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ON THE EXISTENCE OF AFFINE CONNECTIONS WITH RECURRENT PROJECTIVE CURVATURE

Abstract. We find new examples of locally equiaffine connections with parallel or recurrent projective curvature tensor. Certain applications in the theory of totally geodesic affine immersions are also discussed.

1. Preliminaries

Let M be an n-dimensional differentiable manifold endowed with an affine connection ∇ with no torsion. Denote by R, Ric the Riemann–Christoffel curvature tensor and the Ricci curvature tensor of ∇. We adopt the following convention for the definitions of these objects

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$
$$Ric(X,Y) = \mathrm{Trace}\{ Z \mapsto R(Z,X)Y \}$$

for any $X, Y \in \mathfrak{X}(M)$, $\mathfrak{X}(M)$ being the Lie algebra of vector fields on M.

The Weyl projective curvature tensor P of ∇ is defined by (see [1], [7] or [10])

$$P(X,Y)Z = R(X,Y)Z - (L(X,Y) - L(Y,X))Z + L(Y,Z)X - L(X,Z)Y$$

for any $X, Y, Z \in \mathfrak{X}(M)$, where L is the $(0,2)$-tensor field given by

$$L(X,Y) = -(n^2-1)^{-1}(n\,Ric(X,Y) + Ric(Y,X)).$$

It is classical that P is an invariant with respect to projective transformations of affine connections. P vanishes identically when $\dim M = 2$; and in the case when $\dim M \geq 3$, $P = 0$ if and only if ∇ is locally projectively flat.

An affine connection ∇ is said to be with parallel projective curvature if $\nabla P = 0$. And it is called with recurrent projective curvature if P is non-zero and there is a 1-form ψ (called the recurrence form of P) such that

$$\nabla P = \psi \otimes P. \tag{1}$$

In the above definition, "P is non-zero" means that there is a point on the manifold at which P does not vanish. However, it should be added that any tensor field T satisfying the condition $\nabla T = \psi \otimes T$, for a certain 1-form ψ, must vanish either everywhere or nowhere on M ([11], [12]). Thus, for an affine connection with recurrent projective curvature, tensor P is non-zero at each point of the manifold and $\dim M \geq 3$.

Any locally projectively flat affine connection as well as any affine locally symmetric ($\nabla R = 0$) has parallel projective curvature. And similarly, any affine connection with recurrent curvature ($\nabla R = \varphi \otimes R$, $R \neq 0$) and of dimension ≥ 3 is locally projectively flat (see [13] for affine connections of this kind) or of recurrent projective curvature. It was shown in [4] that (a) there are affine connections with parallel projective curvature which are neither locally projectively flat nor locally symmetric; and (b) there are affine connections with recurrent projective curvature which are neither of recurrent curvature nor of parallel projective curvature. Both the above assertions (a) and (b) do not hold in the class of Levi-Civita connections related to (pseudo-)Riemannian metrics (see [3], [2], [5], [4]).

An affine connection ∇ is said to be locally equiaffine if around each point $x \in M$ there is a parallel volume element, that is, a nonvanishing n-form ω such that $\nabla\omega = 0$. An affine connection with no torsion is locally equiaffine if and only if its Ricci tensor is symmetric (see [6]). In the case when ∇ is a locally equiaffine connection, the Weyl projective curvature tensor P can be expressed in the following way

$$P(X,Y)Z = R(X,Y)Z - (n-1)^{-1}(Ric(Y,Z)X - Ric(X,Z)Y)$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

Our purpose is to find new examples of locally equiaffine connections with parallel or recurrent projective curvature.

2. The connection

Throughout the rest of this paper, we always assume that $n \geq 3$ and Latin indices take on values $1, 2, \ldots, n$, while Greek indices vary on range $2, 3, \ldots, n$.

Let $(x^1, x^2, \ldots, x^n)$ be the Cartesian coordinates in the space $\mathbb{R}^n$. Consider an open, connected subset U of $\mathbb{R}^n$ and a non-zero vector field E on U,

$$E = \textstyle\sum_s f^s \partial_s, \quad \text{where} \quad \partial_s = \frac{\partial}{\partial x^s}.$$

We endow U with an affine connection ∇ by assuming

$$\nabla_{\partial_1}\partial_1 = E, \quad \nabla_{\partial_1}\partial_\alpha = 0, \quad \nabla_{\partial_\alpha}\partial_1 = 0, \quad \nabla_{\partial_\alpha}\partial_\beta = 0.$$

