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# SOME METRIC PROPERTIES OF HYPERSPACES

## 1. Introduction

Metric convexity and strong metric convexity are basic notions of distance geometry (see [2]). Let us briefly recall the definitions and summarize some simple facts.

Let $(X, \rho)$ be a metric space. For any pair of distinct points $a, b \in X$, a *metric segment* with the endpoints $a, b$ is a subset of $X$ isometric to the interval $[0, \rho(a, b)]$. For every isometric embedding $f : [0, \rho(a, b)] \to X$ with $f(0) = a$ and $f(\rho(a, b)) = b$, let

$$\triangle_f(a, b) := f([0, \rho(a, b)]). \tag{1.1}$$

For any $a, b \in X$ a point $c \in X$ *lies between* $a, b$ (we write $B_\rho(a, c, b)$) if and only if

$$\rho(a, c) + \rho(c, b) = \rho(a, b).$$

For any $a, b \in X$ a point $c \in X$ such that

$$\rho(a, c) = \rho(c, b) = \frac{1}{2}\rho(a, b)$$

is called a *midpoint* of the pair $\{a, b\}$.

We say that $(X, \rho)$ is (*strongly*) *metrically convex* if every pair of points $a, b \in X$ can be joined by a (unique) metric segment.

**1.1.** *Let $X$ be metrically convex. The union of all metric segments with endpoints $x_1, x_2 \in X$ coincides with the set of points lying between $x_1$ and $x_2$.*

**1.2.** [4, **Lemma 0.1**] *A metric space $(X, \rho)$ is strongly metrically convex if and only if $(X, \rho)$ is metrically convex and every pair of points in $(X, \rho)$ has a unique midpoint.*

For every $a \in X$, $\alpha > 0$ the set

$$\mathbf{B}_\rho(a, \alpha) := \{y \in X; \rho(a, y) \leqslant \alpha\}$$

is called the *ball* with center $a$ and radius $\alpha$.

By $\mathbf{B}(a, \alpha)$ we shall denote the ball with center $a$ and radius $\alpha$ in $\mathbb{R}^n$ with Euclidean metric.

For every $x \in X$ and $A \subset X$, let

$$\rho(x, A) := \inf\{\rho(x, a); a \in A\}$$

and

$$(A)_\alpha := \{x \in X; \rho(x, A) \leqslant \alpha\} \tag{1.2}$$

for any $\alpha > 0$.

Let $\mathcal{C}(X)$ be the set of compact subsets of $X$.

For any nonempty sets $A, B \in \mathcal{C}(X)$ the *Hausdorff distance* is defined by the formula

$$\rho_H(A, B) = \max\{\sup_{a \in A} \rho(a, B), \sup_{b \in B} \rho(b, A)\}. \tag{1.3}$$

It is well known (see [7], p. 48) that

$$\rho_H(A, B) = \inf\{\alpha > 0; A \subset (B)_\alpha \text{ and } B \subset (A)_\alpha\}. \tag{1.4}$$

**1.3 Theorem ([7, Th. 1.8.2.]).** *If $(X, \rho)$ is complete, then so is the metric space $(\mathcal{C}(X), \rho_H)$.*

The present paper concerns the hyperspace $\mathcal{C}^n := \mathcal{C}(\mathbb{R}^n)$ and its subspaces: $\mathcal{K}^n$ consisting of all convex bodies (non-empty, compact, convex subsets) in $\mathbb{R}^n$ and $\mathcal{B}^n$ consisting of all n-balls in $\mathbb{R}^n$.

Evidently $(\mathbb{R}^n, \rho)$ with $\rho$ Euclidean is strongly metrically convex. For distinct $a, b$ the affine segment

$$\triangle(a, b) = \{(1 - t)a + tb; t \in [0, 1]\}$$

is the unique metric segment with the endpoints $a, b$:

$$\triangle(a, b) = \triangle_f(a, b), \tag{1.5}$$

where

$$f(t) = \left(1 - \frac{t}{\rho(a, b)}\right) \cdot a + \frac{t}{\rho(a, b)} \cdot b$$

for every $t \in [0, \rho(a, b)]$.

**1.4.** *Let $(X, \rho)$ be a metrically convex space. Every metric segment in $(X, \rho)$ is strongly metrically convex.*

Proof. Let $x_1, x_2 \in X$ and $\rho(x_1, x_2) = \alpha$ and let $f : [0, \alpha] \to X$ be an isometric embedding.

Evidently, the metric segment $[0, \alpha]$ is strongly metrically convex in $\mathbb{R}$. Since strong metric convexity is invariant under isometries, also $\triangle_f(x_1, x_2)$ is strongly metrically convex in $X$. ■

An *affine segment* in $\mathcal{C}^n$ is defined by means of the Minkowski addition and multiplication: for any distinct $A, B \in \mathcal{C}^n$,

$$\triangle(A, B) = \{(1-t)A + tB; t \in [0, 1]\}.$$

An example of affine segment $\triangle(A, B)$ is presented in Fig. 1

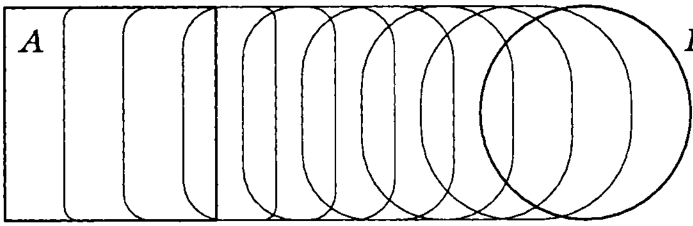


Fig. 1

As we shall see in Section 3, generally $\triangle(A, B)$ is one of many metric segments joining A and B.

