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A GENERAL COINCIDENCE THEOREM FOR COMPATIBLE MULTIVALUED MAPPINGS SATISFYING AN IMPLICIT RELATION

1. Introduction

Let (X, d) be a metric space. We denote by $CB(X)$ the set off all nonempty closed bounded subsets of (X, d) and by H the Hausdorff-Pompeiu metric on $CB(X)$, i.e.

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\},$$

where $A, B \in CB(X)$ and

$$d(x, A) = \inf_{y\in A}\{d(x, y)\}.$$

It is well known that $(CB(X), H)$ is a metric space and the completeness of (X, d) implies the completeness of $(CB(X), H)$.

Let $A, B \in CB(X)$ and $k > 1$. In the sequel the following well known fact will be used [4]: for each $a \in A$, there is $b \in B$ such that $d(a, b) \leq kH(A, B)$.

Let S and T be two self mappings of a metric space (X, d). Sessa [5] defines S and T to be weakly commuting if $d(STx, TSx) \leq d(Tx, Sx)$ for all x in X. Jungck [2] defines S and T to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$$

for some $x \in X$. Clearly, commuting mappings are weakly commuting and

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weakly commuting mappings are compatible, but neither implication is reversible (Ex. 1 [6] and Ex. 2.2 [2]).

Kaneko and Sessa extend the definition of compatibility to include multivalued mappings in the following way.

DEFINITION 1 [3]. The mappings $f : X \to X$ and $T : X \to CB(X)$ are compatible if $fTx \in CB(X)$ for all $x \in X$ and

$$\lim_{n\to\infty} H(Tfx_n, fTx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Tx_n = M \in CB(X)$$

and

$$\lim_{n\to\infty} f(x_n) = t \in M.$$

DEFINITION 2. Consider $f : X \to X$ and $T : X \to CB(X)$. A point $z \in X$ is called a coincidence point of f and T if $fz \in Tz$.

THEOREM A [3]. *Let (X, d) be a complete metric space, $f : X \to X$ and $T : X \to CB(X)$ be compatible continuous mappings such that $T(X) \subset f(X)$ and $H(Tx, Ty) \leq h.\max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}(d(fx, Ty) + d(fy, Tx))\}$ for all x, y in X, where $h \in (0, 1)$. Then there exists a coincidence point of f and T.*

THEOREM B [1]. *Let (X, d) be a complete metric space. Let $f, g : X \to X$ and $S, T : X \to CB(X)$ be continuous mappings such that f is compatible with S and g is compatible with T. Assume that $S(X) \subset g(X), T(X) \subset f(X)$ and that $H(Sx, Ty) \leq rd(fx, gy)$ for all $x, y \in X$, where $0 < r < 1$. Then there is a coincidence point for f and S, as for g and T.*

The purpose of this paper is to extend and improve Theorems A and B.

2. Implicit relations

Let $\widetilde{\mathcal{F}}$ be the set of all functions $f : R_+^6 \to R$ satisfying the following conditions:

$\widetilde{\mathcal{F}}_1 : F(t_1, \ldots, t_6)$ is decreasing in variables $t_2, \ldots, t_6$;

$\widetilde{\mathcal{F}}_2$: there exist $h \in (0, 1)$ and $k > 1$ with $hk < 1$ such that the inequality (F_a): $u \leq kt$ and $F(t, v, v, u, u + v, 0) \leq 0$, or ($F_b$): $u \leq kt$ and $F(t, v, u, v, 0, u + v) \leq 0$ implies $t \leq hv$.

Ex.1. $F(t_1, \ldots, t_6) = t_1 - m \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}$, where $m \in (0, 1)$.

$\widetilde{\mathcal{F}}_1$: Obviously.

$\widetilde{\mathcal{F}}_2$: Let be $u > 0, u \leq kt$ and

$$F(t, v, v, u, u + v, 0) = t - m \max\{u, v, \frac{1}{2}(u + v)\} \leq 0,$$

where $1 < k < \frac{1}{m}$. If $u \geq v$ then $u \leq kt \leq kmu < u$, a contradiction. Thus $v > u$ and $t \leq mv = hv$, where $h = m < 1$. Similarly, $u > 0, u \leq kt$ and $F(t, v, u, v, 0, u+v) \leq 0$ imply $t \leq hv$. If $u = 0$, then $u \leq v$ and $t \leq mv = hv$.

Ex.2. $F(t_1, \ldots, t_6) = t_1 - m[\max\{t_2^2, t_3t_4, t_5t_6, t_3t_5, t_4t_6)\}]^{1/2}$, where $0 < m < \frac{1}{\sqrt{2}}$.

