DEMONSTRATIO MATHEMATICA
Vol. XXXIII No 1 2000

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# ON SEQUENCE SPACES AND $\widehat{S}$-CONVERGENCE

**Abstract.** The purpose of this paper is to introduce some sequence spaces and also give some inclusion relations between sequence spaces and $\widehat{S}$-convergence.

## 1. Introduction

Let $l_\infty$ be the set of all real or complex sequences $x = (x_k)$ with the norm $|||x||| = \sup|x_k| < \infty$. A linear functional $L$ on $l_\infty$ is said to be a Banach limit (see, [1]) if it has the properties,

(i) $L(x) \geq 0$ if $x \geq 0$ (i.e. $x_n \geq 0$ for all $n$)
(ii) $L(e) = 1$, where, $e = (1, 1 \ldots)$
(iii) $L(Sx) = L(x)$,

where the shift operator $S$ is defined by

$$(Sx)_n = x_{n+1}.$$

Let $\mathbf{B}$ be the set of all Banach limits on $l_\infty$. A sequence $x$ is almost convergent to a number $s$ if $L(x) = s$ for all $L \in \mathbf{B}$.

Lorentz [5] has shown that $x$ is almost convergent to $s$ if and only if

$$t_{km} = t_{km}(x) = \frac{x_m + x_{m+1} + \ldots + x_{m+k}}{k+1} \longrightarrow s$$

as $k \to \infty$ uniformly in $m$. We denote the set of all almost convergent sequences by $\widehat{c}$ and we denote the set of all sequences which are almost convergent to zero by $\widehat{c}_0$. Maddox [6, 7] has defined that $x$ is strongly almost convergent to a numbers $s$ if

$$t_{km}(|x-s|) = \frac{1}{k+1}\sum_{i=0}^{k}|x_{i+m} - s| \to 0$$

as $k \to \infty$ uniformly in $m$. We denote the space of all strongly almost convergent sequences by $[\widehat{c}]$ and we denote the space of all sequences which are strongly almost convergent to zero by $[\hat{c}_0]$. It is obvious that

$$[\hat{c}_0] \subset [\hat{c}] \subset \hat{c} \subset l_\infty.$$

Using the concept of almost convergence the following sequence space has been recently introduced and examined by Das and Sahoo [2].

$$[w] = \left\{ x : \frac{1}{n+1} \sum_{k=0}^{n} |t_{km}(x-s)| \to 0 \right. \\ \left. \text{as } n \to \infty \text{ uniformly in } m \text{ for some } s \right\}.$$

Quite recently E. Savaş [17] defined the following sequence space related with the concept of almost convergence

$$[w^1] = \left\{ x : \frac{1}{r+1} \sum_{n=0}^{r} \frac{1}{n+1} \sum_{k=0}^{n} |t_{km}(x-s)| \to 0 \right. \\ \left. \text{as } r \to \infty \text{ uniformly in } m \text{ for some } s \right\}.$$

If we put $s = 0$, we write $[w^0]$ in place of $[w^1]$.

We recall that a function $f : [0, \infty) \to [0, \infty)$ is called a modulus if,

(i) $f(x) = 0$ if and only if $x = 0$;
(ii) $f(x+y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$;
(iii) $f$ is increasing;
(iv) $f$ is continuous from the right at 0.

It is immediate from (ii) and (iv) that $f$ is continuous everywhere on $[0, \infty)$.

A modulus function may be unbounded or bounded. For example, $f(t) = t^p (0 < p \leq 1)$ is unbounded but $f(t) = t/(1+t)$ is bounded.

Ruckle [13] and Maddox [8] and other authors used modulus function to construct new sequence spaces. Recently Mursaleen and Chishti [11] introduced and examined the following sequence spaces

$$[w(f)] = \left\{ x : \frac{1}{n+1} \sum_{k=0}^{n} f(|t_{km}(x-s)|) \to 0 \right. \\ \left. \text{as } n \to \infty \text{ uniformly in } m, \text{ for some } s \right\},$$

where $f$ is a modulus function. Quite recently in [15], E. Savaş defined and studied some sequence spaces by using a modulus $f$.

