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H. G. Ince

CESÁRO CONULL FK-SPACES

Abstract. The purpose of this paper is to study the (strongly) Cesáro conull FK-spaces and to give some characterizations.

1. Introduction

The classification of conservative matrices as conull or coregular, due to Wilansky [14], has been extended by Yurimyae [17] and Snyder [13] to all FK-spaces. Bennett [2] continued work on conull FK-spaces; and improved some results of Sember [10]–[12].

Motivating by Bennett's paper [2] and his talks at the Ankara University during the summer of 1996 we study the (strongly) Cesáro conull FK-spaces.

In Section 2 we introduce the notation and terminology while in Section 3 we study the Cesáro conull FK-spaces and provide some examples to illustrate the differences between the conull and Cesáro conull FK-spaces. Section 4 deals with the strongly Cesáro conull FK-spaces; and gives a relationship between the (Cesáro wedge) weak Cesáro wedge and (strongly) Cesáro conull FK-spaces. In Section 5 we obtain some results for a summability domain E_A to be (strongly) Cesáro conull. Section 6 presents some applications to summability domains.

2. Notation and preliminary results

Let w denote the space of all real or complex-valued sequences. It can be topologized with the seminorms $p_i(x) = |x_i|$, $(i = 1, 2, \ldots)$, and any vector subspace of w is called a sequence space. A sequence space X, with a vector space topology τ, is a K-space provided that the inclusion mapping

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$I : (X,\tau) \to w$, $I(x) = x$, is continuous. If, in addition, τ is complete, metrizable and locally convex then (X,τ) is called an FK-space. So an FK-space is a complete, metrizable locally convex topological vector space of sequences for which the coordinate functionals are continuous. An FK-space whose topology is normable is called a BK-space. The basic properties of such spaces may be found in [15], [16] and [19].

By m, c, c_0 we denote the spaces of all bounded sequences, convergent sequences and null sequences, respectively. These are FK-spaces under $\|x\| = \sup_k |x_k|$. By $\ell^p, (1 \le p < \infty)$, and cs we shall denote the space of all absolutely p-summable sequences, and convergent series, respectively. As usual ℓ^1 is replaced by ℓ. The sequence spaces

$$h = \left\{x \in w : \lim_j x_j = 0 \quad \text{and} \quad \sum_{j=1}^{\infty} j\,|\Delta x_j| < \infty\right\},$$

$$q = \left\{x \in w : \sup_j |x_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} j\,|\Delta^2 x_j| < \infty\right\},$$

and

$$\sigma s = \left\{x \in w : \lim_n \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} x_j \text{ exists}\right\}$$

are BK-spaces with the norms

$$\|x\|_h = \sum_{j=1}^{\infty} j\,|\Delta x_j| + \sup_j |x_j|,$$

$$\|x\|_q = \sum_{j=1}^{\infty} j\,|\Delta^2 x_j| + \sup_j |x_j|,$$

and

$$\|x\|_{\sigma s} = \sup_n \left|\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} x_j\right|$$

respectively, where $\Delta x_j = x_j - x_{j+1}, \Delta^2 x_j = \Delta x_j - \Delta x_{j+1}$. Let $q_0 := q \cap c_0$, and $bv = \{x \in w : \sum_j |x_j - x_{j+1}| < \infty\}$, $bv_0 := bv \cap c_0$ (see [2], [4] and [6]).

Troughout the paper e denotes the sequence of ones, $(1,1,\ldots,1,\ldots)$; δ^j, $(j = 1,2,\ldots)$, the sequence $(0,0,\ldots,0,1,0,\ldots)$ with the one in the j-th position. Let $\phi := \ell.hull\left\{\delta^k : k \in N\right\}$ and $\phi_1 := \phi \cup \{e\}$. The topological dual of X is denoted by X'.

A sequence x in a locally convex sequence space X is said to have the property AK (respectively σK) if $x^{(n)} \to x$ (respectively $\frac{1}{n}\sum_{k=1}^{n} x^{(k)} \to x$) in X where $x^{(n)} = \sum_{k=1}^{n} x_k \delta^k = (x_1,\ldots,x_n,0,0,\ldots)$.

The subspace of a locally convex sequence space X consisting of the sequences with the property AK (respectively σK) is denoted by X_{AK} (respectively $X_{\sigma K}$). Every AK-space is a σK-space, [4]. For example w, h, ℓ, c_0 are AK-spaces while $q_0, \sigma s$ are σK-spaces (see [4], [6]).