$\widetilde{\mathcal{F}}_1$: Obviously.

$\widetilde{\mathcal{F}}_2$: Let be $u > 0$, $u \leq kt$ and $F(t, v, v, u, u+v, 0) = t - m[\max\{v^2, uv, 0, v(u + v)\}]^{1/2} \leq 0$, with $h = \sqrt{2}m \in (0, 1)$, $k > 1$ and $kh < 1$. If $u \geq v$ then $u \leq kt \leq km\sqrt{2}u < u$, a contradiction. Thus $v > u$ and $t \leq m\sqrt{2}v = hv$. Similarly, $u > 0, u \leq kt$ and $F(t, v, u, v, 0, u + v) \leq 0$ imply $t \leq hv$. If $u = 0$, then $u \leq v$ and $t \leq m\sqrt{2}v = hv$.

Ex.3. $F(t_1, \ldots, t_6) = t_1^2 + \frac{t_1}{t_5t_6+1} - (at_2^2 + bt_3^2 + ct_4^2)$, where $a > 0$, $b > 0$, $c > 0$ and $a + b + c < 1$.

$\widetilde{\mathcal{F}}_1$: Obviously.

$\widetilde{\mathcal{F}}_2$: Let be $u > 0$, $v \geq 0$, $u \leq kt$ and $F(t, v, v, u, u + v, 0) \leq 0$, where $1 < k < \frac{1}{\sqrt{a+b+c}}$. Then $t^2 + t - (av^2 + bv^2 + cu^2) \leq 0$ which implies $t^2 \leq av^2 + bv^2 + cu^2$. If $u \geq v$, then $u^2 \leq k^2t^2 \leq k^2(a + b + c)u^2 < u^2$, a contradiction. Thus $u < v$ and $t \leq \sqrt{a + b + c}v = hv$, where $h = \sqrt{a + b + c} \in (0, 1)$ and $kh < 1$. Similarly, $u > 0, v \geq 0, u \leq kt$ and $F(t, v, u, v, 0, u + v) \leq 0$ imply $t \leq hv$. If $u = 0$, then $u \leq v$ and $t \leq \sqrt{a + b + c}v = hv$.

3. Main result

THEOREM. *Let (X, d) be a complete metric space. Let $f, g : X \to X$ and $S, T : X \to CB(X)$ be continuous mappings such that f is compatible with S and g is compatible with T. Assume $S(X) \subset g(X), T(X) \subset f(X)$ and for all $x, y \in X$*

(1) $F(H(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)) \leq 0$,

where $F \in \widetilde{\mathcal{F}}$. Then f and S and g and T have a common coincidence point.

P r o o f. Let x_0 be an arbitrary but fixed element in X. We shall construct two sequences $\{x_n\}$, $\{y_n\}$ of elements in X and sequence $\{A_n\}$ of elements in $CB(X)$. Since $S(X) \subset g(X)$, there exists $x_1 \in X$ such that $y_1 = gx_1 \in Sx_0$.

Then there exists an element $y_2 = f(x_2) \in Tx_1 = A_1$, because $T(X) \subset f(X)$, such that

$$d(y_1, y_2) = d(gx_1, fx_2) \leq kH(Sx_0, Tx_1).$$

Since $S(X) \subset g(X)$, we may choose $x_3 \in X$ such that $y_3 = gx_3 \in Sx_2 = A_2$ and

$$d(y_2, y_3) \leq kH(Tx_1, Sx_2).$$

By induction we produce the sequences $\{x_n\}$, $\{y_n\}$ and $\{A_n\}$ such that

$$y_{2k+1} = gx_{2k+1} \in Sx_{2k} = A_{2k}, \tag{2}$$

$$y_{2k+2} = fx_{2k+2} \in Tx_{2k+1} = A_{2k+1}, \tag{3}$$

$$d(y_{2k+1}, y_{2k}) \leq kH(Sx_{2k}, Tx_{2k-1}), \tag{4}$$

$$d(y_{2k+1}, y_{2k+2}) \leq kH(Sx_{2k}, Tx_{2k+1}) \quad \text{for every } k \in N. \tag{5}$$

Letting $x = x_{2k}, y = x_{2k+1}$ in condition (1), we have succesively

$$F(H(Sx_{2k}, Tx_{2k+1}), d(fx_{2k}, gx_{2k+1}), d(fx_{2k}, Sx_{2k}), d(gx_{2k+1}, Tx_{2k+1}), d(fx_{2k}, Tx_{2k+1}), d(gx_{2k+1}, Sx_{2k})) \leq 0,$$

$$F(H(Sx_{2k+1}, Tx_{2k+1}), d(y_{2k}, y_{2k+1}), d(y_{2k}, y_{2k+1}), d(y_{2k+1}, y_{2k+2}), d(y_{2k}, y_{2k+2}), d(y_{2k+1}, y_{2k+1})) \leq 0.$$

