DEMONSTRATIO MATHEMATICA
Vol. XXXIII No 1 2000

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ON A CLASS OF GENERALIZED FREDHOLM OPERATORS, VII

This note is a continuation of our previous papers [2]–[7]. Our aim is to obtain some perturbation results concerning operators in the class $\Phi_g(X)$. Notations and definitions not explicitly given are taken from [2] and [3]. X always denotes an infinite-dimensional complex Banach space and $\mathcal{A}$ denotes a complex algebra with identity $e \neq 0$. If $\mathcal{B}$ is a complex Banach algebra with identity $e \neq 0$ and $t \in \mathcal{B}$ then we write $\sigma(t)$ and $r(t)$ for the spectrum and the spectral radius of t, respectively. As in [2] and [3] we use the following notations:

$$\begin{aligned}
\mathcal{L}(X) &= \{T : X \to X : T \text{ is linear and bounded}\},\\
\mathcal{F}(X) &= \{T \in \mathcal{L}(X) : \dim T(X) < \infty\},\\
\mathcal{K}(X) &= \{T \in \mathcal{L}(X) : T \text{ is compact}\},\\
\Phi(X) &= \{T \in \mathcal{L}(X) : T \text{ is Fredholm}\},\\
\Phi_g(X) &= \{T \in \mathcal{L}(X) : T \text{ is generalized Fredholm}\},\\
\widehat{\mathcal{L}} &= \mathcal{L}(X)/\mathcal{F}(X), \widetilde{\mathcal{L}} = \mathcal{L}(X)/\mathcal{K}(X),\\
\mathcal{A}^{-1} &= \{r \in \mathcal{A} : r \text{ is invertible}\},\\
\mathcal{A}^{g} &= \{r \in \mathcal{A} : r \text{ is generalized invertible}\}.
\end{aligned}$$

Let $T \in \mathcal{L}(X)$. We write $\widehat{T}$ for the coset $T + \mathcal{F}(X)$ of T in $\widehat{\mathcal{L}}$ and $\tilde{T}$ for the coset $T + \mathcal{K}(X)$ of T in $\widetilde{\mathcal{L}}$.

Recall from [2], Proposition 3.9 that

(1) $\quad t \in \mathcal{A}^g \Leftrightarrow$ there is $s \in \mathcal{A}$ with $tst = t, sts = s$ and $ts = st$.

1991 *Mathematics Subject Classification*: 47 A 11.
Key words and phrases: generalized Fredholm operators.

If $\mathcal{J}$ is a Φ-ideal in $\mathcal{L}(X)$, then it is well-known that

$$T \in \Phi(X) \Leftrightarrow T + \mathcal{J} \in (\mathcal{L}(X)/\mathcal{J})^{-1}.$$

Observe that $\mathcal{F}(X)$ and $\mathcal{K}(X)$ are Φ-ideals in $\mathcal{L}(X)$. Theorem 2.3 in [2] shows that

$$T \in \Phi_g(X) \Leftrightarrow \widehat{T} \in \widehat{\mathcal{L}}^g.$$

The starting point of our investigations in this note is the following

PROPOSITION 1. *Let $t \in \mathcal{A}^g$ and take a pseudo-inverse of t with $e - st - ts \in \mathcal{A}^{-1}$. If $a \in \mathcal{A}t \cap t\mathcal{A}$ and $e - as \in \mathcal{A}^{-1}$ then*

(i) $t - a \in \mathcal{A}^g$,
(ii) $s(e - as)^{-1}$ *is a pseudo-inverse of* $t - a$ *and*
(iii) $e - s(e - as)^{-1}(t - a) - (t - a)s(e - as)^{-1} = e - st - ts \in \mathcal{A}^{-1}$.

Proof. Since $e - as \in \mathcal{A}^{-1}$, we get $e - sa \in \mathcal{A}^{-1}$. Put $b = (e - as)^{-1}$ and $c = (e - sa)^{-1}$. Since $t\mathcal{A} = ts\mathcal{A}$, $\mathcal{A}t = \mathcal{A}st$ and $a \in \mathcal{A}t \cap t\mathcal{A}$, we derive

