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NONLINEAR FUNDAMENTAL SYSTEMS FOR LINEAR DIFFERENTIAL EQUATIONS IN FRÉCHET SPACES

Abstract. Let E be a Fréchet space. We prove that $ex(E) = ex'(E)$, that is that the IVP $u' = Au + f$, $u(0) = u_0$ is always solvable if the homogeneous problem $u' = Au$, $u(0) = u_0$ is always solvable (even if this solution is not unique). Moreover we prove that there is a continuous, in general nonlinear selection of solutions, which can be applied to prove an existence theorem for $u' = Au + g(\cdot, u)$, $u(0) = u_0$.

1. Introduction

Let E be a real Fréchet space and $(p_n)_{n=1}^{\infty}$ a separating family of seminorms corresponding to the topology of E. Fix $T > 0$, and let $C([0,T], E)$ be polynormed by (q_n) with $q_n(u) = \max\{p_n(u(t)) : t \in [0,T]\}$, let E^* denote the topological dual space of E endowed with the weak$*$ topology $\sigma(E^*, E)$, and let $L(E)$ and $L(E^*)$ be the spaces of all continuous endomorphisms of E and E^*, respectively. Moreover $A^* \in L(E^*)$ denotes the adjoint of $A \in L(E)$ with respect to the duality $\langle E, E^* \rangle$.

For $A \in L(E)$, $f \in C([0,T], E)$ and $u_0 \in E$ we consider the initial value problems (IVPs for short)

(1) $\quad u'(t) = Au(t), \quad u(0) = u_0;$

(2) $\quad u'(t) = Au(t) + f(t), \quad u(0) = u_0.$

It is well known that (1) can have no or more than one solution. Following the notation in [1] and [6] we define:

ex$(E) := \{A \in L(E) : (1)$ is solvable on $[0,T]$ $(u_0 \in E)\}$,
ex$'(E) := \{A \in L(E) : (2)$ is solvable on $[0,T]$ $(u_0 \in E, f \in C([0,T], E))\}$,
unex$(E) := \{A \in L(E) : (1)$ is uniquely solvable on $[0,T]$ $(u_0 \in E)\}$,
ln$(E) := \{A \in L(E) : \exp(tA)u_0$ is convergent $(t \in \mathbb{R}, u_0 \in E)\}$.

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The sets $\mathrm{ex}(E^*)$, $\mathrm{unex}(E^*)$ are defined analogously. Obviously $\ln(E) \subseteq \mathrm{unex}(E) \subseteq \mathrm{ex}(E) \subseteq L(E)$, and strict inclusion can occur in all possible situations, see [1], [6]. In case of $E = \omega$, the space of all real sequences, endowed with the topology of coordinatewise convergence, we have $\mathrm{unex}(\omega) \neq \mathrm{ex}(\omega) = L(\omega)$ and $\mathrm{ex}(\omega) = \mathrm{ex}'(\omega)$ (see [3], [5] and [7]). In this paper we prove:

THEOREM 1. *Let* $A \in \mathrm{ex}(E)$. *Then* $A \in \mathrm{ex}'(E)$ *and there is a continuous function* $S : E \times C([0,T], E) \to C([0,T], E)$, *such that* $S(u_0, f)$ *is a solution of* (2).

THEOREM 2. *Let* $A \in \mathrm{ex}(E)$ *and* $g : [0,T] \times E \to E$ *continuous and compact. Then the IVP*

$$u'(t) = Au(t) + g(t, u(t)), \qquad u(0) = u_0 \tag{3}$$

is solvable on $[0,T]$ $(u_0 \in E)$.

REMARKS: 1). In Theorem 1 it is possible to get a selection $\widetilde{S}$ of solutions such that $\widetilde{S}(0,0) = 0$. Just set $\widetilde{S}(u_0, f) = S(u_0, f) - S(0,0)$.

2). Since ω is a Montel space and since $L(\omega)$ can be represented as the space of all row-finite matrices (see for example [3]), Theorem 2 implies the solvability of IVPs for row-finite systems of differential equations $u' = Au + g(\cdot, u)$, $u(0) = u_0$ with g row-finite continuous and bounded. IVPs for nonautonomous row-finite systems $u' = A(t)u$ can be locally unsolvable [4].

The main tool used to obtain the above results is that a continuous linear and surjective operator between Fréchet spaces has a continuous (in general nonlinear) right inverse (see for example Bourbaki, TVS II, Ch. 4, Proposition 12, [2]).

In the sequel, for $A \in ex(E)$ let

$$F := \{u \in C([0,T], E) : \ u \text{ is differentiable and solves } u'(t) = Au(t)\}.$$

The space F is a closed subspace of $C([0,T], E)$, hence F is a Fréchet space. The operator $P : F \to E$ defined by $P(u) = u(0)$ is linear and continuous, and P is surjective since $A \in \mathrm{ex}(E)$. Now P has a continuous right inverse, that is there is a continuous mapping $Q : E \to F$ with $P \circ Q = id_E$.