Thus

$$F(H(Sx_{2k}, Tx_{2k+1}), d(y_{2k}, y_{2k+1}), d(y_{2k}, y_{2k+1}), d(y_{2k+1}, y_{2k+2}), d(y_{2k}, y_{2k+1}) + d(y_{2k+1}, y_{2k+2}), 0) \leq 0. \tag{6}$$

By (5), (6) and (F_a) we have

$$H(Sx_{2k}, Tx_{2k+1}) = H(A_{2k}, A_{2k+1}) \leq hd(y_{2k}, y_{2k+1}), \tag{7}$$

$$d(y_{2k+1}, y_{2k+2}) \leq khd(y_{2k}, y_{2k+1}). \tag{8}$$

Similarly, by (1), (4) and (F_b), we obtain

$$H(Sx_{2k}, Tx_{2k-1}) = H(A_{2k}, A_{2k-1}) \leq hd(y_{2k-1}, y_{2k}), \tag{9}$$

$$d(y_{2k}, y_{2k+1}) \leq khd(y_{2k-1}, y_{2k}). \tag{10}$$

Since $kh \in (0,1)$ it follows from (8) and (10) that $\{y_n\}$ is a Cauchy sequence. Hence there exists z in X such that $y_n \to z$. Therefore, $gx_{2k+1} \to z$ and $fx_{2k+2} \to z$. Moreover, $gfx_{2k+2} \to gz$, $Sfx_{2k+2} \to Sz$, $fgx_{2k+1} \to fz$ and $Tgx_{2k+1} \to Tz$, by the continuity. Also from (7), (9), $h \in (0,1)$ and the fact that $\{y_n\}$ is a Cauchy sequence it follows that $\{A_k\}$ is a Cauchy sequence in the complete metric space $(CB(X), H)$. Thus $A_k \to A \in CB(X)$. This implies $Tx_{2k+1} \to A$, $Sx_{2k+2} \to A$ and therefore $z \in A$, because

$$d(z, A) = \lim_{n\to\infty} d(y_n, A) \leq \lim_{k\to\infty} H(A_{n-1}, A_n) = 0.$$

Observe that then

$$\lim_{k\to\infty} fx_{2k} = z \in A = \lim_{k\to\infty} Sx_{2k}.$$

Hence, by compatibility of f and S, we have

$$\lim_{k\to\infty} H(fSx_{2k}, Sfx_{2k}) = 0.$$

Therefore

$$\lim_{k\to\infty} d(fy_{2k+1}, Sy_{2k}) = 0.$$

Hence $d(fz, Sz) = 0$, so $fz \in Sz$. Similarly $gz \in Tz$.

Theorem and Ex. 1.3 imply the following result.

COROLLARY1. *Let (X, d) be a complete metric space. Let $f, g : X \to X$ and $S, T : X \to CB(X)$ be continuous mappings such that f is compatible with S and g is compatible with T. Assume that $S(X) \subset g(X), T(X) \subset f(X)$ and for all $x, y \in X$ there is*

$$(1.1) \quad H(Sx, Ty) \le m \max\{d(fx, gy), d(fx, Sx), d(gx, Ty), \tfrac{1}{2}(d(fx, Ty) + d(gy, Sx))\}, \quad \text{where} \quad m \in (0,1),$$

or

$$(1.2) \quad H^2(Sx, Ty) \le m^2 \max\{d^2(fx, gy), d(fx, Sx)d(gy, Ty), d(fx, Ty) d(gy, Sx), d(fx, Ty)d(fx, Sx), d(gy, Ty)d(gy, Sx)\}, \quad \text{where} \quad 0 < m^2 < \tfrac{1}{2},$$

or

$$(1.3) \quad H^2(Sx, Ty) + \frac{H(Sx, Ty)}{d(fx, Ty)d(gy, Sx) + 1} - (ad^2(fx, gy) + bd^2(fx, Sx) + cd^2(gy, Ty)) \le 0, \quad \text{where} \quad a, b, c > 0 \quad \text{and} \quad a + b + c < 1.$$

Then f and S as g and T have a common coincidence point.

REMARK. By Corollary 1 and (1.1) for $S = T$ and $f = g$, we obtain Theorem A. Theorem B follows from Corollary (1) and (1.1), because

$$d(fx, gy) \le \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), \tfrac{1}{2}(d(fx, Ty) + d(gy, Sx))\}.$$

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