Denoting by Γ_{ij}^k the components of the connection ∇ with respect to the natural basis, we write

$$\Gamma_{ij}^k = f^k \delta_i^1 \delta_j^1.$$

The components of the curvature tensor R and the Ricci tensor *Ric* of ∇ are as follows

$$R_{hij}{}^k = (\partial_h f^k)\delta_i^1\delta_j^1 - (\partial_i f^k)\delta_h^1\delta_j^1,$$
$$R_{ij} = \textstyle\sum_s(\partial_s f^s)\delta_j^1\delta_j^1 - (\partial_i f^1)\delta_j^1.$$

Hence we see that the Ricci tensor is symmetric, if and only if $\partial_\alpha f^1 = 0$.

In the sequel, we suppose that ∇ is equiaffine, so that $\partial_\alpha f^1 = 0$ is fulfilled. Thus, the possible non-zero components of the curvature tensor, the Ricci tensor and the Weyl projective tensor of ∇ are the following

$$R_{\alpha 11}{}^\beta = \partial_\alpha f^\beta, \tag{2}$$
$$R_{11} = \textstyle\sum_\lambda(\partial_\lambda f^\lambda), \tag{3}$$
$$P_{\alpha 11}{}^\beta = \partial_\alpha f^\beta - (n-1)^{-1}\textstyle\sum_\lambda(\partial_\lambda f^\lambda)\delta_\alpha^\beta. \tag{4}$$

Consequently, the only non-zero components of the covariant derivatives of the curvature tensor, the Ricci tensor and the Weyl projective tensor are the following

$$\nabla_i R_{\alpha 11}{}^\beta = \partial_i\partial_\alpha f^\beta - 2f^1\delta_i^1\partial_\alpha f^\beta, \tag{5}$$
$$\nabla_i R_{11} = \partial_i\left(\textstyle\sum_\lambda(\partial_\lambda f^\lambda)\right) - 2f^1\delta_i^1\textstyle\sum_\lambda(\partial_\lambda f^\lambda), \tag{6}$$
$$\begin{aligned}\nabla_i P_{\alpha 11}{}^\beta = {}&\partial_i\left(\partial_\alpha f^\beta - (n-1)^{-1}\textstyle\sum_\lambda\left(\partial_\lambda f^\lambda\right)\delta_\alpha^\beta\right)\\ &- 2f^1\delta_i^1\left(\partial_\alpha f^\beta - (n-1)^{-1}\textstyle\sum_\lambda\left(\partial_\lambda f^\lambda\right)\delta_\alpha^\beta\right).\end{aligned} \tag{7}$$

THEOREM. *Suppose ∇ is a locally equiaffine connection defined by*

$$\nabla_{\partial_1}\partial_1 = E, \quad \nabla_{\partial_1}\partial_\alpha = 0, \quad \nabla_{\partial_\alpha}\partial_1 = 0, \quad \nabla_{\partial_\alpha}\partial_\beta = 0. \tag{8}$$

on an open, connected subset $U \subset \mathbb{R}^n$, where $E = \sum_s f^s\partial_s$ with arbitrary functions $f^2, \ldots, f^n$ and a function f^1 satisfying additionally the condition

$$\partial_\alpha f^1 = 0. \tag{9}$$

(i) Let U be additionally simply connected. If ∇ is of recurrent projective curvature, then there exist a non-zero, trace-free $(n-1)\times(n-1)$-matrix of real constants $[C_\alpha^\beta]$ and functions $g, h : U \to \mathbb{R}$ such that $h > 0$ everywhere on U and

$$\partial_\alpha f^\beta = g\delta_\alpha^\beta + hC_\alpha^\beta. \tag{10}$$

(ii) The converse to (i) holds good without the additional assumption that U is simply connected.

(iii) In (i) as well as in (ii), the recurrence form ψ of P is given by

$$\psi_i = \partial_i \log h - 2f^1\delta_i^1. \tag{11}$$

(iv) In (i) as well as in (ii), the connection ∇ is of recurrent curvature if and only if $g = Dh$, where D is a constant.

