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HARMONICITY ON NEARLY TRANS-SASAKI MANIFOLDS

Abstract. In this paper we shall study the harmonicity and D-pluriharmonicity of a (φ, J)-holomorphic map from a nearly trans-Sasaki manifold into an almost Hermitian manifold.

1. Introduction

The theory of harmonic maps on Kähler and Riemannian manifolds is very rich in interesting results ([2]–[4]).

In 1995 S. Ianus and A. M. Pastore have introduced the study of harmonic maps into almost contact metric manifolds. Moreover they introduced the concept of φ-pluriharmonicity in analogy with the known one from the geometry of almost Hermitian manifolds ([7]).

In 1985 Oubina introduced a new class of almost contact metric manifold. Thus a trans-Sasaki manifold is an analogue of a locally conformal Kähler manifold ([11]). A nearly trans-Sasaki manifold is a more general concept.

The purpose of this paper is to study the harmonicity and D-pluriharmonicity on nearly trans-Sasaki manifolds. We shall prove that any (φ, J)-holomorphic map from a nearly trans-Sasaki manifold into a quasi-Kähler manifold is harmonic. We also prove that any (φ, J)-holomorphic map from a nearly trans-Sasaki manifold into a Kähler manifold is D-pluriharmonic.

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2. Preliminaries

In this section we recall some notions of the harmonic theory ([3], [4]).

If $E \to M$ is a smooth vector bundle over a smooth manifold, we shall denote by $\Gamma(E)$ the space of smooth sections of E.

Let $f : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds. Then its differential df is a section of the bundle $TM \otimes f^{-1}TN \to M$.

This bundle has a connection ∇' induced from the Levi-Civita connection ∇^M of M and the pull back connection $\widetilde{\nabla}$. Then applying that connection to df gives the second fundamental form of f

$$\alpha_f(X,Y) = \widetilde{\nabla}_X df(Y) - df(\nabla^M_X Y)$$

for any $X, Y \in \Gamma(TM)$. It easy to prove that α_f is symmetric.

The section $\tau(f) \in \Gamma(f^{-1}(TN))$ defined by

$$\tau(f) = \mathrm{Tr}\,\nabla' df = \sum_{i=1}^{m} \nabla' df(e_i, e_i),$$

where $\{e_i\}$ is an orthonormal frame on TM, is called the tension field of f.

We say that a map $f : (M, g) \to (N, h)$ between two Riemannian manifolds M and N is harmonic if and only if $\tau(f) \equiv 0$ ([3]).

We shall give some classical examples:

1. Any totally geodesic map $f : (M, g) \to (N, h)$ is harmonic ([4]).

2. If $f : (M, g) \to (N, h)$ is an isometric immersion then f is harmonic if and only if it is a minimal immersion ([3]).

3. Any holomorphic or antiholomorphic map $f : M \to N$ between Kähler manifolds is harmonic ([9])

4. If $f : R^m \to R^n$ is a harmonic polinomial map then its restriction $f : S^{m-1} \to S^{n-1}$ is harmonic.

5. Let G_1, G_2 be two Lie groups endowed with bi-invariant Riemannian metrics, then any Lie morphism $f : G_1 \to G_2$ is a harmonic map.

3. Harmonicity on nearly trans-Sasaki manifolds

Let M be a smooth manifold of dimension $2m + 1$. We recall that an almost contact structure on M is a triple (φ, ξ, η) where φ is a tensor field of type $(1, 1)$, ξ is a vector field and η is a 1-form which satisfy (see [1])

$$\varphi^2 = -I + \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1,$$

where I is the identity endomorphism on TM. Then we have

$$\varphi(\xi) = 0 \quad \text{and} \quad \eta \circ \varphi = 0.$$

Furthemore, if g is an associated Riamannian metric on M, that is, a metric which satisfies

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

then we say that (φ, ξ, η, g) is an almost contact metric structure . In such a way we obtain an almost contact metric manifold. The fundamental 2-form Φ of an almost contact metric manifold M is defined by

$$\Phi(X, Y) = g(X, \varphi Y), \; X, Y \in \Gamma(TM).$$

We denote by D the distribution orthogonal to ξ.