It is well known that a linear continuous and surjective operator between Fréchet spaces has a *linear* right inverse if and only if its kernel has a topological complement (see for example [8], p. 109). In our case P has a continuous linear right inverse if $A \in \mathrm{unex}(E)$. If $A \in \mathrm{ex}(E)$ but $A \notin \mathrm{unex}(E)$ in general there is no continuous linear right inverse, but there are examples for that a continuous linear right inverse can exist even in this case (see [1], Example 4.15 and [6]).

2. Proofs

Proof of Theorem 1: Fix $f \in C([0,T],E)$ and let $H : [0,T] \times [0,T] \to E$ be defined by $H(t,s) = Q(f(t))(s)$. Observe that:

a) H is continuous.

Indeed. To see this let us fix $(\tilde{t},\tilde{s}) \in [0,T] \times [0,T]$ and a seminorm q_n. Since Q and f are continuous there exists $\delta > 0$ such that

$$t \in [0,T], |t-\tilde{t}| < \delta \Longrightarrow q_n(Q(f(t)) - Q(f(\tilde{t}))) < \varepsilon,$$

hence

$$p_n(Q(f(t))(s) - Q(f(\tilde{t}))(s)) < \varepsilon \quad (s \in [0,T]).$$

Moreover, the function $Q(f(\tilde{t}))(\cdot) : [0,T] \to E$ is continuous. Therefore we can choose $\delta > 0$ such that in addition

$$p_n(Q(f(\tilde{t}))(s) - Q(f(\tilde{t}))(\tilde{s})) < \varepsilon \quad (s \in [0,T], |s-\tilde{s}| < \delta).$$

Thus we have

$$p_n(H(t,s) - H(\tilde{t},\tilde{s})) < 2\varepsilon \quad (t,s \in [0,T], |t-\tilde{t}| < \delta, |s-\tilde{s}| < \delta).$$

b) The derivative $H_s(t,s)$ exists for each $(t,s) \in [0,T] \times [0,T]$, and

$$H_s(t,s) = AH(t,s) \quad (t,s \in [0,T]).$$

Hence $H_s : [0,T] \times [0,T] \to E$ is continuous.

c) The function $v : [0,T] \to E$ defined by

$$v(t) := \int_0^t H(\tau, t-\tau)\, d\tau$$

is differentiable on $[0,T]$ and $v'(t) = Av(t) + f(t)$ $(t \in [0,T])$.

Indeed. The integral in the definition of v exists (as a Riemann integral) since $\tau \mapsto H(\tau, t-\tau)$ is continuous on $[0,t]$, and $v \in C([0,T],E)$ since H is continuous on $[0,T] \times [0,T]$.

Let $\varphi \in E^*$. We have $\varphi \circ H \in C([0,T] \times [0,T], \mathbb{R})$ and $(\varphi \circ H)(t,s)$ is continuously differentiable in s. Hence

$$\begin{aligned}\frac{d}{dt}\Big(\int_0^t \varphi(H(\tau,t-\tau))\,d\tau\Big) &= \varphi(H(t,0)) + \int_0^t \frac{d}{dt}\varphi(H(\tau,t-\tau))\,d\tau \\ &= \varphi(Q(f(t))(0)) + \varphi\Big(A\int_0^t H(\tau,t-\tau)\,d\tau\Big) \\ &= \varphi(f(t) + Av(t)) \quad (t \in [0,T]).\end{aligned}$$

This proves that v is weakly differentiable with weak derivative $v' = Av + f$. Hence $v' \in C([0,T],E)$ and therefore v is differentiable on $[0,T]$. Thus v solves (2) with $u_0 = 0$;

d) The function $u : [0,T] \to E$, defined by

$$u(t) := Q(u_0)(t) + \int_0^t Q(f(\tau))(t-\tau)\,d\tau$$

is a solution of (2).

e) Let $S(u_0, f)$ be the solution of (2) selected in d). This function $S : E \times C([0,T],E) \to C([0,T],E)$ is continuous.

To prove this let $d : C([0,T],E) \times C([0,T],E) \to \mathbb{R}$ denote the usual metric corresponding to (q_n). First note that $S(\cdot, f) : E \to C([0,T],E)$ is continuous since Q is continuous. Next, let (f_k) be a convergent sequence in $C([0,T],E)$ with limit $f \in C([0,T],E)$. Then

$$Q(f_k(t)) \to Q(f(t)) \ (k \to \infty) \text{ in } C([0,T],E) \text{ uniformly on } [0,T]:$$

Otherwise there would be an $\varepsilon_0 > 0$ and a subsequence (f_{k_j}) of (f_k) and corresponding $t_j \in [0,T]$ such that

$$(*) \qquad d(Q(f_{k_j}(t_j)), Q(f(t_j))) \geq \varepsilon_0.$$

Without loss of generality let $t_j \to t_0$ $(j \to \infty)$.