Let $z = (z_k) \in w$ be such that $z_k \neq 0$ for every $k = 1, 2, \ldots$ Then

$$V_0(z) := \left\{ x \in c_0 : \sum_{k=1}^{\infty} |z_k|\,|\Delta x_k| < \infty \right\}$$

is an FK-AK space with the norm $\|x\|_{V_0(z)} = \sum_{k=1}^{\infty} |z_k|\,|\Delta x_k|$, [7].

Finally, $s = \{s_n\}_{n=1}^{\infty}$ always denotes a strictly increasing sequence of non-negative integers with $s_1 = 0$. We shall also be interested in spaces of the form

$$c\,|s| := \left\{ x \in w : \lim_j x_j = 0 \quad \text{and} \quad \sup_n \sum_{j=s_n+1}^{s_{n+1}} j\,|\Delta x_j| < \infty \right\}$$

which becomes an FK-space with the norm $\|x\|_{c|s|} = \sup_n \sum_{j=s_n+1}^{s_{n+1}} j|\Delta x_j|$, [8].

If X is any sequence space then,

$$\begin{aligned} X^{\sigma} &= \left\{ x \in w : \lim_n \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} x_j y_j \quad \text{exists for all } y \in X \right\} \\ &= \{ x \in w : x.y \in \sigma s \quad \text{for all } y \in X \} \end{aligned}$$

where $x.y = (x_n y_n)$, [6]. For example $\sigma s^{\sigma} = q$, [3].

Using the fact that the space $z^{-1}.X := \{x : z.x \in X\}$ is an FK-space (see [16], Theorem 4.3.6) one can get immediately the following:

PROPOSITION 2.1. *Let (X, u) be an FK-σK space and $z \in w$, then $z^{-1}.X$ is also a σK-space.*

Taking $X = \sigma s$ in Proposition 2.1. we get

$$z^{-1}.\sigma s = \{x : z.x \in \sigma s\} = \left\{ x : \lim_n \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} z_j x_j \text{ exists} \right\} = z^{\sigma}.$$

So we have

THEOREM 2.2. *If $z \in w$, then z^{σ} is a σK-space.*

Following Yurimyae [17] and Snyder [13] we say that an FK-space (X, τ) containing ϕ_1 is a conull space if $e - e^{(n)} = (0, 0, \ldots, 0, 1, 1, \ldots) \to 0$ (weakly) in X. It is strongly conull space if $e - e^{(n)} \to 0$ in X, [2].

A relationship between (strongly) conull and (wedge) weak wedge FK-spaces is given by Bennett in [2]. Recall that if (X, τ) is a K-space containing

ϕ, and $\delta^k \to 0$ in X then (X, τ) is called a wedge space; and if $\delta^k \to 0$ (weakly) in X then (X, τ) is called a weak wedge space [2].

In [8] we have introduced the concept of a Cesáro wedge FK-space and given some characterizations.

We recall that if $\frac{e^{(n)}}{n} = \frac{1}{n}\sum_{k=1}^{n} \delta^k = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}, 0, \ldots) \to 0$ in a K-space (X, τ) containing ϕ then (X, τ) is called a Cesáro wedge space; and if $\frac{e^{(n)}}{n} \to 0$ (weakly) in X then (X, τ) is called a weak Cesáro wedge space. In [8] some examples are also provided to illustrate the differences between (weak) wedge and (weak) Cesáro wedge FK-spaces.

3. Cesáro conull FK-spaces

In a seminar held at the Ankara University during the summer of 1996, Prof. G. Bennett of Indiana University (USA) introduced the concept of Cesáro conullity for an FK-space X containing ϕ_1; and suggested the related topic to work on.

DEFINITION 3.1. Let X be an FK-space containing ϕ_1. If

$$\mu^n := e - \frac{1}{n}\sum_{k=1}^{n} e^{(k)} = (\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}, 1, \ldots) \to 0 \text{ in } X \tag{1}$$

then X is called strongly C_1-conull FK-space, where $e^{(k)} := \sum_{j=1}^{k} \delta^j$. If the convergence holds in the weak topology in (1) then X is called C_1-conull. Hence X is C_1-conull iff

$$f(e) = \lim_n \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k} f(\delta^j), \quad \forall f \in X'.$$

We shall now present two examples of C_1-conull FK-spaces which are not conull. First we need some further notations.