Now let V be a C^∞-almost Hermitian manifold with the metric h and almost complex structure J. The Kähler form Ω is given by $\Omega(X,Y) = g(X,JY)$, and the Lee form is the 1-form θ defined by $\theta(X) = \frac{1}{n-1}\delta\Omega(JX)$, where δ the coderivative, $\dim V = 2n$ and $X, Y \in \Gamma(TV)$.

We recall that V is said to be Kähler if $d\Omega = 0$ and $N_J = 0$ and locally conformal Kähler if $d\Omega = \theta \wedge \Omega$ and $N_J = 0$, where N_J denotes the Nijenhuis tensor of J ([12]).

An almost contact manifold $M(\varphi, \xi, \eta)$ is said to be normal if the almost complex structure J on $M \times R$ given by

$$J\left(X, a\frac{d}{dt}\right) = \left(\varphi X - a\xi, \eta(X)\frac{d}{dt}\right),$$

where a is a C^∞ function on $M \times R$, is integrable, which is equivalent to the condition $N_\varphi + 2d\eta \otimes \xi = 0$ where N_φ denotes the Nijenhuis tensor of φ ([1]).

Now let (φ, ξ, η, g) be an almost contact metric structure on M. We define an almost Hermitian structure (J, h) on $M \times R$, where J is the above almost complex structure J and h is the Hermitian metric

$$h\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) = g(X,Y) + ab.$$

An almost contact metric structure (φ, ξ, η, g) is said to be trans-Sasaki if the almost Hermitian structure (J, h) on $M \times R$ is locally conformal Kähler ([11]).

In ([11]) the author proves that if $M(\varphi, \xi, \eta, g)$ is an almost contact metric manifold, then M is trans Sasaki if and only if

$$(\nabla_X \varphi)Y = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\},$$

where $\alpha, \beta \in C^\infty(M)$.

THEOREM 1. *Let $M(\varphi, \xi, \eta, g)$ be an almost contact metric manifold, $N(J,h)$ be an almost Hermitian manifold and $f : M \to N$ be a (φ, J)-holomorphic map. Then we have the following formula*

$$J(\tau(f)) = df(\operatorname{div}\varphi) - \operatorname{Tr}_g \beta, \tag{3.1}$$

where $\beta(X,Y) = (\widetilde{\nabla}_X J)(dfY)$, for any $X, Y \in \Gamma(TM)$.

Proof. We see that df is an 1-form on M which takes values in $f^{-1}(TN)$, so that also $df \circ \varphi$ and $J \circ df$. As f is (φ, J)-holomorphic we have

$$\nabla'(df \circ \varphi) = \nabla'(J \circ df). \tag{3.2}$$

We recall that for an 1-form ω on M we have

$$(\nabla\omega)(X,Y) = (\nabla_X\omega)(Y) = \nabla_X\omega(Y) - \omega(\nabla_X Y) \tag{3.3}$$

for any $X, Y \in \Gamma(TM)$. By using the above relation for $\omega = df \circ \varphi$ we have

$$\begin{aligned}(\nabla'(df \circ \varphi))(X,Y) &= \tilde{\nabla}_X df(\varphi Y) - (df \circ \varphi)(\nabla^M_X Y) \\ &= (\tilde{\nabla}_X df)(\varphi Y) + df(\nabla^M_X \varphi Y) - (df \circ \varphi)(\nabla^M_X Y) \\ &= (\tilde{\nabla}_X df)(\varphi Y) + df((\nabla_X \varphi)(Y)).\end{aligned}$$

Thus we get

$$(\nabla'(df \circ \varphi))(X,Y) = \alpha_f(X, \varphi Y) + df((\nabla_X \varphi)(Y)). \tag{3.4}$$