Now we extend the spaces $[w^1]$ and $[w^0]$ to the spaces $[w^1(f)]$ and $[w^0(f)]$. Then we extend the relationship between the $\widehat{S}$-null sequences and the sequence space $[w^0(f)]$.

Let $f$ be a modulus. We define,

$$[w^1(f)] = \{x : \frac{1}{r+1}\sum_{n=0}^{r}\frac{1}{n+1}\sum_{k=0}^{n} f(|t_{km}(x-s)|) \to 0 \quad \text{as } r \to \infty \text{ uniformly in } m\}.$$

If we put $s = 0$, then we obtain $[w^0(f)]$. Note that, if we put $f(x) = x$ then $[w(f)] = [w]$ and $[w^1(f)] = [w^1]$ which were studied by Das and Sahoo [2] and E. Savaş [17] respectively.

## 2. Main results

We now establish a number of theorems about the sequence spaces which were defined above. We now have

THEOREM 1. *Let $f$ be a modulus. Then $[w(f)]$ and $[w^1(f)]$ are linear spaces.*

Proof. We consider only $[w^1(f)]$. Suppose that $x_k \to s$ in $[w^1(f)]$, $y_k \to s'$ in $[w^1(f)]$ and $\alpha$, $\beta$ are in $C$. Then there exists integers $T_\alpha$ and $R_\beta$ such that $|\alpha| \leq T_\alpha$ and $|\beta| \leq R_\beta$.

We therefore have, uniformly in $m$

$$\frac{1}{r+1}\sum_{n=0}^{r}\frac{1}{n+1}\sum_{k=0}^{n} f(|t_{km}(\alpha x + \beta y - (\alpha s + \beta s'))|)$$

$$\leq T_\alpha \frac{1}{r+1}\sum_{n=0}^{r}\frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{n+1} f(|t_{km}(x-s)|)$$

$$+ R_\beta \frac{1}{r+1}\sum_{n=0}^{r}\frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{n+1} f(|t_{km}(y-s')|)$$

This implies that $\alpha x + \beta y \to \alpha s + \beta s'$ in $[w^1(f)]$.

We have

THEOREM 2. *Let $f$ be any modulus. If $\beta = \lim_{t\to\infty} f(t)/t > 0$, then $[w^1(f)] = [w^1]$.*

Before we proceed to prove Theorem 2 we first state a Lemma.

LEMMA. *Let $f$ be modulus. Let $0 < \delta < 1$. Then for each $x \geq \delta$ we have $f(x) \leq 2f(1)\delta x$.*

The proof follows on the line on Maddox [8].

Proof of Theorem 2. We note that the limit exists for any modulus $f$ by proposition of Maddox [9]. Then $x \in [w^1]$ implies that $\alpha(r,m) = \frac{1}{r+1}\sum_{n=0}^{r}\frac{1}{n+1}\sum_{k=0}^{n}|t_{km}(x-s)| \to 0$ as $r \to \infty$ uniformly in $m$ for some $s$.

For arbitrary $\varepsilon > 0$, choose $\delta$ with $0 < \delta < 1$, such that $f(u) < \varepsilon$ for every $u$ with $0 \le u \le \delta$. We can write for each $m$

$$\begin{aligned}\frac{1}{r+1}\sum_{n=0}^{r}\frac{1}{n+1}\sum_{k=0}^{n} f(|t_{km}(x-s)|) \\ &= \frac{1}{r+1}\sum_{n=0}^{r}\frac{1}{n+1}\sum_{\substack{k=0\\ |t_{km}(x-s)\le\delta}}^{n} f(|t_{km}(x-s)|) \\ &+ \frac{1}{r+1}\sum_{n=0}^{r}\frac{1}{n+1}\sum_{\substack{k=0\\ |t_{km}(x-s)>\delta}}^{n} f(|t_{km}(x-s)|) \\ &\le \varepsilon + 2f(1)\delta^{-1}\alpha(r,m) \to 0,\end{aligned}$$

by the lemma as $r \to \infty$ uniformly in $m$. Therefore $x \in [w^1(f)]$.

Note that in this part of the proof we do not need $\beta > 0$.