Let $A = (a_{ij})$ be an infinite matrix. The matrix A may be considered as a linear transformation of sequences $x = (x_k)$ by the formula $y = Ax$, where $y_i = \sum_{j=1}^{\infty} a_{ij}x_j$, $(i = 1, 2, \ldots)$. A is called conservative if $Ax \in c$ for all $x \in c$.

For an FK-space (E, u) we consider the summability domain $E_A := \{x \in w : Ax \in E\}$. Then E_A is an FK-space under the seminorms $p_i(x) = |x_i|$, $(i = 1, 2, \ldots)$ $h_i(x) = \sup_m |\sum_{j=1}^{m} a_{ij}x_j|$, $(i = 1, 2, \ldots)$ and $(u \circ A)(x) = u(Ax)$ (see [16] and [18]).

Now we present the examples promised in this section.

EXAMPLE 3.2. Define the seqeunce Ax by $(Ax)_j = x_j - x_{j-1}, (x_0 = 0)$ if j is a square, and 0 otherwise. Then ℓ_A is C_1-conull.

To see this, consider an $f \in \ell'_A$. Since A is triangular, it follows from ([16], p. 66) that $f(x) = t(Ax)$ for some $t \in m$, and for all $x \in \ell_A$. Then $\mu^n \to 0$ (weakly) in ℓ_A if and only if, for every $f \in \ell'_A$, we have

$$(2) \qquad f(\mu^n) = \sum_{j=1}^{\infty} t_j\Big(\sum_{k=1}^{j} a_{jk}\mu_k^{(n)}\Big) \to 0, (n \to \infty)$$

for every $t \in m$. Now define the matrix $B = (b_{nj})$ by $b_{nj} = \sum_{k=1}^{j} a_{jk}\mu_k^{(n)}$. Then (2) holds if and only if B maps m into c_0, which, is equivalent to $\lim_n \sum_{j=1}^{\infty} |b_{nj}| = 0$. On the other hand we have $\sum_{k=1}^{j} a_{jk}\mu_k^{(n)} = \frac{1}{n}$, if $j = m^2 \leq n$, and 0 otherwise. Since the set $M = \{m^2 : m \in N\}$ has density zero, we get

$$\sum_{j=1}^{\infty}\left|\sum_{k=1}^{j} a_{jk}\mu_k^{(n)}\right| = \frac{1}{n}\sum_{j=1}^{n}\chi_M(j) \to 0, \quad (n \to \infty),$$

where χ_M is the characteristic function of M.

In order to show that ℓ_A is not conull we first observe that if

$$\psi^n := e - e^{(n)} = (0, 0, \ldots, 0, 1, 1, \ldots)$$

then

$$\sum_{k=1}^{j} a_{jk}\psi_k^n = 1, \quad \text{if } j = m^2 = n, \text{ and 0 otherwise.}$$

Hence $\lim_n \sum_{j=1}^{\infty} |\sum_{k=1}^{j} a_{jk}\psi_k^{(n)}| \neq 0$, which proves the claim.

The next example is provided by G. Bennett:

EXAMPLE 3.3. Define the sequence Ax by $(Ax)_n = \sqrt{n}x_n - \sqrt{n}x_{n-1}$ $(x_0 = 0)$. Then $(\ell_2)_A$ is also C_1-conull but not conull. The proof uses the same technique as in Example 1, so, is therefore omitted.

Unexpectedly we have the following

THEOREM 3.4. *Let A be a conservative matrix. Then c_A is conull if and only if it is C_1-conull.*

Proof. Only the sufficiency part needs to be proved. Assume c_A is C_1-conull. It follows from the Banach–Steinhaus theorem that $f := \lim_A \in c'_A$. A few calculation yields that

$$(3) \qquad \lim_A e - \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k}\lim_A \delta^j = \chi(A) + \frac{1}{n}\sum_{k=1}^{n}\sum_{j=k+1}^{\infty}\lim_A \delta^j$$

where $\chi(A) = \lim_A e - \sum_{j=1}^{\infty}\lim_A \delta^j$. Since A is conservative we have $(\lim_A \delta^j) \in cs$, so, the second term on the right hand side in (3) tends to zero as $n \to \infty$. The left hand side must also tend to zero as $n \to \infty$ because of C_1-conullity of c_A. This implies that $\chi(A) = 0$, i.e, A is conull. Now a result due to Snyder [13] gives the conclusion.

Our next result follows immediately from Theorem 3.4.