**1.5. Definition.** A set $\mathcal{X} \subset \mathcal{C}^n$ is *convex* if and only if $\triangle(A, B) \subset \mathcal{X}$ for every $A, B \in \mathcal{X}$.

The problem of convexity and metric convexity of $\mathcal{K}^n$, $\mathcal{C}^n$ were considered in [1], [3] and [5] - [7]. Most of the authors deal mainly with the space $\mathcal{K}^n$. From their results it follows that $\mathcal{K}^n$ and $\mathcal{C}^n$ are convex and metrically convex, but not strongly metrically convex.

L. Montejano in [6] introduced the notions of hypersegment and hyperconvexity in $\mathcal{K}^n$; these notions differ essentialy from the notion of segment (metric segment) and convexity (metric convexity).

In the present paper we consider the convexity and the (strong) metric convexity of $\mathcal{C}^n$, $\mathcal{K}^n$, $\mathcal{B}^n$ and some of their subsets (Section 2). In Section 3 we give some examples of metric segments in these spaces.

The main results are contained in Section 4. We introduce a partial order in the set of all metric segments with given endpoints in $\mathcal{C}^n$ ($\mathcal{K}^n$, $\mathcal{B}^n$). We find the greatest segment in this set for $\mathcal{C}^n$, $\mathcal{K}^n$ and $\mathcal{B}^n$ and the least segment for $\mathcal{B}^n$. We prove that for $\mathcal{K}^n$ and $\mathcal{C}^n$ generally the least segment does not exist.

## 2. Convexity and metric convexity in $(\mathcal{C}^n, \rho_H)$

We start with some examples of convex subsets of $\mathcal{C}^n$. Evidently

**2.1.** *Every affine segment in $\mathcal{C}^n$ is convex.*
Proof is analogous to that for affine segment in $\mathbb{R}^n$.

We shall prove that every ball in $\mathcal{C}^n$ with convex center is convex.

**2.2 Proposition.** *For every convex $A \in \mathcal{C}^n$ and $\alpha > 0$ the set $\mathbf{B}_{\rho_H}(A, \alpha)$ is convex in $\mathcal{C}^n$.*

Proof. By [5, 1.6 p.238], for every $X, Y \in \mathcal{C}^n$ and $Z \in \mathcal{K}^n$,

$$\rho_H(tX + (1-t)Y, Z) \leqslant t\rho_H(X, Z) + (1-t)\rho_H(Y, Z)$$

for every $t \in [0, 1]$.

Let $C_1, C_2 \in \mathbf{B}_{\rho_H}(A, \alpha)$ and $C = (1-t)C_1 + tC_2$ for some $t \in [0, 1]$. Then

$$\rho_H(A, C) \leqslant (1-t)\rho_H(A, C_1) + t\rho_H(A, C_2) \leqslant \alpha. \quad \blacksquare$$

The following is evident.

**2.3.** *The subspaces $\mathcal{K}^n$ and $\mathcal{B}^n$ are convex in $(\mathcal{C}^n, \rho_H)$.*

Notice that for any $A, B \in \mathcal{C}^n$ the affine segment $\Delta(A, B)$ is also a metric segment. Thus every convex subset of $\mathcal{C}^n$ is metrically convex. In particular,

**2.4.** *$(\mathcal{B}^n, \rho_H)$, $(\mathcal{K}^n, \rho_H)$, and $(\mathcal{C}^n, \rho_H)$ are metrically convex.*

Let us now pass to the notion of strong metric convexity. We start with the following:

**2.5 Lemma.** *The space $(\mathcal{B}^n, \rho_H)$ is isometric to $(\mathbb{R}^n \times \mathbb{R}^+, \bar{\rho})$, where*

$$\bar{\rho}((x_1, t_1), (x_2, t_2)) = \rho(x_1, x_2) + |t_1 - t_2| \tag{2.1}$$

*for every $(x_i, t_i) \in \mathbb{R}^n \times \mathbb{R}^+$, $i = 1, 2$.*

Proof. It is easy to check that for arbitrary $x_1, x_2 \in \mathbb{R}^n$ and $r_1, r_2 > 0$

$$\rho_H(\mathbf{B}(x_1, r_1), \mathbf{B}(x_2, r_2)) = \rho(x_1, x_2) + |r_1 - r_2|. \tag{2.2}$$

Let $h : \mathcal{B}^n \to \mathbb{R}^n \times \mathbb{R}^+$ be defined by the formula

$$h(\mathbf{B}(x, r)) := (x, r). \tag{2.3}$$

Then, by (2.2), the function $h$ is an isometry. $\blacksquare$

Since, evidently, $(\mathbb{R}^n \times \mathbb{R}^+, \bar{\rho})$ is not strongly metrically convex, as a direct consequence of Lemma 2.5 we obtain the following.

**2.6 Theorem.** *The space $(\mathcal{B}^n, \rho_H)$ is not strongly metrically convex.*

This yields the well known result (see [7], p. 59):

**2.7 Corollary.** *$(\mathcal{K}^n, \rho_H)$ and $(\mathcal{C}^n, \rho_H)$ are not strongly metrically convex.*

Every midpoint of a pair $\{A, B\}$ in $\mathcal{C}^n$ belongs to a metric segment with endpoints $A, B$. Hence, to show two different metric segments joining $A$ and $B$, it suffices to show two different midpoints of $\{A, B\}$.

