\}$ and $Q_x^M = \{q \in M \mid \text{for all } u \in M : u \leq q \text{ iff } u \leq x\}$. By definition of $\widehat{M}$ we have $x = \bigvee_{\widehat{E}} U_x^M = \bigvee_{\widehat{M}} U_x^M$. Further, by part (i) $\bigwedge_{\widehat{M}} Q_x^M$ exists and by definition of $\widehat{M}$ and $Q_x^M$ we obtain

$$\begin{aligned}\bigwedge_{\widehat{M}} Q_x^M &= \bigvee_{\widehat{M}}\{a \in \widehat{M} \mid a \le Q_x^M\} \\ &= \bigvee_{\widehat{M}}\{u \in M \mid a \in \widehat{M} \text{ and } u \le a \le Q_x^M\} \\ &= \bigvee_{\widehat{M}}\{u \in M \mid u \le Q_x^M\} = \bigvee_{\widehat{M}}\{u \in M \mid u \le x\} = x\,.\end{aligned}$$

We conclude that $\widehat{M}$ is the MacNeille completion of $M$ by Schmidt [25].

Finally let $x \in \widehat{E}$. Then there is $U \subseteq E$ with $\bigvee_{\widehat{E}} U = x$. Let $\mathcal{E} = \{\alpha \subseteq U \mid \alpha \text{ is finite}\}$ be directed by set inclusion. For every $\alpha \in \mathcal{E}$ we put $u_\alpha = \bigvee \alpha$. Evidently $\bigvee_{\widehat{E}} U = \bigvee\{u_\alpha \mid \alpha \in \mathcal{E}\}$. By Lemma 4.2 there exist $k_1 \in \{1, 2, \dots, n\}$ and a cofinal subset $\mathcal{E}_1 \subseteq \mathcal{E}$ such that $u_\alpha \in M_{k_1}$ for all $\alpha \in \mathcal{E}_1$. Moreover, $\bigvee_{\widehat{E}}\{u_\alpha \mid \alpha \in \mathcal{E}_1\} = \bigvee_{\widehat{E}}\{u_\alpha \mid \alpha \in \mathcal{E}\} = \bigvee_{\widehat{E}} U = x$, which implies that $x \in \widehat{M_{k_1}}$. This proves that $\widehat{E} = \cup_{k=1}^n \widehat{M_k}$.

THEOREM 4.5. *A block-finite lattice effect algebra $(E; \oplus, 0, 1)$ has a MacNeille completion which is a complete effect algebra $(\widehat{E}; \oplus, 0, 1)$ containing $E$ as a sub-effect algebra iff $E$ is Archimedean.*

Proof. Let $E = \bigcup_{k=1}^{n} M_k$ where $M_k$ are blocks of $E$.

(1) Assume that the MacNeille completion $\widehat{E}$ is a complete effect algebra. Then by Theorem 4.3 $\widehat{E} = \bigcup_{k=1}^{n} \widehat{M_k}$ where $\widehat{M_k}$ are blocks of $\widehat{E}$, hence complete $MV$-algebras. Moreover, $M_k \subseteq \widehat{M_k}$ for $k = 1, 2, \dots, n$. We conclude that $E$ is Archimedean, since $e \in E$ implies $e \in M_k \subseteq \widehat{M_k}$ for some $k \in \{1, \dots, n\}$ and $\widehat{M_k}$ are Archimedean.

(2) Conversely, assume that $E$ is Archimedean. Then every block $M \subseteq E$ is Archimedean. If $U, Q \subseteq E$ are such that $U \le Q$ then for every finite $\alpha \subseteq U \cup Q$ such that $\alpha \cap U \neq \emptyset$ and $\alpha \cap Q \neq \emptyset$ we put $u_\alpha = \bigvee_E \alpha \cap U$ and $q_\alpha = \bigwedge \alpha \cap Q$. By Lemma 4.2 there exist $k_1 \in \{1, 2, \dots, n\}$ and a cofinal subset $\mathcal{E}_1 \subseteq \mathcal{E} = \{\alpha \subseteq U \cup Q \mid \alpha \cap U \neq \emptyset \neq \alpha \cap Q, \alpha \text{ finite}\}$, such that $u_\alpha \in M_{k_1}$, for all $\alpha \in \mathcal{E}_1$. Similarly there exist $k_2 \in \{1, 2, \dots, n\}$ and a cofinal subset $\mathcal{E}_2 \subseteq \mathcal{E}_1$ such that $q_\alpha \in M_{k_2}$, for all $\alpha \in \mathcal{E}_2$. Let $U_1 = \{u_\alpha \mid \alpha \in \mathcal{E}_2\}$, $Q_1 = \{q_\alpha \mid \alpha \in \mathcal{E}_2\}$. Then $U_1 \le Q_1$ and $\bigvee_{\widehat{E}} U_1 = \bigvee_{\widehat{E}} U$, $\bigwedge_{\widehat{E}} Q_1 = \bigwedge_{\widehat{E}} Q$. Moreover, $U_1 \cup Q_1$ is a set of mutually compatible elements. It follows that there is a block $M \subseteq E$ such that $U_1 \cup Q_1 \subseteq M$. Since $M$ is Archimedean, by Corollary 3.5 the MacNeille completion $\widehat{M}$ of $M$ is a complete effect algebra. It follows by [23] that $M$ is strongly $D$-continuous. Since $U_1 \le Q_1$ and $U_1, Q_1 \subseteq M$ we have $\bigwedge_M (Q_1 \ominus U_1) = 0$ iff $\bigvee_{\widehat{M}} U_1 = \bigwedge_{\widehat{M}} Q_1$. Moreover, for $c \in E$, $c \le q \ominus u$ for all $u \in U$ and $q \in Q$ iff $c \le q_\alpha \ominus u_\alpha$ for all $\alpha \in \mathcal{E}$. Thus