Now we rewrite (3.3) for $\omega = J \circ df$:

$$\begin{aligned}(\nabla'(J \circ df))(X,Y) &= \tilde{\nabla}_X J(dfY) - (J \circ df)(\nabla^M_X Y) \\ &= (\tilde{\nabla}_X J)(dfY) + J(\tilde{\nabla}_X dfY) - J(df(\nabla^M_X Y)) \\ &= (\tilde{\nabla}_X J)(dfY) + J((\tilde{\nabla} df)(X,Y)).\end{aligned}$$

Thus we get

$$(\nabla'(J \circ df))(X,Y) = (\tilde{\nabla}_X J)(dfY) + J((\alpha_f(X,Y)). \tag{3.5}$$

From (3.2), (3.4) and (3.5) we have

$$J((\alpha_f(X,Y)) + (\tilde{\nabla}_X J)(dfY) = df((\nabla_X \varphi)(Y)) + \alpha_f(X, \varphi Y). \tag{3.6}$$

Now let $\{e_1, \dots, e_m, \varphi e_1, \dots, \varphi e_m, \xi\}$ be an orthonormal local frame on TM. By using the symmetry of the second fundamental form of f we get

$$\begin{aligned}&\sum_{i=1}^{m} \{\alpha_f(e_i, \varphi e_i) + \alpha_f(\varphi e_i, \varphi^2 e_i)\} + \alpha_f(\xi, \varphi \xi) \\ &\qquad = \sum_{i=1}^{m} \{\alpha_f(e_i, \varphi e_i) - \alpha_f(\varphi e_i, e_i)\} = 0.\end{aligned}$$

Finally by taking the trace in (3.6) we get the formula (3.1).

The above result is proved more generally for f-structures (see [6]).

We recall that an almost Hermitian manifold $N(J,h)$ is quasi-Kähler if

$$(\nabla^N_X J)Y + (\nabla^N_{JX} J)JY = 0$$

for any $X, Y \in \Gamma(TN)$. Any Riemannian 3-symmetric space and thus the sphere S^6 is a quasi-Kähler manifold.

Let $M(\varphi, \xi, \eta, g)$ be an almost contact metric manifold. Then M is called nearly trans-Sasaki if

$$\begin{aligned}(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = \alpha\{2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\} \\ -\beta\{\eta(Y)\varphi X + \eta(X)\varphi Y\}.\end{aligned} \tag{3.7}$$

It is clear that any trans-Sasaki, and thus any Sasaki manifold satisfies the above relation.

THEOREM 2. *Let $M(\varphi,\xi,\eta,g)$ be a nearly trans-Sasaki manifold, $N(J,h)$ be a quasi-Kähler manifold and $f: M \to N$ be a (φ, J)-holomorphic map. Then the tension field $\tau(f)$ vanishes and thus f is a harmonic map.*

Proof. Let $\{e_1,\dots,e_m,\varphi e_1,\dots,\varphi e_m,\xi\}$ be an orthonormal φ-adapted local frame on TM. Then

$$\operatorname{div}\varphi = \sum_{i=1}^{m}\{(\nabla_{e_i}\varphi)e_i + (\nabla_{\varphi e_i}\varphi)\varphi e_i\} + (\nabla_\xi\varphi)\xi.$$

From (3.7) we have $(\nabla_{e_i}\varphi)e_i = \alpha\xi$, $(\nabla_{\varphi e_i}\varphi)\varphi e_i = \alpha\xi$, and $(\nabla_\xi\varphi)\xi = 0$ and thus

$$\operatorname{div}\varphi = 2m\alpha\xi.$$

On the other hand as f is (φ, J)-holomorphic we have $J(df\xi) = df(\varphi\xi) = 0$, and thus $df(\xi) = 0$. Now as N is a quasi-Kähler manifold we obtain

$$\begin{aligned}\operatorname{Tr}_g\beta &= \sum_{i=1}^{m}\{(\tilde\nabla_{e_i}J)dfe_i + (\tilde\nabla_{\varphi e_i}J)df\varphi e_i\} + (\tilde\nabla_\xi J)df\xi \\ &= (\nabla^N_{dfe_i}J)dfe_i + (\nabla_{Jdfe_i}J)Jdfe_i = 0.\end{aligned}$$

Finally by using Theorem 1, as $df(\xi) = 0$, we obtain that $\tau(f) = 0$ and hence f is harmonic.