To illustrate Corollary 2.7, we shall now give an example of a pair $\{A, B\}$ with two different midpoints in $\mathcal{C}^n$.

**2.8 Example.** Let $A, B \in \mathcal{B}^2$

$$A = \mathbf{B}((-x, 0), r), \ B = \mathbf{B}((x, 0), r).$$

Let $\rho_H(A, B) = \alpha$, $M := (A)_{\frac{\alpha}{2}} \cap (B)_{\frac{\alpha}{2}}$,

$$M_+ = \{(x, y) \in M; y \geqslant -r\}, \text{ and } M_- = \{(x, y) \in M; y \leqslant r\}.$$

Then

$$\rho_H(A, M_+) = \rho_H(M_+, B) = \frac{\alpha}{2} \text{ and } \rho_H(A, M_-) = \rho_H(M_-, B) = \frac{\alpha}{2}$$

and $M_+ \neq M_-$. Hence $M_+, M_-$ are two different midpoints of the pair $\{A, B\}$.

By 2.2, every ball with a convex center in $\mathcal{C}^n$ is convex; hence, it is metrically convex in $(\mathcal{C}^n, \rho_H)$. However, generally, balls in $(\mathcal{C}^n, \rho_H)$ are not strongly metrically convex.

**2.9 Example.** Let $X = \mathbf{B}(x_0, r)$, $A, B \in \mathbf{B}_{\rho_H}(X, 3r)$, $A = \mathbf{B}(x_1, r)$, $B = \mathbf{B}(x_2, r)$ and $\rho(x_0, x_1) = \rho(x_0, x_2) = r$, $\rho(x_1, x_2) = 2r$.

Evidently, all the metric segments joining $A, B$ are contained in $\mathbf{B}_{\rho_H}(X, 3r)$.

Let $M_+, M_-$ be two midpoints described as in Example 2.8. For these midpoints we can find isometric embeddings $f, g : [0, \rho_H(A, B)] \to \mathcal{C}^n$ such, that

$$M_+ = f(\frac{1}{2}\rho_H(A, B)) \text{ and } M_- = g(\frac{1}{2}\rho_H(A, B)).$$

Since $M_+ \neq M_-$, we found two different metric segments joining $A, B$ and contained in $\mathbf{B}_{\rho_H}(X, 3r)$ (compare 4.2).

In view of 1.4, every metric segment in $\mathcal{C}^n$ is strongly metrically convex.

Schneider in [8] was concerned with metric segments in $\mathcal{K}^n$. His theorem can be formulated as follows:

**2.10 Theorem ([8]).** *For every $K_1, K_2 \in \mathcal{K}^n$ the following conditions are equivalent:*

(i) *there exists a unique metric segment joining $K_1$ and $K_2$;*

(ii) *either $K_1 = (K_2)_r$ or $K_2 = (K_1)_r$ with some $r \geqslant 0$, or else $K_1, K_2$ lie in parallel hyperplanes and $K_1 = K_2 + t$ with some vector $t$ orthogonal to these hyperplanes.*

This theorem provides next examples of strongly metrically convex subsets of $(\mathcal{C}^n, \rho_H)$:

**2.11 Example.** Let $K \in \mathcal{K}^n$. If $S$ is a connected subset of $\mathbb{R}^+$, then the set $\{(K)_\alpha; \alpha \in S\}$ is strongly metrically convex.

**2.12 Example.** Let $K \in \mathcal{K}^n$ and let $K$ lie in a hyperplane $H$ orthogonal to $u \neq o$. If $T$ is a connected subset of $\mathbb{R}$ then the set $\{K + t \cdot u; t \in T\}$ is strongly metrically convex.

## 3. Metric segments in $(\mathcal{C}^n, \rho_H)$

We shall first consider some examples of metric segments in $(\mathcal{B}^n, \rho_H)$.

**3.1 Example.** Let $B_i = \mathbf{B}(x_i, r_i)$ for $i = 1, 2$ and $\lambda := \rho(x_1, x_2) > 0$. We can assume, that $r_2 > r_1$. Let $\delta := r_2 - r_1 > 0$.

Evidently, the following formulae define isometric embeddings $f_j : [0, \rho_H(B_1, B_2)] \to \mathcal{B}^n$ for $j = 1, 2, 3$.

$$f_1(t) = \begin{cases} \mathbf{B}\left(x_1 \cdot \left(1 - \frac{t}{\lambda}\right) + x_2 \cdot \frac{t}{\lambda},\ r_1\right) & \text{for } t \in [0, \lambda] \\ \mathbf{B}\left(x_2,\ \left(1 - \frac{t-\lambda}{\delta}\right) \cdot r_1 + \frac{t-\lambda}{\delta} \cdot r_2\right) & \text{for } t \in [\lambda, \lambda + \delta] \end{cases},$$

$$f_2(t) = \begin{cases} \mathbf{B}\left(x_1,\ r_1 + t\right) & \text{for } t \in [0, \delta] \\ \mathbf{B}\left(x_1 \cdot \left(1 - \frac{t-\delta}{\lambda}\right) + x_2 \cdot \frac{t-\delta}{\lambda},\ r_2\right) & \text{for } t \in [\delta, \lambda + \delta] \end{cases},$$

$f_3(t) = \mathbf{B}\left(s\left(t\right), r\left(t\right)\right)$, where $r\left(t\right) = r_1 + t \cdot \frac{\delta}{\lambda+\delta}$, $s\left(t\right) = \frac{\lambda+\delta-t}{\lambda+\delta} x_1 + \frac{t}{\lambda+\delta} x_2$.

