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DUAL REFLEXIVITY THEOREM IN C^*-MODULES

Introduction

Suppose A is a unital C^*-algebra and $m : A \to L(X)$ is unital bounded algebra homomorphism where $L(X)$ is the algebra of all continuous linear operators on a Banach space X. Our main result is that whenever A is a separable GCR C^*-algebra and X is a reflexive Banach space, then $m^*(A)$ is weak$*$-reflexive. It is a dual version of Theorem 6 in [1]. For unexplained notion and terminology we refer to [2] or [3].

Suppose A is a unital C^*-algebra. We say that a Banach space X is a Banach C^*-module if there is a bilinear mapping from $A \times X$ into X, $(a, x) \to a.x$, satisfying the following conditions:

(i) $1.x = x$;
(ii) $(ab).x = a.(b.x)$;
(iii) $\|a.x\| \leq \|a\|\|x\|$ for each $a, b \in A$, and $x \in X, 1 \in A$.

We let X' denote the dual of a Banach space X. We denote by X'' the second dual of X. Throughout this paper we assume that X is a reflexive Banach space.

THEOREM 1 [1, Theorem 6]. *Suppose X is a reflexive Banach space and A is a separable GCR C^*-algebra and $m : A \to L(X)$ is bounded unital. Then*

$$AlgLat(m(A)) = \overline{m(A)}^{SOT}$$

where SOT is the strong operator topology.

By the assumption in Theorem 1 we can represent A'' as a von Neumann algebra on a separable Hilbert space so that the weak operator topology and the weak $*$ topology coincide. Hence, it is accomplished in the following bilinear mappings:

$$(A) \qquad X \times X' \to A', (x, x') \to (x.x')(a) = x'(a.x), \quad a \in A;$$

$$(B) \qquad A'' \times X' \to X', (a, x') \to (a.x')(x) = a(x.x'), \quad x \in X.$$

Furthermore, (B) defines a Banach A''-module structure on X'. The bilinear map $A \times X \to X, (a, x) \to a.x$, gives the bounded unital algebra homomorphism $m : A \to L(X)$. Also, the bilinear map (B) gives a homomorphism $m^* : A'' \to L(X')$ defined by $m^*(a)x' = a.x'$.

LEMMA 2. *The following assertions are true:*

(i) *For each $a \in A$, $m^*(a)$ is the adjoint in $L(X')$ of the operator $m(a)$ in $L(X)$.*

(ii) *m^* is $w^* - w^*$ operator continuous.*

(iii) *For each x' in X' the linear map from A'' into X' that sends a to $a.x'$ is $w^* - w^*$ continuous.*

Suppose S is a unital subalgebra of $L(X)$ and Y is a subspace of X, then Y is an invariant subspace of S if $TY \subseteq Y$ for every T in S. The set of all invariant subspaces of S is $LatS$, and

$$AlgLatS = \{T : LatS \subseteq LatT\}.$$

PROPOSITION 3. *$T \in AlgLatm^*(A) \Rightarrow T' \in AlgLatm(A)$ where T' is the adjoint of T.*

Proof. Suppose $T \in AlgLatm^*(A)$. Let Y be a closed subspace of X, and $m(a)Y \subseteq Y$ for all $a \in A$. Taking polar both sides we have $m^*(a)Y^\circ \subseteq Y^\circ$, ($Y^\circ$ is the polar of Y). By the hypothesis, $TY^\circ \subseteq Y^\circ$. If we take polar of both sides, we obtain $T'Y \subseteq Y$, (bipolar theorem). Hence, $T' \in AlgLat(m(A))$.

THEOREM 4. *Suppose X is a reflexive Banach space and A is a separable GCR C^*-algebra and $m : A \to L(X)$ is bounded unital. Then*

$$AlgLatm^*(A) = \overline{m^*(A)}^{w^*.O.T}$$

where $w^*.O.T.$ is the weak $*$ operator topology.

Proof. Suppose that $T \in m^*(A)$, Y is a closed subspace in X', and $m^*(a)Y \subseteq Y$ for all $a \in A$. Then $TY \subseteq Y$, i.e., $T \in AlgLatm^*(A)$. Let us take $T \in \overline{m^*(A)}^{w^*OT}$ and Y be a closed subspace in X' such that $m^*(a)Y \subseteq Y$ for all $a \in A$. Then there exists a net (a_α) in A such that $xm^*(a_\alpha)x' \to xTx'$ for all $x' \in X'$ and $x \in X$. Since $m^*(a_\alpha)Y \subseteq Y$ for each α and Y is closed, it follows that $Tx' \in Y$,i.e., $TY \subseteq Y$. Therefore, $T \in AlgLatm^*(A)$.

Let $T \in AlgLatm^*(A)$. By Proposition 3, we have $T' \in AlgLatm(A)$. By Theorem 1 there exists a net (a_α) in A such that $m(a_\alpha) \to T'$ in the strong operator topology. Hence, $x'm(a_\alpha)x \to x'T'x$ for all $x' \in X'$ and $x \in X$. $x'm(a_\alpha)x = (m^*(a_\alpha)x')x$ implies that $T \in \overline{m^*(A)}^{w^*OT}$.

References

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