COROLLARY 3.5. *Let X be a C_1-conull FK-space but not conull space. Then, for any conservative matrix B, $c_B \neq X$.*

4. Strongly C_1-conull FK-spaces

First note that we have the following implications:

C_1-conull

↗ ↘

Strongly C_1-conull   Weak C_1-wedge

↘ ↗

C_1-wedge

None of the above implications can be reversed.

We, however, establish a relationship between (strongly) Cesáro conull and (Cesáro wedge) weak Cesáro wedge FK-spaces. To see this, consider the one-to-one and onto mapping $T : w \to w$, $Tx = (x_1, x_1+x_2, \ldots, \sum_{k=1}^{n} x_k, \ldots)$, and $T^{-1}x = (x_1, x_2 - x_1, \ldots, x_n - x_{n-1}, \ldots)$, [2] . Now we have

LEMMA 4.1. *Let (X, τ) be an FK-space. Then*

(i) *X is strongly C_1-conull if and only if $T^{-1}X$ is C_1-wedge space;*

(ii) *X is C_1-conull if and only if $T^{-1}X$ is weak C_1-wedge space.*

Proof. Note that the FK-topology of X can be given by a sequence of seminorms $\{d_n\}$, say. Then $T^{-1}(X)$ can be topologized by $d'_n(x) = d_n(Tx)$, $(n = 1, 2, \ldots)$, so that it too becomes an FK-space as well, ([9] , p. 253).

We just prove (ii) and leave (i) to the reader. Observe that

$$T : (T^{-1}(X), \tau') \to (X, \tau)$$

is a topological isormophism (see [9] , p. 254). If X is C_1-conull, then $\mu^r \to 0$ (weakly) in X. Since $T^{-1} : (X, \tau) \to (T^{-1}(X), \tau')$ is continuous, it is weakly continuous. Hence $T^{-1}(\mu^r) = T^{-1}\left(e - \frac{1}{r}\sum_{k=1}^{r} e^{(k)}\right) = \frac{e^{(r)}}{r} \to 0$ (weakly) in $T^{-1}(X)$, and so $T^{-1}(X)$ is weak C_1-wedge space.

To prove the sufficiency it is enough to observe that

$$T : (T^{-1}(X), \tau') \to (X, \tau)$$

is weakly continuous and $T\left(\frac{e^{(r)}}{r}\right) = e - \frac{1}{r}\sum_{k=1}^{r} e^{(k)}$.

THEOREM 4.2. *Let (X, τ) be an FK-space. Then the following conditions are equivalent:*

(i) *X is strongly C_1-conull;*
(ii) *for some $z \in w$ such that $z_n = o(n)$,*

$$T(V_0(z)) = \Big\{x \in w : \lim_n \Delta x_{n-1} = 0 \text{ and } \sum_{n=1}^{\infty} |z_n||\Delta^2 x_{n-1}| < \infty, x_0 = 0\Big\} \subset X;$$

(iii) *for some sequence s,*

$$T(c|s|) = \Big\{x \in w : \lim_n \Delta x_{n-1} = 0 \text{ and } \sup_n \sum_{j=s_n+1}^{s_{n+1}} j|\Delta^2 x_{j-1}| < \infty, x_0 = 0\Big\} \subset X$$

and the inclusion mapping $I : T(c\,|s|) \to X$ is compact;

(iv) *$q \subset X$ and the inclusion mapping $I : q \to X$ is compact.*

Proof. (i) $\Rightarrow$(ii). If X is strongly C_1-conull, then by Lemma 4.1,(i), $T^{-1}(X)$ is C_1-wedge space. So, by Theorem 3.3. of [8], $V_0(z) \subset T^{-1}(X)$ for some z such that $z_n = o(n)$. It follows that $T(V_0(z)) \subset T(T^{-1}(X)) = X$. On the other hand one can easily show that

$$T(V_0(z)) = \Big\{x \in w : \lim_n \Delta x_{n-1} = 0 \text{ and } \sum_{n=1}^{\infty} |z_n|\,|\Delta^2 x_{n-1}| < \infty\Big\},$$

what gives (ii).

(ii)$\Rightarrow$(iii) Let $T(V_0(z)) \subset X$ for some $z \in w$ such that $z_n = o(n)$. Then $V_0(z) \subset T^{-1}(X)$. By Theorem 3.6 of [8], $V_0(z)$ is a C_1-wedge space. Now Theorem 3.8,(i), of [8] implies that $T^{-1}(X)$ is C_1-wedge space. It follows from Theorem 3.3 of [8] that $c\,|s| \subset T^{-1}(X)$ and the inclusion mapping $I : c\,|s| \to T^{-1}(X)$ is compact. Hence $T(c\,|s|) \subset X$, and the mapping $T \circ I : c\,|s| \to X$ is compact. Since T^{-1} is continuous, the inclusion mapping $I = T \circ I \circ T^{-1} : T(c\,|s|) \to X$ is also compact. The first part of the claim is even more clear.