COROLLARY 1. *Let $M(\varphi,\xi,\eta,g)$ be a Sasaki manifold, $N(J,h)$ be a Kähler manifold and $f: M \to N$ be a (φ, J)-holomorphic map. Then f is a harmonic map.*

EXAMPLE ([7]) A Sasaki manifold $M(\varphi,\xi,\eta,g)$ is regular if every point $x \in M$ has a cubical coordinate neighborhood $\mathcal{U}$ such that the integral curves of ξ passing through $\mathcal{U}$ pass through the neighborhood only once. A compact regular Sasaki manifold is a circle bundle $S^1 \to M \to N \simeq M/S^1$ called the Boothby–Wang fibration and N has an induced Kähler structure (J,h) such that the projection map $\pi: M \to N$ is (φ, J)-holomorphic.

A classical example is the Hopf fibration $S^1 \to S^{2n+1} \to P^n(C)$.

4. D-pluriharmonicity on nearly trans-Sasaki manifolds

Let M be Kähler manifold with complex structure J and N a Riemannian manifold. A smooth map $f: M \to N$ is called pluriharmonic if the second fundamental form α_f satisfies

$$\alpha_f(X,Y) + \alpha_f(JX,JY) = 0$$

for any $X, Y \in \Gamma(TM)$. Any pluriharmonic map is a harmonic map ([10]).

In ([7]) S. Ianus and A. M. Pastore considered an analogue concept for the almost contact metric manifolds. If $M(\varphi,\xi,\eta,g)$ is an almost contact metric manifold and N is a Riemannian manifold, a smooth map $f : M \to N$ is called φ-pluriharmonic if

$$\alpha_f(X,Y) + \alpha_f(\varphi X, \varphi Y) = 0 \tag{4.1}$$

for any $X, Y \in \Gamma(TM)$. Furthermore f is said to be D-pluriharmonic if (4.1) holds for any $X, Y \in \Gamma(D)$. It is also known that φ-pluriharmonicity implies harmonicity ([7]).

PROPOSITION 1. *Let $M(\varphi, \xi, \eta, g)$ be a nearly trans-Sasaki manifold, $N(J, h)$ be a Kähler manifold and $f : M \to N$ be a (φ, J)-holomorphic map. Then f is D-pluriharmonic.*

Proof. We recall that for any $X, Y \in \Gamma(TM)$ we have ([2])

$$\tilde{\nabla}_X df(Y) - \tilde{\nabla}_Y df(X) = df([X,Y]).$$

It is obvious that as M is a nearly trans-Sasaki manifold we have

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = \alpha 2g(X,Y)\xi$$

for every $X, Y \in \Gamma(D)$.

As f is (φ, J)-holomorphic and N is a Kähler manifold we get

$$\begin{aligned} \tilde{\nabla}_{\varphi X} df(\varphi Y) &= \tilde{\nabla}_{\varphi X} J df(Y) = J(\tilde{\nabla}_{\varphi X} df(Y)) \\ &= J(\tilde{\nabla}_Y df(\varphi X) + df([\varphi X, Y])) \\ &= -\tilde{\nabla}_Y df(X) + df(\varphi[\varphi X, Y]). \end{aligned}$$

Now by using the above relations and the symmetry of the second fundamental form α_f of f we have

$$2\alpha_f(X,Y) + 2\alpha_f(\varphi X, \varphi Y) = df\{(\nabla_Y \varphi)\varphi X + (\nabla_X \varphi)\varphi Y\}$$

for all $X, Y \in \Gamma(D)$.

We have seen that for a (φ, J)-holomorphic map from an almost contact metric manifold into an almost Hermitian manifold we have $df(\xi) = 0$. Now by using this relation and (3.7) we have

$$\alpha_f(X,Y) + \alpha_f(\varphi X, \varphi Y) = 0$$

for all $X, Y \in \Gamma(D)$, and thus f is D-pluriharmonic.

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