Proof. At first, we note that in view of (1), (4) and (7), connection ∇ has recurrent projective curvature tensor P if and only if

$$\partial_i F_\alpha^\beta - 2f^1\delta_i^1 F_\alpha^\beta = \psi_i F_\alpha^\beta \tag{12}$$

for a certain 1-form ψ, where we have assumed

$$F_\alpha^\beta = \partial_\alpha f^\beta - (n-1)^{-1}\textstyle\sum_\lambda(\partial_\lambda f^\lambda)\delta_\alpha^\beta. \tag{13}$$

(i) Suppose that U is simply connected and ∇ is of recurrent projective curvature. Because P is non-zero at every point of U, by (4), the matrix of functions $[F_\alpha^\beta]$ is a non-zero matrix at every point of U. Denoting

$$\widetilde{\psi}_i = \psi_i + 2f^1\delta_i^1, \tag{14}$$

we rewrite system (12) in the following way

$$\partial_i F_\alpha^\beta = \widetilde{\psi}_i F_\alpha^\beta. \tag{15}$$

As an integrability condition of (15), we derive

$$\partial_j\widetilde{\psi}_i - \partial_i\widetilde{\psi}_j = 0. \tag{16}$$

Indeed, by (15), we have

$$0 = \partial_j\partial_i F_\alpha^\beta - \partial_i\partial_j F_\alpha^\beta = (\partial_j\widetilde{\psi}_i - \partial_i\widetilde{\psi}_j)F_\alpha^\beta.$$

Condition (16) means that form $\widetilde{\psi} = \sum_s\widetilde{\psi}_s dx^s$ is closed. By the famous Poincaré theorem (U is here simply connected), there exists a function $\widetilde{h}U :\to \mathbb{R}$ such that

$$\widetilde{\psi}_i = \partial_i\widetilde{h}. \tag{17}$$

Consequently, the only solutions of the system (15) are of the form

$$F_\alpha^\beta = hC_\alpha^\beta, \tag{18}$$

where $h = e^{\widetilde{h}} > 0$ and the real constants C_α^β form a non-zero matrix. The matrix $[C_\alpha^\beta]$ is trace-free since $\sum_\lambda F_\lambda^\lambda = 0$ (cf. (13)). Using (18) and (13), we obtain immediately (10) with

$$g = (n-1)^{-1}\textstyle\sum_\lambda(\partial_\lambda f^\lambda).$$

(ii) It is a straightforward verification, that (10) always implies (12), which is the recurrence of P.

(iii) By (14) and (17), the components of the recurrence form are given by (11).

(*iv*) In view of (2) and (5), the curvature tensor R of ∇ is recurrent if and only if there exists a 1-form φ such that

$$\partial_i(\partial_\alpha f^\beta) - 2f^1\delta_i^1(\partial_\alpha f^\beta) = \varphi_i(\partial_\alpha f^\beta). \tag{19}$$

But by (10), condition (19) holds if and only if

$$(\partial_i g)\delta_\alpha^\beta + (\partial_i h)C_\alpha^\beta = (2f^1\delta_i^1 + \varphi_i)(g\delta_\alpha^\beta + hC_\alpha^\beta),$$

which is clearly equivalent to

$$\partial_i g = (2f^1\delta_i^1 + \varphi_i)g \quad \text{and} \quad \partial_i h = (2f^1\delta_i^1 + \varphi_i)h.$$

The last condition is fulfilled with a certain 1-form φ if and only if there is a constant D such that $g = Dh$. This completes the proof.

In view of the above theorem, we see that to construct examples of locally equiaffine connections with recurrent projective curvature it is suffcient to seek solutions of the system of partial differential equations (10) and (9). We consider the following cases:

CASE 1. Let $U = (a,b) \times V$, where (a,b) is an open interval, $-\infty \le a < b \le +\infty$, V is an open, connected subset of $\mathbb{R}^{n-1}$ and the functions g, h depend on the variable $x^1 \in (a,b)$ only. It is obvious that in this case, functions f^β satisfy (10) if and only if they are of the form

$$f^\beta(x^1, x^2, \ldots, x^n) = g(x^1)x^\beta + h(x^1)\textstyle\sum_\lambda C_\lambda^\beta x^\lambda + k^\beta(x^1), \tag{20}$$

where $k^\beta : (a,b) \to \mathbb{R}$.

It can be easily verified that Mikesh's examples stated in [4] are just of the type desribed in the above case with suitable specified functions g, h and k^β.