(iii)$\Rightarrow$(iv). It is known that $h \subset c\,|s|$, hence $T(h) \subset T(c\,|s|)$. We now claim that $T(h) = q$. First observe that

$$T(h) = \Big\{x : \lim_n \Delta x_{n-1} = 0 \text{ and } \sum_{n=1}^{\infty} n\,|\Delta^2 x_{n-1}| < \infty,\ x_0 = 0\Big\}.$$

It follows from a result of Buntinas [4] that $q \subset bv \subset c$, thus $q \subset T(h)$. We now prove the reverse inclusion. If $x \in T(h)$, then

$$|\Delta x_n| \le \sum_{k=n}^{\infty} |\Delta^2 x_{k-1}| < \infty,$$

and

$$\sum_{n=1}^{\infty} |\Delta x_n| \le \sum_{n=1}^{\infty} n\,|\Delta^2 x_n| < \infty,$$

which yields that $x \in bv$ and hence $x \in m$. So we have $T(h) \subset q$ and therefore $T(h) = q$. Hence the inclusion mapping $I : q \to T(c\,|s|)$ is continuous. So, by (iii), the inclusion mapping $I : q \to X$ is compact.

(iv)$\Rightarrow$(i). First observe that $U := \{e - \frac{1}{n}\sum_{k=1}^{n} e^{(k)} : n = 1, 2, \ldots\}$ is a bounded subset of q and so must be relatively compact in X. Thus, it is easy to see that, for each i, $p_i(\mu^n) = \frac{i}{n}$ if $i < n$, and 1 if $i \geq n$. Hence we have, for each i, that $p_i(\mu^n) \to 0$ as $n \to \infty$. Now Theorem 2.3.11 of [9] implies that $\mu^n \to 0$ in (X, τ), giving (i).

THEOREM 4.3. *If $z \in \sigma s$, then z^σ is a strongly C_1-conull FK-space.*

Proof. If $z \in \sigma s$, then $e \in z^{-1}.\sigma s = z^\sigma$, which is by Theorem 2.2 a σK-space. So we must have that $e - \frac{1}{n}\sum_{k=1}^{n} e^{(k)} \to 0, (n \to \infty)$, whence the result.

The next result deals with C_1-conullity.

THEOREM 4.4. *An FK-space X is C_1-conull if and only if $q \subset X$ and the inclusion mapping $I : q \to X$ is weakly compact.*

Proof. Assume that X is C_1-conull. Then by Lemma 4.1(ii), $T^{-1}(X)$ is weak C_1-wedge. It follows from Theorem 4.2 of [8] that $h \subset T^{-1}(X)$ and the inclusion mapping $I : h \to T^{-1}(X)$ is weakly compact. Hence $T(h) = q \subset X$. Furthermore $I = T \circ I \circ T^{-1} : T(h) \to X$ is an inclusion mapping where $h \xrightarrow{I} T^{-1}(X) \xrightarrow{T} X$ and $T^{-1} : T(h) \to h$ is continuous. It follows that $T^{-1} : T(h) \to h$ is weakly continuous; and since $I : h \to T^{-1}(X)$ is weakly compact, the inclusion mapping $I = T \circ I \circ T^{-1} : T(h) \to X$ is weakly compact, that proves the necessity.

Conversely assume that $q \subset X$ and $I : q \to X$ is weakly compact. Hence the unit ball $B = \{x \in q : \|x\|_q \leq 1\}$ in q is $\sigma(X, X')$-realtively compact. Observe that $p_i(\mu^n) = \frac{i}{n}$ if $i < n$. Hence, for each i, $p_i(\mu^n) \to 0$ as $n \to \infty$. The same is also true in $\sigma(X, X')$ by Theorem 2.3.11 of [9]. This proves the theorem.

Now we have the following

COROLLARY 4.5. *The intersection of all (strongly) C_1-conull FK-spaces is q.*

Proof. Let the intersection of all C_1-conull FK–spaces be Y. By Theorems 4.3 and 4.4 we have

$$q \subset Y \subset \cap\{z^\sigma : z \in \sigma s\} = \sigma s^\sigma = q,$$

hence the result.