The metric segments $\triangle_{f_1}(B_1, B_2)$, $\triangle_{f_2}(B_1, B_2)$, and $\triangle_{f_3}(B_1, B_2)$ are presented in Fig. 2, 3 and 4, respectively.

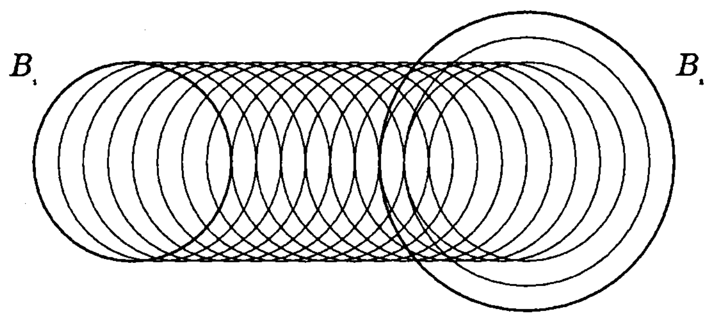


Fig. 2

Notice that $\triangle_{f_3}(B_1, B_2) = \triangle(B_1, B_2)$.

Example 3.1 can be generalized as follows.

**3.2 Example.** Let $B' = \mathbf{B}(x', t')$ and $B'' = \mathbf{B}(x'', t'')$ with $x' \neq x''$ and $t' \leqslant t''$.

Let, further, $\pi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ be the projection, $\pi(x, t) = x$, and let $h$ be defined by (2.3).

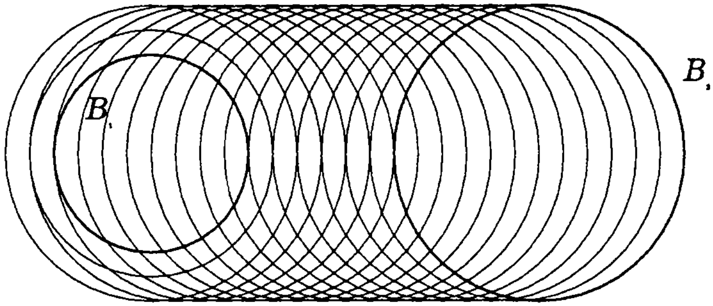


Fig. 3

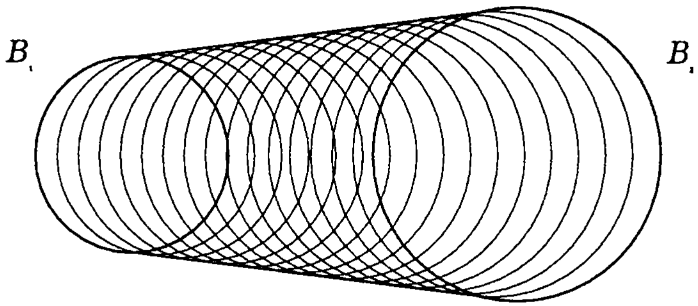


Fig. 4

Consider an arc $L \subset \mathbb{R}^n \times \mathbb{R}^+$ with endpoints $(x', t')$, $(x'', t'')$, satisfying the following conditions:
(i) $\pi(L) = \triangle(x', x'')$,
(ii) $L = \bigcup_{i=1}^{m} L_i$, where $L_i$ is an arc with endpoints $(x_i, t_i)$, $(x_{i+1}, t_{i+1})$, for $i = 1, \ldots, m-1$, $x_1 = x'$, $x_m = x''$, $t_1 = t'$, $t_m = t''$, $t_1 \leqslant t_2 \ldots \leqslant t_m$ and each $L_i$ is either the graph of weakly increasing function or $x_i = x_{i+1}$.

Then $L$ is a metric segment in $(\mathbb{R}^n \times \mathbb{R}^+, \bar{\rho})$, and the set $h^{-1}(L)$ is a metric segment in $\mathcal{B}^n$ with endpoints $B'$, $B''$ (see Fig. 5).

By 1.1, the union of all metric segments in $\mathcal{B}^n$ with endpoints $B_1, B_2$ coincides with the set of balls lying between $B_1$ and $B_2$ in $(\mathcal{B}^n, \rho_H)$:

$$(3.1) \qquad \bigcup_{f \in F} \triangle_f (B_1, B_2) = \{X \in \mathcal{B}^n; B_{\rho_H}(B_1, X, B_2)\},$$

where $F$ is the set of all isometric embeddings of $[0, \rho_H(B_1, B_2)]$ into $\mathcal{B}^n$. We can describe this set as follows:

**3.3.** *A ball* $\mathbf{B}(x, r)$ *lies between* $B_1$ *and* $B_2$ *in* $(\mathcal{B}^n, \rho_H)$ *if and only if* $x \in \triangle(x_1, x_2)$ *and* $r \in [r_1, r_2]$.