$\bigwedge_E(Q \ominus U) = 0$ iff $\bigwedge_E(Q_1 \ominus U_1) = 0$. It follows that $\bigwedge_E(Q \ominus U) = 0$ implies $\bigwedge_M(Q_1 \ominus U_1) = 0$, which implies $\bigwedge_{\widehat{M}} Q_1 = \bigvee_{\widehat{M}} U_1$ and hence $\bigwedge_{\widehat{E}} Q = \bigvee_{\widehat{E}} U$. By Theorem 3.2, also conversely, $\bigwedge_{\widehat{E}} Q = \bigvee_{\widehat{E}} U$ implies $\bigwedge_E(Q \ominus U) = 0$. We conclude that $E$ is strongly D-continuous, which by [23] implies that $\oplus$ can be extended on $\widehat{E}$ such that $\widehat{E}$ becomes a complete effect algebra.

REMARK 4.6. All results of this section may be formulated also for $D$-lattices (introduced by Kôpka-Chovanec [16]). It is because blocks of a lattice effect algebra $(E; \oplus, 0, 1)$ (or derived $D$-lattice $(E; \leq, \ominus, 0, 1)$) have the property: If from elements $a, b, c \in E$ with $a \oplus b = c$ $(b = c \ominus a)$ defined in $E$ at least two are in a block $M$ then $a, b, c \in M$. Moreover, $1 \in M$. Thus $M$ is simultaneously a sub-effect algebra and a sub-$D$-lattice in $E$. Moreover, if $\widehat{E}$ is a MacNeille completion of a poset $(E; \leq)$ then the embedding $\varphi : E \to \widehat{E}$ has the property $\varphi(a \oplus_E b) = \varphi(a) \oplus_{\widehat{E}} \varphi(b)$ for all $a, b \in E$, $a \leq b'$ iff $\varphi(d \ominus_E c) = \varphi(d) \ominus_{\widehat{E}} \varphi(c)$ for all $c, d \in E$, $c \leq d$.

## 5. Block-finite orthomodular lattices

It is well known that a lattice effect algebra $(E; \oplus, 0, 1)$ in which $e \wedge e' = 0$ for all $e \in E$ (equivalently, $e \vee e' = 1$) is an orthomodular lattice, which means that $a \leq b \Rightarrow b = a \vee (a' \wedge b)$ for all $a, b \in E$. In such case blocks of $E$ become Boolean algebras. It is because an MV-algebra $M$ is a Boolean algebra iff $e \wedge e' = 0$ for all $e \in M$. Conversely, in every orthomodular lattice $(L; \vee, \wedge, ', 0, 1)$ the partial binary operation $\oplus$ defined by $a \oplus b = a \vee b$ iff $a \leq b'$ satisfies effect algebra axioms. In fact $(L; \oplus, 0, 1)$ becomes an Archimedean lattice effect algebra ([12], [16]). In such case, compatibilities of two elements $a, b \in L$ considered in the orthomodular lattice $L$ and in the effect algebra $L$ coincide. This is because $a \leftrightarrow b$ implies that $a \leftrightarrow b'$ and by Jenča-Riečanová [11], $a = a \wedge (b \vee b') = (a \wedge b) \vee (a \wedge b')$ which implies that $a \vee b = b \vee (a \wedge b') = b \oplus (a \wedge b') = b \oplus (a \ominus (a \wedge b))$. Moreover, in [23] it has been shown that the MacNeille completion $\widehat{L}$ of an orthomodular lattice $L$ is orthomodular iff $\widehat{L}$ is a complete effect algebra containing $L$ as a sub-effect algebra (under which orthomodular lattices $L$ and $\widehat{L}$ and effect algebras $L$ and $\widehat{L}$ are mutually corresponding as stated above). We may now apply Theorems of Section 4 to orthomodular lattices.