Considering Theorem 4.2 one can get the same result for strongly C_1-conull FK-spaces.

We recall in that the intersection of all (strongly) conull FK-spaces is bv, (see [2]) ; and observe that $q \subset bv$; [4].

THEOREM 4.6. (i) *An FK-space that contains a (strongly) C_1-conull FK-space must be a (strongly) C_1-conull FK-space.*

(ii) *A closed subspace, containing ϕ_1, of a (strongly)C_1-conull FK-space is a (strongly) C_1-conull FK-space.*

(iii) *A countable intersection of (strongly) C_1-conull FK-spaces is a (strongly) C_1-conull FK-space.*

The proof is easily obtained from the elementary properties of FK-spaces (see, e.g, [16]).

We note that q is not a (strongly) C_1-conull space. Hence it follows from Corollary 4.5 that there is no smallest (strongly) C_1-conull space.

We now show that if X is C_1-conull, then X contains a summable sequence which is not of bounded variation; and also it contains a summable sequence which is not absolutely p-summable.

THEOREM 4.7. (i) *If X is C_1-conull, then $X \cap (cs \setminus bv)$ is non-empty.*

(ii) *If X is C_1-conull, then $X \cap (cs \setminus \ell^p), (p \geq 1)$, is non-empty.*

Proof. (i). Since bv is not C_1-conull, then by Theorem 4.6,(i), $bv \cap X$ is not C_1-conull either. Theorem 4.6(ii), implies that $bv \cap X$ is not closed in X, and the desired result follows from Theorem 2 of [1].

(ii) The proof uses same technique, so we omit the details.

Bennett, in [1], has shown that if X is a conull space, then $m \cap X$ is non-separable in m. We show that the same conclusion remains true if conullity is replaced by C_1-conullity. More precisely we have

THEOREM 4.8. *If X is C_1-conull, then $m \cap X$ is a non-seperable subspace of m.*

Proof. It is clear that c is not a C_1-conull space, and hence, by Theorem 4.6(i), nor is $c \cap X$. Theorem 4.6(ii), implies that $c \cap X$ is not closed in X. Now Theorem 8 of [1] yields the result.

5. Summability domains

In this section we give simple conditions for a summability domain E_A to be (strongly) Cesáro conull. The conditions will depend on the choice of the FK-space E and the matrix A.

The sequence $\{a_{ij}\}_{j=1}^{\infty}$ is called the i-th row of A and is denoted by r^i, $(i = 1, 2, \ldots)$; similarly, the j-th column of the matrix A, $\{a_{ij}\}_{i=1}^{\infty}$ is denoted by k^j, $(j = 1, 2, \ldots)$.

THEOREM 5.1. *Let E be an FK-space and A be a matrix such that $\phi_1 \subset E_A$. Then E_A is a C_1-conull space if and only if*

$$A\left(e - \frac{1}{r}\sum_{k=1}^{r} e^{(k)}\right) \to 0(weakly) \text{ in } E.$$

Proof. **Necessity:** Let E_A be C_1-conull space. Then for all $f \in E_A'$,

(4) $$f(\mu^r) \to 0, (r \to \infty).$$

Let $f(x) = g(Ax)$, for $g \in E'$, so $f \in E_A'$ by Theorem 4.4.2 of [16]. Since $f(\mu^r) = g(A\mu^r)$, the result follows from (4).

Sufficiency: Let $f \in E_A'$. By Theorem 4.4.2 of [16] $f \in E_A'$, if and only if

$$f(x) = \sum_k \alpha_k x_k + g(Ax),$$

for all $x \in E_A$, where $\alpha \in w_A^\beta = \{x : \sum_n x_n y_n \text{ convergent for all } y \in w_A\}$, and $g \in E'$. Thus we get the following

(5) $$f(\mu^r) = \frac{1}{r}\sum_{k=1}^{r}\sum_{j=k+1}^{\infty} \alpha_j + g(A\mu^r).$$

By hypothesis $e \in E_A \subset w_A$. Then $\alpha \in w_A^\beta \subset e^\beta = cs$ which implies $\lim_r \frac{1}{r}\sum_{k=1}^{r}\sum_{j=k+1}^{\infty} \alpha_j = 0$. By hypothesis, the second term on the right hand side of (5) tends to zero too, whence the result.