Now suppose that $\beta > 0$ and $x \in [w^1(f)]$, since this $\beta > 0$, we have $f(t) \ge \beta t$ for all $t \ge 0$. It follows that $x \in [w^1(f)]$ implies that $x \in [w^1]$.

THEOREM 3. *Let $f$ be any modulus. Then $[w(f)] \subset [w^1(f)]$.*

Proof. $x \in [w(f)]$ and let $\frac{1}{n+1}\sum_{k=0}^{n} f(|t_{km}(x-s)|) \to 0$ as $n \to \infty$ uniformly in $m$, this implies that its arithmetic mean also converges to 0 as $r \to \infty$ uniformly in $m$. This completes the proof.

Recall that if $x$ is a sequence of complex number, we say that $x$ is statistically convergent to the number $s$ provided that for each $\varepsilon > 0$,

$$\lim_{n\to\infty}\frac{1}{n}|\{k \le m : |x_k - s| \ge \varepsilon\}| = 0 \quad \text{for each } \varepsilon > 0,$$

where the larger vertical bars indicate the number of elements in the enclosed set. The set of all statistical convergent sequences is denoted by $s$, [see, 4].

Over the years, statistical convergence has been studied in number theory [3] and trigonometric series [18]. It has also been considered in locally convex spaces [10].

Quite recently E. Savaş [16] defined $\widehat{S}$-convergence as follows:

A sequence $x = (x_k)$ is said to be $\widehat{S}$-convergence to 0 if for every $\varepsilon > 0$ $\lim_{n\to\infty}\frac{1}{n}|\{0 \le k \le n : |t_{km}(x)| \ge \varepsilon\}| = 0$, uniformly in $m$. The set of all $\widehat{S}$-convergence is denoted by $\overline{S}_0$.

We now have

THEOREM 4. *$\overline{S}_0 \subset [w^0(f)]$ if $f$ is bounded.*

Proof. Suppose that $f$ is bounded and that $x \in \overline{S}_0$. Since $f$ is bounded there exists an integer $K$ such that $f(t) < K$ for all $t \ge 0$. Let $\varepsilon > 0$. Then for each $m$ we have

$$\frac{1}{r+1}\sum_{n=0}^{r}\frac{1}{n+1}\sum_{k=0}^{n}f(|t_{km}(x)|)$$
$$=\frac{1}{r+1}\sum_{n=0}^{r}\frac{1}{n+1}\sum_{\substack{k=0\\|t_{km}(x)|\geq\varepsilon}}^{n}f(|t_{km}(x)|)$$
$$+\frac{1}{r+1}\sum_{n=0}^{r}\frac{1}{n+1}\sum_{\substack{k=0\\|t_{km}(x)|<\varepsilon}}^{n}f(|t_{km}(x)|)$$
$$\leq\frac{1}{r+1}\sum_{\substack{n=0\\|t_{km}(x)|\geq\varepsilon}}^{r}\frac{1}{n+1}\cdot K|\{0\leq k\leq n:|t_{km}(x)|\geq\varepsilon\}|+f(\varepsilon).$$

We now select $N_\varepsilon$ such that

$$\frac{1}{n+1}|\{0\leq k\leq n:|t_{km}(x)|\geq\varepsilon\}|<\frac{\varepsilon}{K}$$

for each $m$ and $n>N_\varepsilon$. Now for $n>N_\varepsilon$ we have

$$\frac{1}{r+1}\sum_{n=0}^{r}\frac{1}{n+1}\sum_{k=0}^{n}f(|t_{km}(x)|)\leq\frac{1}{r+1}\sum_{n=0}^{r}\cdot K\cdot\frac{\varepsilon}{K}+f(\varepsilon)=\varepsilon+f(\varepsilon)$$

and so letting $\varepsilon\to 0$, the result follows. This completes the proof.

**Acknowledgements.** This paper was written when the author was at University of Florida of U.S.A. He expresses his gratitude to the Department of Mathematics, University of Florida and World Bank whose financial support made this visit possible and he is also grateful to the referee for many useful comments that improved the presentation of the paper.

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*Received August 5, 1998; revised version September 13, 1999.*