$$tsa = a = ast. \tag{2}$$

It follows from $c^{-1}s = (e - sa)s = s(e - as) = sb^{-1}$ that

$$cs = sb. \tag{3}$$

Use (2) to obtain

$$(t - a)s = ts - as = ts - tsas = ts(e - as) = tsb^{-1},$$

thus

$$(t - a)sb = ts. \tag{4}$$

Use again (2) to derive

$$s(t - a) = st - sa = st - sast = (e - sa)st = c^{-1}st,$$

hence

$$st = cs(t - a).$$

From (3) we get

$$st = sb(t - a). \tag{5}$$

By (4) and (2),

$$(t - a)sb(t - a) = ts(t - a) = tst - tsa = t - a,$$

hence $sb = s(e - as)^{-1}$ is a pseudo-inverse of $t - a$. Considering (4) and (5), we obtain

$$e - sb(t - a) - (t - a)sb = e - st - ts \in \mathcal{A}^{-1},$$

thus $t - a \in \mathcal{A}^g$. ■

COROLLARY 1. *Let t, s and a as in Proposition 1. If $t \in \mathcal{A}^{-1}$ then $t - a \in \mathcal{A}^{-1}$.*

Proof. If $t \in \mathcal{A}^{-1}$ then $s = t^{-1}$. It follows from (4) and (5) that

$$(t-a)sb = ts = e = st = sb(t-a). \blacksquare$$

PROPOSITION 2. *Let $t \in \mathcal{A}^g$ and $s \in \mathcal{A}$ with the properties in (1). If $a \in \mathcal{A}$ and $ta = at$ then $sa = as$.*

Proof. Since $sta = sat = satst = sat^2s = st^2as = tstas = tas$, it follows that

$$s^2(ta) = s(sta) = s(tas) = (sta)s = (tas)s = (ta)s^2,$$

thus

$$sa = stsa = s^2ta = tas^2 = ats^2 = asts = as. \blacksquare$$

COROLLARY 2. *Suppose that t and s are as in Proposition 2. If $a \in \mathcal{A}t \cup t\mathcal{A}$, $ta = at$ and $e - as \in \mathcal{A}^{-1}$ then*

$$a \in \mathcal{A}t \cap t\mathcal{A}$$

and

$$t - a \in \mathcal{A}^g.$$

Proof. Let $a \in t\mathcal{A} = ts\mathcal{A}$. Then $a = tsa = ats$, by Proposition 2. Thus $a = ats = ast \in \mathcal{A}t$. Similar arguments show that if $a \in \mathcal{A}t$ then $a \in t\mathcal{A}$. Since $(e - st - ts)^2 = (e - 2st)^2 = e$, we have $e - st - ts \in \mathcal{A}^{-1}$. Proposition 1 shows now that $t - a \in \mathcal{A}^g$. $\blacksquare$

Now we turn to the operator situation. Recall from [3], (3.6) that if $T \in \Phi_g(X)$ then there is $S \in \mathcal{L}(X)$ such that

$$TST = T,\ STS = S \text{ and } TS - ST \in \mathcal{F}(X).$$

In this case we call S a *$\mathcal{F}$-pseudo-inverse* of T. If $T \in \Phi_g(X)$ and S is a pseudo-inverse of T with $I - ST - TS \in \Phi(X)$ then S is called a *Φ-pseudo-inverse* of T. If S is a $\mathcal{F}$-pseudo-inverse of T, then there is $F \in \mathcal{F}(X)$ with $TS = ST + F$. Thus $I - ST - TS = I - 2ST - F$. Since $(I - 2ST)^2 = I$, $I - 2ST$ is invertible in $\mathcal{L}(X)$ thus $I - ST - TS \in \Phi(X)$. *This shows that each $\mathcal{F}$-pseudo-inverse is a Φ-pseudo-inverse.*

COROLLARY 3. *Suppose that $T \in \Phi_g(X)$ and S is a Φ-pseudo-inverse of T. If $A \in \mathcal{L}(X)$, $I - AS \in \Phi(X)$, $(I - TS)A \in \mathcal{F}(X)$ and $A(I - ST) \in \mathcal{F}(X)$ then*

$$T - A \in \Phi_g(X).$$

Proof. Since $\widehat{T} \in \widehat{\mathcal{L}}^g$, $\hat{I} - \widehat{A}\widehat{S} \in \widehat{\mathcal{L}}^{-1}$, $\widehat{A} = \widehat{T}\widehat{S}\widehat{A} = \widehat{A}\widehat{S}\widehat{T} \in \widehat{T}\widehat{\mathcal{L}} \cap \widehat{\mathcal{L}}\widehat{T}$, we get from Proposition 1 that $\widehat{T} - \widehat{A} \in \widehat{\mathcal{L}}^g$, hence $T - A \in \Phi_g(X)$. $\blacksquare$