THEOREM 5.2. *Let (E, u) be an FK-space and A be a matrix such that $\phi_1 \subset E_A$. Then E_A is strongly C_1-conull space if and only if*

$$A\left(e - \frac{1}{r}\sum_{k=1}^{r} e^{(k)}\right) \to 0 \quad in\ E.$$

Proof. The necessity follows at once from observing that the matrix mapping $A : E_A \to E$ is continuous ([15], Corollary 11.3).

Sufficiency: By Theorem 4.3.8 of [16] $(w_A, p \cup h)$ is an AK-space, hence it is a σK-space. Hence, for each n, $p_n(\mu^r) \to 0$ and $h_n(\mu^r) \to 0$. By hypothesis $(uoA)(\mu^r) = u(A\mu^r) \to 0$, which proves the theorem.

6. Applications

In this section we apply some of our previous results, to summability domains.

The following theorem is an application of Theorem 4.2 to summability domains.

THEOREM 6.1. *Let E be an FK-space and A be a matrix. Then the following conditions are equivalent:*

(i) *E_A is a strongly C_1-conull space;*

(ii) *$q \subset E_A$ and the mapping $A : q \to E$ is compact;*

(iii) *$k^j \in E$, for all j, and the sequence $\{(Ae - \frac{1}{r}\sum_{k=1}^{r}\sum_{j=1}^{k} a_{ij})_{i=1}^{\infty} : r \geq 1\}$ converges to zero in E.*

Proof. (i)⇒(ii). From Theorem 4.2, (i)⇔(iv), $q \subset E_A$ and the inclusion mapping $I : h \to E_A$ is compact. Also by Corollary 11.3 of [15] $A : E_A \to E$ is continuous. Then $A : q \to E$, which may be regarded as a composition of $I : q \to E_A$ with $A : E_A \to E$, must be compact.

(ii)⇒(iii). Observe that $\delta^j \in q$, for all j, and $q \subset E_A$, we get $k^j = A(\delta^j) \in E$, for all j. Since $e \in q \subset E_A \subset w_A$ and is a σK-space, $\mu^r \to 0$ in w_A. The fact that $A : w_A \to w$ is continous, implies $A(\mu^r) \to 0$ in w. On the other hand $U = \{\mu^r : r = 1, 2, \ldots\}$ is bounded subset in q and $A : q \to E$ is compact then $A(U) = \{A(\mu^r) : r = 1, 2, \ldots\}$ is relatively compact in E. Thus, by Theorem 2.3.11 of [9] $A(\mu^r) \to 0$ in w implies that $A(\mu^r) \to 0$ in E.

(iii)⇒(i). This is Theorem 5.2.

The following theorem is an application of Theorem 4.4 to summability domains.

THEOREM 6.2. *Let E be an FK-space and A be a matrix. Then the following conditions are equivalent:*

(i) *E_A is a C_1-conull space;*

(ii) *$q \subset E_A$ and the mapping $A : q \to E$ is weakly compact;*

(iii) *$k^j \in E$, for all j, and the sequence $\{(Ae - \frac{1}{r}\sum_{k=1}^{r}\sum_{j=1}^{k} a_{ij})_{i=1}^{\infty} : r \geq 1\}$ converges weakly to zero in E.*

Proof. (i)⇒(ii). From Theorem 4.4, $q \subset E_A$ and the inclusion mapping $I : q \to E_A$ is weakly compact. Also $A : E_A \to E$ is weakly continuous. Thus $A : q \to E$, where $A = A \circ I$, is weakly compact.

(ii)⇒(iii). As in the proof of (ii)⇒(iii) of Theorem 6.1, $k^j \in E$, for all j; and $A(\mu^r) \to 0$ in w. Combining this with $A(U) = \{A(\mu^r) : r = 1, 2, \ldots\}$ is weakly relatively compact in E we get, by Theorem 2.3.11 of [9] that $A(\mu^r) \to 0$ (weakly) in E.

(iii)⇒(i). This is Theorem 5.1.

COROLLARY 6.3. *m_A is C_1-conull if and only if the following conditions hold:*

(i) $\sup_{i,n} \left|\frac{1}{n}\sum_{p=1}^{n}\sum_{j=1}^{p} a_{ij}\right| < \infty$,

(ii) *for any given* $\epsilon > 0$ *and an increasing sequence* $\{n_k\}_{k=1}^{\infty}$ *positive integers, there exists* L *such that*

$$\sup_i \min_{1\le r\le L} \left| \frac{1}{n_{k_r}} \sum_{p=1}^{n_{k_r}} \sum_{j=p+1}^{\infty} a_{ij} \right| < \epsilon.$$

Proof. This follows by putting $E = m$ in Theorem 6.2, the equivalence (i)⇔(iii), and using the characterization of weak sequential convergence in m given in [5], IV, 6.31, p. 281.