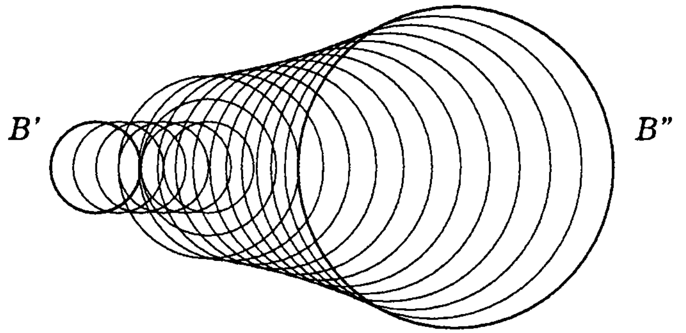


Fig. 5

The set of all balls lying between $B_1$ and $B_2$ in $(\mathcal{B}^2, \rho_H)$ is presented in Fig. 6.

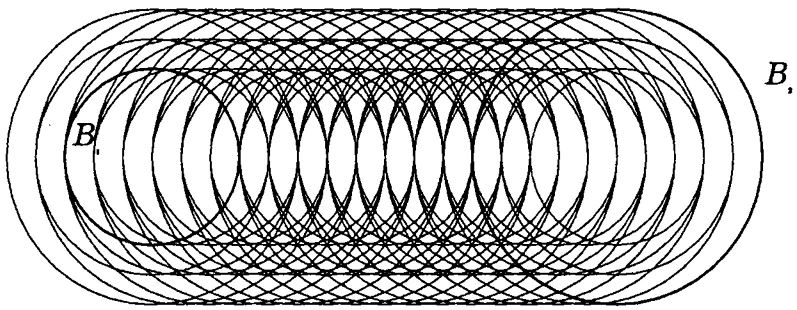


Fig. 6

Let us now consider some examples of metric segments in $(\mathcal{C}^n, \rho_H)$. The first example is provided by the following result of Jongmans.

**3.4 (see [5], p. 241).** *Let $A, B, C \in \mathcal{C}^n$. If $C$ lies between $A$ and $B$ in $(\mathcal{C}^n, \rho_H)$, then the set $\triangle(A, C) \cup \triangle(C, B)$ is a metric segment joining $A$ and $B$.*

**3.5 Example.** Let $A = B_1$, $B = B_2$ as in Example 3.1, and let $C$ be a midpoint of the pair $\{A, B\}$. The metric segment joining $A$ and $B$ described in 3.4 is presented in Fig. 7.

**3.6 Lemma.** *Let $A, B \in \mathcal{C}^n$, $\rho_H(A, B) = \alpha$, and let*

$$M(t) := (A)_t \cap (B)_{\alpha - t} \tag{3.2}$$

*for every $t \in [0, \alpha]$. Then $\rho_H(A, M(t)) = t$ and $\rho_H(M(t), B) = \alpha - t$.*

Proof. Since $A, B$ are compact and $\rho_H$ is continuous, there exist points $a \in A$ and $b \in B$ such that $\rho(a, b) = \rho_H(A, B)$. It is easy to see that $\frac{\alpha - t}{\alpha} a + \frac{t}{\alpha} b \in M(t)$. Thus $M(t) \neq \emptyset$.

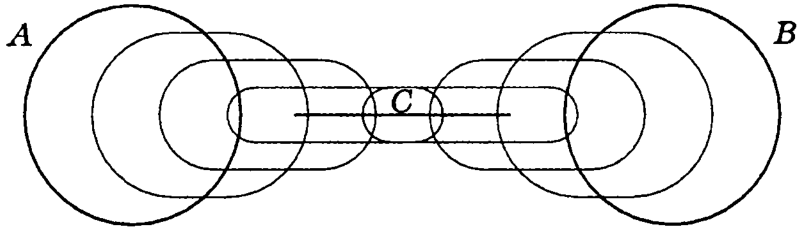


Fig. 7

Let us show that $\rho_H(A, M(t)) = t$ and $\rho_H(M(t), B) = \alpha - t$.

By (3.2)

$$M(t) \subset (A)_t. \tag{3.3}$$

We shall prove the inclusion

$$A \subset (M(t))_t. \tag{3.4}$$

Let $x \in A$. Since $B$ is compact, there is $y \in B$ such that $\rho(x, B) = \rho(x, y)$. By the assumption $\rho(x, y) \leqslant \alpha$. Take $s \in bd\mathbf{B}(x, t) \cap \Delta(x, y)$. Thus $s \in M(t)$ and $x \in \mathbf{B}(s, t)$ which implies $x \in (M(t))_t$.

By (1.4), (3.3) and (3.4)

$$\rho_H(A, M(t)) \leqslant t. \tag{3.5}$$

Analogously we obtain

$$\rho_H(M(t), B) \leqslant \alpha - t. \tag{3.6}$$

By (3.5), (3.6), and triangle inequality,

$$\rho_H(A, M(t)) + \rho_H(M(t), B) = \rho_H(A, B).$$

Finally $\rho_H(A, M(t)) = t$ and $\rho_H(M(t), B) = \alpha - t$. ■

**3.7 Lemma.** *Let $A, B \in C^n$, $\rho_H(A, B) = \alpha$, and let $0 \leqslant t_1 \leqslant t_2 \leqslant \alpha$. Then*

$$M(t_1) = (A)_{t_1} \cap (M(t_2))_{t_2 - t_1} \tag{3.7}$$

*and*

$$M(t_2) = (B)_{\alpha - t_2} \cap (M(t_1))_{t_2 - t_1}, \tag{3.8}$$

*where $M(t_1)$, $M(t_2)$ are defined by* (3.2).