Remark. If $A, S \in \mathcal{L}(X)$ then it is well-known that

$$I - AS \in \Phi(X) \Leftrightarrow I - SA \in \Phi(X).$$

COROLLARY 4. *Suppose that $T \in \Phi_g(X)$ and $S \in \mathcal{L}(X)$ is a $\mathcal{F}$-pseudo-inverse of T. If $A \in \mathcal{L}(X)$, $I - AS \in \Phi(X)$, $TA - AT \in \mathcal{F}(X)$ and $A(I - ST) \in \mathcal{F}(X)$ then*

$$T - A \in \Phi_g(X).$$

Proof. We have $\widehat{T} \in \widehat{\mathcal{L}}^g$, $\widehat{T}\widehat{S}\widehat{T} = \widehat{T}$, $\widehat{S}\widehat{T}\widehat{S} = \widehat{S}$, $\widehat{T}\widehat{S} = \widehat{S}\widehat{T}$, $\widehat{I} - \widehat{A}\widehat{S} \in \widehat{\mathcal{L}}^{-1}$, $\widehat{T}\widehat{A} = \widehat{A}\widehat{T}$ and $\widehat{A} = \widehat{A}\widehat{S}\widehat{T} \in \widehat{\mathcal{L}}\widehat{T}$. Corollary 2 gives $\widehat{T} - \widehat{A} \in \widehat{\mathcal{L}}^g$, thus $T - A \in \Phi_g(X)$. ∎

Although Proposition 1, Corollary 1, Corollary 3 and Corollary 4 seem rather special, they contain the classical perturbation results for Fredholm operators. We look at the following two corollaries.

COROLLARY 5. *Suppose that $T \in \Phi(X)$ and S is a pseudo-inverse of T. If $A \in \mathcal{L}(X)$ and $\|A\| < \|S\|^{-1}$ then*

$$T - A \in \Phi(X).$$

Proof. Proposition 1.3 in [2] shows that S is a Φ-pseudo-inverse of T. From $\|A\| < \|S\|^{-1}$ we get

$$I - AS \in \mathcal{L}(X)^{-1} \subseteq \Phi(X).$$

Since $I - ST$ and $I - TS$ are finite-dimensional projections, we get $A(I - ST), (I - TS)A \in \mathcal{F}(X)$. Therefore we have $\widehat{T} \in \widehat{\mathcal{L}}^{-1}$, $\widehat{I} - \widehat{A}\widehat{S} \in \widehat{\mathcal{L}}^{-1}$, $\widehat{A} \in \widehat{\mathcal{L}}\widehat{T} \cap \widehat{T}\widehat{\mathcal{L}}$. Corollary 1 gives therefore that $\widehat{T} - \widehat{A} \in \widehat{\mathcal{L}}^{-1}$, thus $T - A \in \Phi(X)$. ∎

COROLLARY 6. *If $\mathcal{J}$ is a Φ-ideal in $\mathcal{L}(X)$ then*

$$T - A \in \Phi(X) \text{ for each } T \in \Phi(X) \text{ and each } A \in \mathcal{J}.$$

Proof. Take $T \in \Phi(X)$, $A \in \mathcal{J}$ and $S \in \mathcal{L}(X)$ with $TST = T$. Then $AS \in \mathcal{J}$, hence $I - AS \in \Phi(X)$. Now proceed as in the proof of Corollary 5. ∎

We have seen in [2] that $\Phi_g(X) + \mathcal{K}(X) \not\subseteq \Phi_g(X)$. The next corollary contains a perturbation result for compact operators.

COROLLARY 7. *Let $T \in \Phi_g(X)$ with Φ-pseudo-inverse S. If $\mathcal{J}$ is a Φ-ideal in $\mathcal{L}(X)$, $A \in \mathcal{J}$, $(I - TS)A \in \mathcal{F}(X)$ and $A(I - ST) \in \mathcal{F}(X)$ then*

$$T - A \in \Phi_g(X).$$

Proof. Since $AS \in \mathcal{J}$, $I - AS \in \Phi(X)$. Corollary 3 gives $T - A \in \Phi_g(X)$. ∎