Our next result follows immediately from Theorems 6.1 and 6.2.

THEOREM 6.4. *Let* E *be an* FK*-space such that weakly convergent sequences are convergent in the* FK*-topology and let* A *be a matrix. Then* E_A *is* C_1*-conull space if and only if it is strongly* C_1*-conull space.*

In particular Theorem 6.4 holds when $E = \ell$, and $E = bv$.

COROLLARY 6.5. *The following conditions are equivalent for any matrix* A*:*

(i) ℓ_A *is (strongly)* C_1*-conull,*
(ii) $\lim_{n\to\infty} \sum_{i=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \sum_{j=k+1}^{\infty} a_{ij} \right| = 0.$

Proof. This is just Theorem 6.1, (i)⇔(iii) and Theorem 6.4 with $E = \ell$.

COROLLARY 6.6. *The following conditions are equivalent for any matrix* A*:*

(i) bv_A *is (strongly)* C_1*-conull,*

(ii) $$\lim_{n\to\infty} \left\{ \sum_{i=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \sum_{j=k+1}^{\infty} (a_{ij} - a_{i+1,j}) \right| + \lim_i \left| \frac{1}{n} \sum_{k=1}^{n} \sum_{j=k+1}^{\infty} a_{ij} \right| \right\} = 0.$$

Proof. This follows at once from Theorem 6.1, (i)⇔(iii) and Theorem 6.4 with $E = bv$.

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References

[1] G. Bennett, *The glinding humps technique for FK-spaces*, Trans. Amer. Math. Soc. 166 (1972), 285–292.

[2] G. Bennett, *A new class of sequence spaces with applications in summability theory*, J. Reine Angew. Math. 266 (1974), 49–75.

[3] L. S. Bosanquet, *Note on convergence and summability factors* J. London Math. Soc. 20 (1945), 39–48.

[4] M. Buntinas, *Convergent and bounded Cesáro sections in FK-spaces*, Math. Z. 121 (1971), 191–200.

[5] N. Dunford and J. T. Schwartz, *Linear Operators*, New York 1958.

[6] G. Goes and S. Goes, *Sequences of bounded variation and sequences of Fourier coefficients, I*, Math. Z. 118 (1970), 93–102.
[7] G. Goes, *Sequences of bounded variation and sequences of Fourier coefficients, II*, J. Math. Anal. Appl. 39 (1972), 477–494.
[8] H. G. Ince, *Cesáro wedge and weak Cesáro wedge FK-spaces*; submitted for publication.
[9] P. K. Kamthan and M. Gupta, *Sequence Spaces and Series*, Marcell Dekker, I, New York and Basel 1981.
[10] J. J. Sember, *A note on conull FK-spaces and variation matrices*, Math. Z. 108 (1968), 1–6.
[11] J. J. Sember, *Summability matrices as compact-like operators*, J. Lond. Math. Soc. (2) 2 (1970), 530–534.
[12] J. J. Sember, *Variational FK-spaces and two-norm convergence*, Math. Z. 119 (1971), 153–159.
[13] A. K. Snyder, *Conull and coregular FK-spaces*, Math. Z. 90 (1965), 376–381.
[14] A. Wilansky, *An application of Banach linear functionals to summability*, Trans. Amer. Math. Soc. 67 (1949), 59–68.
[15] A. Wilansky, *Functional Analysis*, Blaisdell Press, 1964.
[16] A. Wilansky, *Summability Through Functional Analysis*, North Holland, 1984.
[17] E. Yurimyae, *Einige Fragen über verallgemeinerte Matrixverfahren, co-regulär und co-null Verfahren*, Eesti Tead. Akad. Toimetised Tehn. Füüs. Mat. 8 (1959), 115–121.
[18] K. Zeller, *Allgemeine Eigenschaften von Limitierungsverfahren*, Math. Z. 53 (1951), 463–487.
[19] K. Zeller, *Theorie der Limitierungsverfahren*, Berlin-Heidelberg-New York 1958.

ANKARA UNIVERSITY, FACULTY OF SCIENCE,
DEPARTMENT OF MATHEMATICS,
06100 TANDOGAN, ANKARA, TURKEY
E-mail: ince@science.ankara.edu.tr

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