Proof. The inclusions

$$(A)_{t_1} \cap (M(t_2))_{t_2 - t_1} \subset M(t_1)$$

and

$$M(t_1) \subset (A)_{t_1}$$

are obvious. We shall prove the inclusion

$$M(t_1) \subset (M(t_2))_{t_2 - t_1}. \tag{3.9}$$

Let $x \in M(t_1)$. Then there exist $a \in A$ and $b \in B$ such that $x \in \mathbf{B}(a, t_1) \cap \mathbf{B}(b, \alpha - t_1)$.

Take $s \in bd\mathbf{B}(x, t_2 - t_1) \cap \Delta(x, b)$. Thus $\rho(a, s) \leqslant t_2$ and $\rho(s, b) \leqslant \alpha - t_2$. Hence $s \in M(t_2)$ and $x \in \mathbf{B}(s, t_2 - t_1)$. Therefore $x \in (M(t_2))_{t_2 - t_1}$.

Analogously we prove

$$M(t_2) \subset (M(t_1))_{t_2 - t_1}. \tag{3.10}$$

This together with obvious inclusions

$$(B)_{\alpha - t_2} \cap (M(t_1))_{t_2 - t_1} \subset M(t_2)$$

and

$$M(t_2) \subset (B)_{\alpha - t_2}$$

completes the proof of (3.8). ∎

**3.8 Proposition.** *Let $A, B \in C^n$, $\rho_H(A, B) = \alpha$, and let $M : [0, \alpha] \to C^n$ be defined by* (3.2)*. Then $\Delta_M(A, B)$ is a metric segment in $C^n$.*

Proof. Let $0 \leqslant t_1 \leqslant t_2 \leqslant \alpha$. Then $M(t_1) := (A)_{t_1} \cap (B)_{\alpha - t_1}$ and $M(t_2) := (A)_{t_2} \cap (B)_{\alpha - t_2}$. To prove that $M$ is an isometry we have to verify the condition

$$\rho_H(M(t_1), M(t_2)) = t_2 - t_1. \tag{3.11}$$

By (1.4), (3,7) and (3.8)

$$\rho_H(M(t_1), M(t_2)) \leqslant t_2 - t_1. \tag{3.12}$$

By Lemma 3.6

$$\rho_H(A, M(t_i)) = t_i \tag{3.13}$$

for $i = 1, 2$.

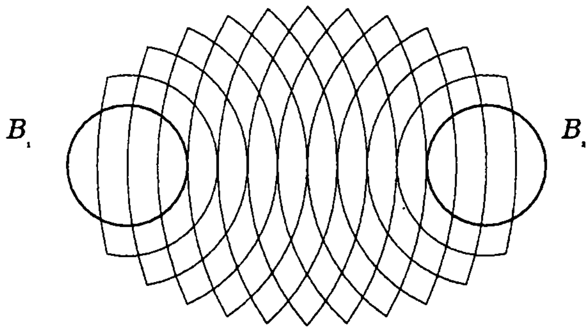

Fig. 8

The triangle inequality, combined with (3.12) and (3.13) yields

$$\rho_H(A, M(t_2)) \leqslant \rho_H(A, M(t_1)) + \rho_H(M(t_1), M(t_2)) \leqslant t_2.$$

Hence, by (3.16), $\rho_H(M(t_1), M(t_2)) = t_2 - t_1$. ■

**3.9 Example.** Let $B_1$, $B_2$ be two balls in $\mathbb{R}^2$ with equal radii. The segment $\triangle_M(B_1, B_2)$ is presented in Fig. 8.

## 4. A partial order in the set of metric segments

We introduce the following relation $\preceq$ in the set of metric segments with given endpoints.

**4.1. Definition.** Let $\mathcal{X}$, $\mathcal{Y}$ be metric segments with endpoints $A, B \in \mathcal{C}^n$ and $\alpha := \rho_H(A, B) > 0$. Then $\mathcal{X} \preceq \mathcal{Y}$ if and only if there exist isometric embeddings $f, g : [0, \alpha] \to \mathcal{C}^n$ such that $f([0, \alpha]) = \mathcal{X}$, $g([0, \alpha]) = \mathcal{Y}$, $f(0) = A = g(0)$, $f(\alpha) = B = g(\alpha)$, and $f(t) \subset g(t)$ for every $t \in [0, \alpha]$.

**4.2 Proposition.** *The relation $\preceq$ is a partial order but is not a linear order.*

Proof. It is easy to check that $\preceq$ is a partial order. We shall show that $\preceq$ is not connected (see Fig. 9).

Let $A = \mathbf{B}(x, r)$, $B = \mathbf{B}(y, r)$, $x = (x_1, \cdots x_{n-1}, 0)$, $y = (y_1, \cdots y_{n-1}, 0)$, and $\alpha := \rho(x, y) > 0$ (then $\alpha := \rho_H(A, B)$). Using Proposition 3.8 we construct two metric segments, $\triangle_{M_+}(A, B)$ and $\triangle_{M_-}(A, B)$, as follows.