COROLLARY 8. *Let T and S as in Corollary 7. If $A \in \mathcal{L}(X)$, $I-AS \in \Phi(X)$, $N(T) \subseteq N(A)$ and $A(X) \subseteq T(X)$ then*

$$T - A \in \Phi_g(X).$$

Proof. Since $(I - ST)(X) = N(T) \subseteq N(A)$, we get $A(I - ST) = 0$. From $A(X) \subseteq T(X) = TS(X)$ we derive $TSA = A$, thus $(I - TS)A = 0$. Corollary 3 gives then $T - A \in \Phi_g(X)$. ∎

COROLLARY 9. *Let $T \in \Phi_g(X)$ and S a $\mathcal{F}$-pseudo-inverse of T. If $A \in \mathcal{L}(X)$, $I - AS \in \Phi(X)$, $TA - AT \in \mathcal{F}(X)$ and $N(T) \subseteq N(A)$ or $A(X) \subseteq T(X)$ then*

$$T - A \in \Phi_g(X).$$

Proof. Case 1: Suppose that $N(T) \subseteq N(A)$, then, as above, $A(I - ST) = 0 \in \mathcal{F}(X)$, hence $\widehat{A} \in \widehat{\mathcal{L}}\widehat{T}$. Corollary 2 gives $T - A \in \Phi_g(X)$.

Case 2: If $A(X) \subseteq T(X)$, then $(I - TS)A = 0 \in \mathcal{F}(X)$, thus $\widehat{A} \in \widehat{T}\widehat{\mathcal{L}}$. Use again Corollary 2 to get $T - A \in \Phi_g(X)$. ∎

COROLLARY 10. *If $T \in \Phi_g(X)$ and if S is a $\mathcal{F}$-pseudo-inverse of T then*

$$T - \lambda S \in \Phi_g(X) \quad \textit{for } |\lambda| < \|S^2\|^{-1}.$$

Proof. Since $\widehat{T}\widehat{S} = \widehat{S}\widehat{T}$, we have $\widehat{S} = \widehat{S}\widehat{T}\widehat{S} = \widehat{S}^2\widehat{T} \in \widehat{\mathcal{L}}\widehat{T}$ and $\widehat{S} = \widehat{T}\widehat{S}^2 \in \widehat{T}\widehat{\mathcal{L}}$. Thus

$$\lambda\widehat{S} \in \widehat{T}\widehat{\mathcal{L}} \cap \widehat{\mathcal{L}}\widehat{T} \text{ for all } \lambda \in \mathbb{C}. \tag{6}$$

If $|\lambda| < \|S^2\|^{-1}$ then $I - \lambda S^2 \in \mathcal{L}(X)^{-1} \subseteq \Phi(X)$, thus

$$\hat{I} - (\lambda\widehat{S})\widehat{S} \in \widehat{\mathcal{L}}^{-1} \text{ for } |\lambda| < \|S^2\|^{-1}. \tag{7}$$

By Corollary 2, (6) and (7) we get $\widehat{T} - \lambda\widehat{S} \in \widehat{\mathcal{L}}^g$, thus $T - \lambda S \in \Phi_g(X)$ if $|\lambda| < \|S^2\|^{-1}$. ∎

Our next result in this paper is a mapping property for operators in $\Phi_g(X)$. Recall that if $T \in \Phi_g(X)$ then, by [2], Proposition 1.3,

$$\operatorname{dist}(0, \sigma_\Phi(T)\backslash\{0\}) > 0.$$

Furthermore, by [3], Theorem 3.10, if $T \in \Phi_g(X)\backslash\mathcal{F}(X)$ and S is a $\mathcal{F}$-pseudo-inverse of T then

$$\operatorname{dist}(0, \sigma_\Phi(T)\backslash\{0\}) = r(\tilde{S})^{-1}. \tag{8}$$

COROLLARY 11. *Let $T \in \Phi_g(X)$ and S a $\mathcal{F}$-pseudo-inverse of T. Suppose that $D \subseteq \mathbb{C}$ is a Cauchy-domain, $0 \in \sigma(T) \subseteq D$, $h : D \to \mathbb{C}$ is holomorphic, $h(0) = 0$ and*

$$\max_{\lambda \in \sigma_\Phi(T)} |\lambda - h(\lambda)| < \operatorname{dist}(0, \sigma_\Phi(T) \setminus \{0\}). \tag{9}$$

Then $h(T) \in \Phi_g(X)$.