CASE 2. Let $n \ge 4$. Suppose that $U = \mathbb{R}^n$ and at every point $x \in \mathbb{R}^n$ not each of $\partial_\alpha h(x)$ vanishes. Then functions f^β satisfy (10) if and only if they are of the form

$$f^\beta(x^1, x^2, \ldots, x^n) = p(x^1)x^\beta + u\left(x^1, \textstyle\sum_\lambda a_\lambda x^\lambda\right) b^\beta + k^\beta(x^1), \tag{21}$$

where a_α, b_β are constants with not each of a_α and b^β vanishing, u is a function of two variables x^1, y, for which $\frac{\partial u}{\partial y}$ is positive everywhere, and p, k^β are functions of one variable. Functions h, g are given here by

$$h(x^1, x^2, \ldots, x^n) = \frac{\partial u}{\partial y}\left(x^1, \textstyle\sum_\lambda a_\lambda x^\lambda\right), \tag{22}$$

$$g(x^1, x^2, \ldots, x^n) = qh(x^1, x^2, \ldots, x^n) + p(x^1), \quad q = \text{const}. \tag{23}$$

Proof. Let f^β be functions fulfilling (10). Then we have

$$\partial_\gamma\partial_\alpha f^\beta = (\partial_\gamma g)\delta_\alpha^\beta + (\partial_\gamma h)C_\alpha^\beta,$$

and therefore

$$(\partial_\gamma g)\delta^\beta_\alpha - (\partial_\alpha g)\delta^\beta_\gamma + (\partial_\gamma h)C^\beta_\alpha - (\partial_\alpha h)C^\beta_\gamma = 0. \tag{24}$$

Since $[C^\beta_\alpha]$ is trace-free, contracting (24) with respect to α and β, we obtain

$$(n-2)\,(\partial_\gamma g) = \textstyle\sum_\lambda (\partial_\lambda h)\, C^\lambda_\gamma.$$

Transvecting (24) with $\partial_\beta h$ and using the last relation, we find

$$(n-3)\,((\partial_\alpha g)(\partial_\beta h) - (\partial_\alpha h)(\partial_\beta g)) = 0.$$

Hence, since $n \geq 4$, we obtain

$$\partial_\alpha g = q\partial_\alpha h \tag{25}$$

for a certain function q. By virtue of (25), relation (24) can be rewritten in the following form

$$(\partial_\gamma h)(q\delta^\beta_\alpha + C^\beta_\alpha) - (\partial_\alpha h)(q\delta^\beta_\gamma + C^\beta_\gamma) = 0,$$

whence it follows that there are functions k^β such that

$$q\delta^\beta_\alpha + C^\beta_\alpha = (\partial_\alpha h)k^\beta. \tag{26}$$

Note that every point of $x \in \mathbb{R}^n$, not each of $k^\beta(x)$ vanishes. Indeed, otherwise we would have $q(x)\delta^\beta_\alpha + C^\beta_\alpha = 0$, which should be an obvious cotradiction. Differentiating (26) partially, we have

$$(\partial_i q)\delta^\beta_\alpha = (\partial_i\partial_\alpha h)k^\beta + (\partial_\alpha h)(\partial_i k^\beta). \tag{27}$$

Note that under any fixed index i, the algebraic rank (with respect to α, β) of the right hand side of (27) does not exceed 2. Therefore, (27) obviously implies $\partial_i q = 0$ and $\partial_i\left((\partial_\alpha h)k^\beta\right) = 0$, i.e., q and $(\partial_\alpha h)k^\beta$ are constants. Choose constants a_α and b^β such that $(\partial_\alpha h)k^\beta = a_\alpha b^\beta$. Thus, (26) takes the form

$$q\delta^\beta_\alpha + C^\beta_\alpha = a_\alpha b^\beta. \tag{28}$$

Moreover, we have $\partial_\alpha h = m a_\alpha$ for a certain function m. Consequently, there is a function $\widetilde{u}$ of two variables x^1 and say y such that $\widetilde{u} > 0$ everywhere and

$$h(x^1, x^2, \ldots, x^n) = \widetilde{u}\left(x^1, \textstyle\sum_\lambda a_\lambda x^\lambda\right). \tag{29}$$

Since q is constant, by (25), the function g takes the form (23), where p is a function of x^1 only. In this situation, with the help of (23) and (28), we modify (10) to the following form

$$\begin{aligned}\partial_\alpha f^\beta &= g\delta^\beta_\alpha + hC^\beta_\alpha = (qh+p)\delta^\beta_\alpha + hC^\beta_\alpha \\ &= p\delta^\beta_\alpha + h(q\delta^\beta_\alpha + C^\beta_\alpha) = p\delta^\beta_\alpha + ha_\alpha b^\beta,\end{aligned}$$

or precisely, using also (29),

$$\partial_\alpha f^\beta(x^1, x^2, \dots, x^n) = p(x^1)\delta_\alpha^\beta + \tilde{u}\left(x^1, \textstyle\sum_\lambda a_\lambda x^\lambda\right) a_\alpha b^\beta.$$

This system has solutions of the form mentioned in (21), where u is a function of two variables x^1, y such that $\frac{\partial u}{\partial y} = \tilde{u}$. Finally, by (29), the function h satisfies (22). The converse is also true. Thus, the proof is complete.