Let $M : [0, \alpha] \to \mathcal{C}^n$ be defined by (3.2).
We define $M_+, M_- : [0, \alpha] \to \mathcal{C}^n$ by the formulae

$$M_+(t) = \{(z_1, \cdots z_n) \in M(t); z_n \geqslant -r\}$$

$$M_-(t) = \{(z_1, \cdots z_n) \in M(t); z_n \leqslant r\}.$$

Then $M_+$ and $M_-$ are isometric embeddings. Moreover,
neither $\triangle_{M_+}(A, B) \preceq \triangle_{M_-}(A, B)$ nor $\triangle_{M_-}(A, B) \preceq \triangle_{M_+}(A, B)$. ■

**4.3 Proposition.** *The relation $\preceq$ restricted to $\mathcal{K}^n$ or $\mathcal{B}^n$ is a partial order but is not a linear order.*

Proof. We shall show that $\preceq$ is not connected.

Notice that if $A, B \in \mathcal{K}^n$, then $M_+(t)$, $M_-(t) \in \mathcal{K}^n$ for every $t \in [0, \rho_H(A, B)]$. Hence

$$\triangle_{M_+}(A, B),\ \triangle_{M_-}(A, B) \subset \mathcal{K}^n.$$

Now we shall show that $\preceq$ restricted to $\mathcal{B}^n$ is disconnected.
Let $A, B \in \mathcal{B}^n$, $A = \mathbf{B}(x, r)$, $B = \mathbf{B}(y, R)$, $R > r > 0$ and $r + R < \rho(x, y)$. Let $s := \frac{R+r}{2}$,

$$L = \triangle\left((x, r), (x, s)\right) \cup \triangle\left((x, s), (y, s)\right) \cup \triangle\left((y, s), (y, R)\right),$$

and let $h$ be defined by (2.3).

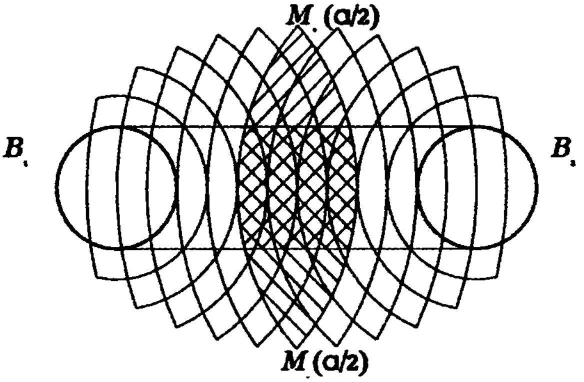


Fig. 9

Then, by 3.2, the set $h^{-1}(L)$ is a metric segment joining $A$ and $B$.

Let us take isometric embedding $f_3$ from Example 3.1.

It is easy to see that neither $\Delta_{f_3}(A,B) \preceq h^{-1}(L)$ nor $h^{-1}(L) \preceq \Delta_{f_3}(A,B)$. ■

**4.4 Proposition.** *Let $M$ be defined by* (3.2). *For every $A, B \in \mathcal{C}^n$ the metric segment $\Delta_M(A,B)$ is the greatest in the sense of relation $\preceq$.*

Proof. Let $A, B \in \mathcal{C}^n$, $\alpha := \rho_H(A,B) > 0$ and let $\Delta_g(A,B)$ be an arbitrary metric segment with the endpoints $A, B$. We shall prove, that $g(t) \subset M(t)$ for every $t \in [0,\alpha]$.

Evidently,

$$\rho_H(A, g(t)) = t, \tag{4.1}$$

and

$$\rho_H(g(t), B) = \alpha - t. \tag{4.2}$$

By (1.4) and (4.1) we obtain

$$g(t) \subset (A)_t;$$

by (1.4) and (4.2) we obtain

$$g(t) \subset (B)_{\alpha-t}.$$

Thus $g(t) \subset (A)_t \cap (B)_{\alpha-t} = M(t)$. ■

Restricting our consideration to the subspace of convex bodies we obtain analogous result:

**4.5 Proposition.** *Let $M$ be defined by* (3.2). *For every $A, B \in \mathcal{K}^n$ the metric segment $\Delta_M(A,B)$ is the greatest in the sense of relation $\preceq$ restricted to $\mathcal{K}^n$.*

Proof. If $A, B \in \mathcal{K}^n$, then $(A)_t, (B)_{\alpha-t} \in \mathcal{K}^n$, whence $(A)_t \cap (B)_{\alpha-t} \in \mathcal{K}^n$. Thus the proof is analogous to that of Proposition 4.4. ■

**4.6 Proposition.** *For every $B_1, B_2 \in \mathcal{B}^n$ the metric segment $\Delta_{f_1}(B_1, B_2)$ from Example* 3.1 *is the least in the sense of relation $\preceq$ restricted to $\mathcal{B}^n$.*

Proof. Let $B_1 = \mathbf{B}(x_1, r_1)$, $B_2 = \mathbf{B}(x_2, r_2)$, $\lambda := \rho(x_1, x_2) > 0$. We can assume that $r_1 < r_2$. Let $\delta := r_2 - r_1 > 0$.

We shall prove that, for any isometry $g : [0, \rho_H(B_1, B_2)] \to \Delta_g(B_1, B_2)$ with $g(0) = B_1$ and $g(\alpha) = B_2$,

$$f_1(t) \subset g(t)$$

for every $t \in [0, \rho_H(B_1, B_2)]$.