Proof. If $T \in \mathcal{F}(X)$, then $\operatorname{dist}(0, \sigma_\Phi(T)\setminus\{0\}) = \infty$. Since $h(0) = 0$, $h(\lambda) = \lambda k(\lambda)$ for some holomorphic $k : D \to \mathbb{C}$. Thus $h(T) = Tk(T) \in \mathcal{F}(X) \subseteq \Phi_g(X)$. Now consider the case where $T \in \Phi_g(X)\setminus\mathcal{F}(X)$. Define $f : D \to \mathbb{C}$ by $f(\lambda) = \lambda - h(\lambda)$. Then $f(0) = 0$, thus $f(\lambda) = \lambda g(\lambda)$ for some holomorphic $g : D \to \mathbb{C}$. This gives $f(T) = Tg(T) = g(T)T$, thus

$$\widehat{f(T)} = \widehat{T}\widehat{g(T)} = \widehat{g(T)}\widehat{T} \in \widehat{T}\widehat{\mathcal{L}} \cap \widehat{\mathcal{L}}\widehat{T}. \tag{10}$$

Furthermore we have

$$\widehat{T}\widehat{f(T)} = \widehat{f(T)}\widehat{T}. \tag{11}$$

Since $\tilde{T}\tilde{S} = \tilde{S}\tilde{T}$ and $\widetilde{f(T)} = f(\tilde{T})$, it follows that

$$r(\widetilde{f(T)}\tilde{S}) \le r(\widetilde{f(T)})r(\tilde{S}) = r(f(\tilde{T}))r(\tilde{S}). \tag{12}$$

From [1] we get

$$r(f(\tilde{T})) = \max_{\lambda \in \sigma_\Phi(T)} |f(\lambda)| = \max_{\lambda \in \sigma_\Phi(T)} |\lambda - h(\lambda)|.$$

Thus by (8), (9) and (12), $r(f(\tilde{T})) < 1$, hence $\tilde{I} - \widetilde{f(T)}\tilde{S} \in \tilde{\mathcal{L}}^{-1}$, therefore $I - f(T)S \in \Phi(X)$ and so $\hat{I} - \widehat{f(T)}\widehat{S} \in \widehat{\mathcal{L}}^{-1}$. Corollary 2 and (11) show now that $\widehat{T} - \widehat{f(T)} \in \widehat{\mathcal{L}}^g$, thus $h(T) = T - f(T) \in \Phi_g(X)$. ■

PROPOSITION 3. *For $t_1, t_2 \in \mathcal{A}$ we have*

$$e - t_1t_2 \in \mathcal{A}^g \Leftrightarrow e - t_2t_1 \in \mathcal{A}^g.$$

Proof. Let $e - t_1t_2 \in \mathcal{A}^g$ and $s \in \mathcal{A}$ such that

$$(e - t_1t_2)s(e - t_1t_2) = e - t_1t_2 \tag{13}$$

and

$$(e - t_1t_2)s = s(e - t_1t_2). \tag{14}$$

From (13) and (14) we get

$$e - t_1 t_2 = s - s t_1 t_2 - t_1 t_2 s + t_1 t_2 s t_1 t_2 \tag{15}$$

and

$$t_1 t_2 s = s t_1 t_2. \tag{16}$$

Put $r = e + t_2 s t_1$. Then we obtain

$$(e - t_2 t_1) r (e - t_2 t_1) == e - t_2(2e - s + s t_1 t_2 - t_1 t_2 + t_1 t_2 s - t_1 t_2 s t_1 t_2) t_1.$$

Hence, by (15),

$$(e - t_2 t_1) r (e - t_2 t_1) == e - t_2(2e - t_1 t_2 - (e - t_1 t_2)) t_1 = e - t_2 t_1,$$

thus r is a pseudo-inverse of $e - t_2 t_1$. (16) shows that

$$\begin{aligned} t_2 t_1 r &= t_2 t_1 (e + t_2 s t_1) = t_2 t_1 + t_2 (t_1 t_2 s) t_1 \\ &= t_2 t_1 + t_2 (s t_1 t_2) t_1 = t_2 t_1 + t_2 s t_1 t_2 (t_2 t_1) \\ &= (e + t_2 s t_1) t_2 t_1 = r t_2 t_1. \end{aligned}$$

Therefore $(e - t_2 t_1) r = r (e - t_2 t_1)$. Theorem 3.3 in [2] gives then $e - t_2 t_1 \in \mathcal{A}^g$. ∎

Corollary 12. *Let $T_1, T_2 \in \mathcal{L}(X)$. Then*

$$I - T_1 T_2 \in \Phi_g(X) \Leftrightarrow I - T_2 T_1 \in \Phi_g(X).$$

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Received December 4, 1997.