THEOREM 5.1. *Let $(L; \vee, \wedge, ', 0, 1)$ be a block-finite orthomodular lattice and let $(\widehat{L}, \leq)$ be the MacNeille completion of the poset $(L; \leq)$. Then*

(i) *$\widehat{L}$ is orthomodular,*

(ii) *For every block $B$ of $L$ the set $\widehat{B} = \{x \in \widehat{L} \mid U \subseteq B,\ x = \bigvee_{\widehat{L}} U\}$ is the MacNeille completion of $B$.*

(iii) *If* $L = \bigcup_{k=1}^{n} B_k$, *where* $B_k$ *are blocks of* $L$ *then* $\widehat{L} = \bigcup_{k=1}^{n} \widehat{B}_k$, *where* $\widehat{B}_k$ *is the MacNeille completion of* $B_k$ *for* $k = 1, 2, \dots, n$.

(iv) $\widehat{L}$ *is a block-finite orthomodular lattice*

## References


[1] L. P. Belluce, *Semi-simple and complete MV-algebras*, Algebra Univ. 29 (1992), 1–9.

[2] G. Birkhoff, *Lattice Theory*, Providence, Rhode Island, 1967.

[3] G. Bruns, *Block-finite orthomodular lattices*, Canad. J. Math. 30 (1978), 315–320.

[4] G. Bruns and R. Greechie, *Orthomodular lattices which can be covered by finitely many blocks*, Canad. J. Math. 34 (1982), 696–699.

[5] C. C. Chang, *Algebraic analysis of many-valued logics*, Trans. Amer. Math. Soc. 88 (1958), 467–490.

[6] C. C. Chang, *A new proof of the completeness of the Lukasiewicz axioms*, Trans. Amer. Math. Soc. 89 (1959), 74–80.

[7] D. J. Foulis, M. K. Bennett, *Effect algebras and unsharp quantum logics*, Found. Phys. 24 (1994), 1331–1352.

[8] R. Giuntini, H. Greuling, *Toward a formal language for unsharp properties*, Found. Phys. 20 (1989), 931–945.

[9] R. J. Greechie, D. Foulis, S. Pulmannová, *The center of an effect algebra*, Order 12 (1995), 91–106.

[10] J. Jakubík, *On complete MV-algebras*, Czech. Math. J. 45 (120) (1995), 473–480.

[11] G. Jenča, Z. Riečanová, *On sharp elements in lattice ordered effect algebras*, BUSEFAL 80 (1999), 24–29.

[12] G. Kalmbach, *Orthomodular Lattices*, Academic Press, London, 1983.

[13] G. Kalmbach, Z. Riečanová, *An axiomatization for abelian relative inverses*, Demonstratio Math. 27 (1994), 769–780.

[14] F. Kôpka, *D-posets of fuzzy sets*, Tatra Mt. Math. Publ. 1 (1992), 83–87.

[15] F. Kôpka, *On compatibility in D-posets*, Inter. J. Theor. Physics 8 (1995), 1525–1531.

[16] F. Kôpka, F. Chovanec, *D-posets*, Math. Slovaca 44 (1994), 21–34.

[17] F. Kôpka, F. Chovanec, *Boolean D-posets*, Tatra Mt. Math. Publ. 10 (1997), 183–197.

[18] D. Mundici, *Interpretation of AFC*-algebras in Lukasiewicz sentential calculus*, J. Funct. Anal. 65 (1986), 15–65.

[19] S. Pulmannová, Z. Riečanová, *Block-finite atomic orthomodular lattices*, J. Pure Appl. Algebra 89 (1993), 295–304.

[20] Z. Riečanová, *Compatibility and central elements in effect algebras*, Tatra Mt. Math. Publ. 16 (1999), 1–8.

[21] Z. Riečanová, *On proper orthoalgebras, difference posets and abelian relative inverse semigroups*, Tatra Mt. Math. Publ. 10 (1997), 119–128.

[22] Z. Riečanová, *Subalgebras, intervals and central elements of generalized effect algebras*, Int. J. Theor. Phys. 38 (1999), 3209–3220.

[23] Z. Riečanová, *MacNeille completions of D-posets and effect algebras*, Int. J. Theor. Phys. 39 (2000), 855–865.

[24] Z. Riečanová, *Generalization of blocks for D-lattices and lattice ordered effect algebras*, Int. J. Theor. Phys. 39 (2000), 231–237.

[25] J. Schmidt, *Zur Kennzeichnung der Dedekind-Mac Neilleschen Hulle einer Geordneten Menge*, Arch. Math. 7 (1956), 241–249.

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