Notice that if $g(t) = B(x, r)$, then

$$x \in \Delta(x_1, x_2) \text{ and } r \in [r_1, r_2]. \tag{4.3}$$

Let $a$ and $s$ be the center and the radius of the ball $f_1(t)$. Then

$$a \in \Delta(x_1, x_2) \text{ and } s \in [r_1, r_2]. \tag{4.4}$$

Notice also that, for $t \in [0, \lambda + \delta]$,

$$\rho_H(B_1, f_1(t)) = \rho_H(B_1, g(t)) = t \tag{4.5}$$

and

$$\rho_H(f_1(t), B_2) = \rho_H(g(t), B_2) = \lambda + \delta - t. \tag{4.6}$$

We shall consider two cases.

Case 1: $t \in [0, \lambda]$. Then, by (4.5), we obtain

$$\rho(x_1, a) = \rho(x_1, x) + |r - r_1|,$$

whence the balls $f_1(t)$ and $g(t)$ are internally tangent and from (4.3) it follows that $r \geqslant r_1$. Thus $f_1(t) \subset g(t)$.

Case 2: $t \in [\lambda, \lambda + \delta]$. Then, by (4.3), (4.4) and (4.6) we obtain

$$\rho(x, a) = \rho(x, x_2) = r - s.$$

Hence the balls $f_1(t)$ and $g(t)$ are internally tangent and $r \geqslant s$. Thus $f_1(t) \subset g(t)$. ∎

Analogously we prove the following:

**4.7 Proposition.** *For every $B_1, B_2 \in \mathcal{B}^n$, the metric segment $\Delta_{f_2}(B_1, B_2)$ from Example* 3.1 *is the greatest in the sense of relation $\preceq$ restricted to $\mathcal{B}^n$.*

As a direct consequence of Propositions 4.6 and 4.7 we obtain

**4.8 Corollary.** *For every $B_1, B_2 \in \mathcal{B}^n$*
(i) *there exists the greatest metric segment in $\mathcal{B}^n$ with endpoints $B_1, B_2$;*
(ii) *there exists the least metric segment in $\mathcal{B}^n$ with endpoints $B_1, B_2$.*

By Propositions 4.4 and 4.5, the statement 4.8 (i) remains valid if $\mathcal{B}^n$ is replaced by either $\mathcal{C}^n$ or $\mathcal{K}^n$. We shall now prove that for 4.8 (ii) the situation is opposite.

**4.9 Theorem.** *There exist $A, B \in \mathcal{C}^n$ such that the least metric segment in $\mathcal{C}^n$ with endpoints $A, B$ does not exist.*

Proof. Let $A = \mathbf{B}(x, r)$, $B = \mathbf{B}(y, R)$, $0 < r < R$ and $\rho(x, y) > r + R$. Let $\alpha := \rho_H(A, B) > 0$.

Suppose that $\triangle_g(A, B)$ is the least metric segment joining $A$ and $B$, and $g(0) = A$, $g(\alpha) = B$. Then, by Definition 4.1, for every isometric embeddings $h_1, h_2 : [0, \alpha] \to \mathcal{C}^n$ with $h_i(0) = A$ and $h_i(\alpha) = B$, where $i = 1, 2$,

$$g(t) \subset h_1(t) \cap h_2(t) \tag{4.7}$$

for every $t \in [0, \alpha]$. Since every midpoint of $\{A, B\}$ in $\mathcal{C}^n$ is the value $h(\frac{\alpha}{2})$ of an isometric embedding $h : [0, \alpha] \to \mathcal{C}^n$, it follows that $g(\frac{\alpha}{2})$ is contained in the intersection of arbitrary two midpoints of $\{A, B\}$.

On the other hand, there exist $a, b, c \in \mathbb{R}^n$ such that

$$\begin{aligned} \rho(x,a) &= \tfrac{\alpha}{2} - 2R + r, & \rho(y,a) &= \tfrac{\alpha}{2} + r,\\ \rho(x,b) &= \tfrac{\alpha}{2} - r, & \rho(y,b) &= \tfrac{\alpha}{2} - R + 2r,\\ \rho(x,c) &= \tfrac{\alpha}{2} + r, & \rho(y,c) &= \tfrac{\alpha}{2} - R. \end{aligned}$$

Then the sets $\{a, c\}$ and $\{b, c\}$ are midpoints of $\{A, B\}$, and the set $\{c\} = \{a, c\} \cap \{a, c\}$ does not contain any midpoint of $\{A, B\}$, so, in particular, it does not contain $g(\frac{\alpha}{2})$. ■

**4.10 Theorem.** *There exist $A, B \in \mathcal{K}^n$ such that the least metric segment in $\mathcal{K}^n$ with endpoints $A, B$ does not exist.*

Proof. We follow the idea of proof of Theorem 4.9.

Let $A, B$, and $\alpha$ be as above. Let $a, b, c \in \mathbb{R}^n$ and

$$\begin{aligned} \rho(x,a) &= \tfrac{\alpha}{2} - 2R + r, & \rho(y,a) &= \tfrac{\alpha}{2} + r,\\ \rho(x,b) &= \tfrac{\alpha}{2} - r, & \rho(y,b) &= \tfrac{\alpha}{2} + R,\\ \rho(x,c) &= \tfrac{\alpha}{2} + r, & \rho(y,c) &= \tfrac{\alpha}{2} - R. \end{aligned}$$

Then the sets $\triangle(a, c)$ and $\triangle(b, c)$ are midpoints of $\{A, B\}$, and the set $\{c\} = \triangle(a, c) \cap \triangle(b, c)$ does not contain any midpoint of $\{A, B\}$. ■

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