Since $\lim_{j\to\infty} f_{k_j}(t_j) = f(t_0) = \lim_{j\to\infty} f(t_j)$ we have

$$\lim_{j\to\infty} Q(f_{k_j}(t_j)) = Q(f(t_0)) \text{ in } C([0,T],E)$$

which is a contradiction to $(*)$. This means that for $H_k, H : [0,T] \times [0,T] \to E$, defined by $H_k(t,s) = Q(f_k(t))(s)$, $H(t,s) = Q(f(t))(s)$,

$$H_k \to H (k \to \infty) \text{ uniformly on } [0,T] \times [0,T].$$

Hence

$$\int_0^t H_k(\tau, t-\tau)\,d\tau \to \int_0^t H(\tau, t-\tau)\,d\tau \quad (k \to \infty)$$

uniformly on $[0,T]$. ∎

Proof of Theorem 2. Let $S : E \times C([0,T],E) \to C([0,T],E)$ be as in Theorem 1 and consider the operator $K : C([0,T],E) \to C([0,T],E)$ defined by

$$K(v)(t) := S(u_0, g(\cdot, v))(t) = Q(u_0)(t) + \int_0^t Q(g(\tau, v(\tau)))(t-\tau)\,d\tau.$$

K is continuous, since $v_k \to v$ implies $g(\cdot, v_k) \to g(\cdot, v)$ $(k \to \infty)$ in $C([0,T],E)$. According to the assumptions, there is a compact $M \subseteq E$

such that

$$g([0,T] \times E) \subseteq M.$$

Fix $t \in [0,T]$. For each $v \in C([0,T],E)$ and $\tau \in [0,t]$ we have

$$Q(g(\tau, v(\tau))) \in Q(M),$$

(where $Q(M) \subseteq C([0,T],E)$ is compact), and therefore there is a compact $M_1 \subseteq E$ such that

$$Q(g(\tau, v(\tau)))(t-\tau) \in M_1 \quad (0 \leq \tau \leq t \leq T).$$

The mean value theorem gives

$$\begin{aligned} K(v)(t) &\in Q(u_0)(t) + t \cdot \overline{\mathrm{conv}}(M_1 \cup \{0\}) \\ &\subseteq Q(u_0)([0,T]) + T \cdot \overline{\mathrm{conv}}(M_1 \cup \{0\}) \\ &= : M_2, \end{aligned}$$

where $M_2 \subseteq E$ is compact, so

$$\overline{\{K(v)(t) : v \in C([0,T],E)\}}$$

is a compact subset of E for each $t \in [0,T]$. Moreover

$$\{K(v) : v \in C([0,T],E)\}$$

is equicontinuous since the derivatives $(K(v))'(t)$ are contained in the bounded (even compact) set $AM_2 + M$. Hence

$$\overline{\{K(v) : v \in C([0,T],E)\}}$$

is a compact subset of $C([0,T],E)$ according to Arzelà-Ascoli's theorem. Tychonov's fixed point theorem now implies that there is a function $u \in C([0,T],E)$ such that $K(u) = u$, and u solves (3). ∎

3. Example

Beside row-finite systems, as outlined in the introduction, our results can be applied for example to the following equation:

Consider the Montel–Fréchet space $E = C^\infty([0,1],\mathbb{R})$ (for example $p_n(x) = \max\{|x^{(n)}(s)| : s \in [0,1]\}$ for $n \geq 0$), and let $A \in L(E)$ defined by $Ax = x'$. It is known [6] that $A \in \mathrm{ex}(E)$ but $A \notin \mathrm{unex}(E)$.

For $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ continuous and bounded let $g : [0,1] \times E \to E$ be the continuous and compact function defined by

$$g(t,x)(s) = \int_0^t h(t, x''(\sigma))\, d\sigma.$$

Note that for each (t,x) the function $g(t,x)$ is constant.

According to Theorem 2 each IVP $u' = Au + g(\cdot, u)$, $u(0) = u_0$ is solvable on $[0,1]$, which implies that the problem

$$v_t(t,s) = v_s(t,s) + \int_0^t h(t, v_{ss}(t,\sigma))\, d\sigma, \quad v(0,s) = v_0(s)$$

with $v_0 \in C^\infty([0,1],\mathbb{R})$ has a solution $v : [0,1] \times [0,1] \to \mathbb{R}$ such that $v(t,s)$ is C^1 in t, C^∞ in s and all partial derivatives are continuous on $[0,1] \times [0,1]$.

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