In the above case, we have assumed for simplicity that $U = \mathbb{R}^n$. One can remark that this is not necessary. However, without this assumption, functions realizing (10) can be described by (21) only locally.

CASE 3. Consider the case $n = 3$ and assume that $U = (a, b) \times V$, where (a, b) is an open interval, $-\infty \leq a < b \leq +\infty$, and V is an open, connected and simply connected subset of $\mathbb{R}^2$. In this case, system (10) has solutions if and only if the function h fulfils the following second order partial differential equation

$$C_3^2 \partial_2^2 h + (C_3^3 - C_2^2)\partial_2\partial_3 h - C_2^3 \partial_3^2 h = 0 \tag{30}$$

and the function g fulfils the following system of first order partial differential equations

$$\begin{aligned} &(a)\ \ \partial_2 g = C_2^3 \partial_3 h - C_3^3 \partial_2 h, \\ &(b)\ \ \partial_3 g = C_3^2 \partial_2 h - C_2^2 \partial_3 h. \end{aligned} \tag{31}$$

Then the procedure of finding solution of (10) steps in following way: take a function $h > 0$ realizing (30); next find a function g realizing (31); and finally solve the system (10) finding f^2 and f^3.

Proof. At first, rewrite (10) as follows

$$\partial_2 f^2 = g + hC_2^2, \quad \partial_3 f^2 = hC_3^2, \tag{32}$$

$$\partial_2 f^3 = hC_2^3, \qquad \partial_3 f^3 = g + hC_3^3. \tag{33}$$

The system (32) of two equations with one unknown function f^2 has solution if and only if the following integrability condition

$$\partial_3(g + hC_2^2) - \partial_2(hC_3^2) = 0 \tag{34}$$

is satisfied on V. Similarly, the system (33) of two equations with one unknown function f^3 has solution if and only if the following integrability condition

$$\partial_3(hC_2^3) - \partial_2(g + hC_3^3) = 0 \tag{35}$$

is satisfied on V. Conditions (34), (35) are just the same as (31)(a), (31)(b), respectively. (30) is now a sufficient and necessary condition for (31) to have solution with respect to g with given h. The rest of the assertion is obvious, which completes the proof.

3. Certain applications

One of the authors has proven the following theorem (see [8]): Let $(M, \nabla) \mapsto (\overline{M}, \overline{\nabla})$ be a totally geodesic affine immersion and $(\overline{M}, \overline{\nabla})$ be an affine manifold of recurrent curvature, say $\overline{\nabla}\overline{R} = \overline{\varphi} \otimes \overline{R}$. Then (M, ∇) is (a) locally flat; or (b) of recurrent curvature, precisely $\nabla R = \varphi \otimes R$, φ being the pull-back of the recurrence form $\overline{\varphi}$. Examples connected with this result have been also given in [8].

Moreover, it is proved (see [9]): Let $(M, \nabla) \longmapsto (\overline{M}, \overline{\nabla})$ be a totally geodesic affine immersion and $(\overline{M}, \overline{\nabla})$ be an affine manifold of recurrent projective curvature, say $\overline{\nabla}\overline{P} = \overline{\varphi} \otimes \overline{P}$. Then (M, ∇) is (a) locally projectively flat; or (b) of recurrent projective curvature, precisely $\nabla P = \varphi \otimes P$, φ being the pull-back of the recurrence form $\overline{\varphi}$.

Additional examples illustrating both of the above cited theorems can be constructed in the following way.

Namely, we take the set U and the connection defined in (8) as the ambient affine manifold $(\overline{M}, \overline{\nabla})$. Next we consider its affine submanifold defined by the equation $x^1 =$ const. with $\xi = \partial/\partial x^1$ as the transversal vector field. By (8), it is obvious that the submanifold has vanishing second fundamental form (i.e., it is totally geodesic) and the induced connection is flat. Using our Theorem and functions described in Cases 1–3, the ambient connection can be specified to be of recurrent curvature or of recurrent projective curvature. This gives the desired examples.

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