DEMONSTRATIO MATHEMATICA
Vol. XXXIII No 1 2000

Suyalatu

ON SOME GENERALIZATION OF LOCAL UNIFORM SMOOTHNESS AND DUAL CONCEPTS

Abstract. In this paper, the conception of local k-uniform smoothness ($LkUS$) is introduced on the base of the concept of k-uniform smoothness that introduced by Suyalatu and Wu Congxin. It is proved that the local k-uniform smoothness and Sullivan's local k-uniform rotundity ($LkUR$) are the dual notions. X is a $LkUS$ space, then X is a $L(k+1)US$ space. However, the converse need not be true. In addition, we also obtain two important results about $LkUS$ space.

Throughout this paper, X will denote a real Banach space and X^* will denote its conjugate space. Set $S(X) \equiv \{x : x \in X, \|x\| = 1\}$, $U(X) \equiv \{x : x \in X, \|x\| \leq 1\}$, $S_x \equiv \{x^* : x^* \in S(X^*), x^*(x) = 1\}$, $x \in S(X)$. Wu Congxin and Li Yongjin defined the notion of local uniform smoothness in [1], which is the dual notion of local uniform convexity introduced by Lovaglia in [2]. In this paper, we introduce the notion of locally k-uniformly smooth space ($LkUS$), which is the extension of notion of locally uniformly smooth space and dual of locally k-uniformly rotund space that introduced by Sullivan in [3].

In [3], Sullivan defined the k-uniformly rotund space.By "fixing" one variable he defined the notion of a locally k-uniformly rotund space. k-uniformly rotund space is dual notion of k-uniformly smooth space that introduced by us in [4]. Corresponding to the k-uniformly smooth space, by "fixing" one variable we can define the notion of a locally k-uniformly smooth space.

DEFINITION 1 [3]. A Banach space X is said to be a $LkUR$ space if for any $\epsilon > 0$, $x \in S(X)$, there is a $\delta = \delta(x, \epsilon) > 0$ such that for $x_1, \ldots, x_k \in S(X)$,

1991 *Mathematics Subject Classification*: 46B09.
Key words and phrases: $LkUR$ space, $LkUS$ space, LkS space, $CLkS$ space.
This work was supported by the Natural Science Foundation of Inner Mongolia, 1999.

if $\|x+x_1+\ldots+x_k\|>(k+1)-\delta$, then

$$A(x,x_1,\ldots,x_k) = \sup\left\{\begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1^*(x) & x_1^*(x_1) & \ldots & x_1^*(x_k) \\ \vdots & \vdots & & \vdots \\ x_k^*(x) & x_k^*(x_1) & \ldots & x_k^*(x_k)\end{vmatrix} : x_1^*,\ldots,x_k^* \in S(X^*)\right\} < \epsilon.$$

DEFINITION 2 [4]. A Banach space X is said to be a k-uniformly smooth space if for each $\epsilon>0$, there is a $\delta(\epsilon)>0$ such that for $x_1^*,\ldots,x_{k+1}^*\in S(X^*)$, if $\|x_1^*+\ldots+x_{k+1}^*\|>(k+1)-\delta$, then

$$B(x_1^*,\ldots,x_{k+1}^*) = \sup\left\{\begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1^*(x_1) & x_2^*(x_1) & \ldots & x_{k+1}^*(x_1) \\ \vdots & \vdots & & \vdots \\ x_1^*(x_k) & x_2^*(x_k) & \ldots & x_{k+1}^*(x_k)\end{vmatrix} : x_1,\ldots,x_k \in S(X)\right\} < \epsilon.$$

By "fixing" one variable we define the notion of locally k-uniformly smooth space ($LkUS$). i.e. The definition of $LkUS$ is as follows:

DEFINITION 3. A Banach space X is said to be a $LkUS$ space if for any $\epsilon>0$, $x\in S(X)$, $x^*\in S_x$, there is a $\delta=\delta(x,x^*,\epsilon)>0$ such that for $x_1^*,\ldots,x_k^*\in S(X^*)$, if $\|x^*+x_1^*+\ldots+x_k^*\|>(k+1)-\delta$, then $B(x^*,x_1^*,\ldots,x_k^*)<\epsilon$.

THEOREM 1 (Dual Theorem):

(a) *If X^* is $LkUS$ space, then X is $LkUR$ space,*
(b) *If X^* is $LkUR$ space, then X is $LkUS$ space.*

Proof. (a) If for any $\epsilon>0$, $x\in S(X)$, there is a $\delta=\delta(x,\epsilon)>0$ such that for $x_1,x_2,\ldots,x_k\in S(X)$, if $\|x+x_1+\ldots+x_k\|>(k+1)-\delta$, then, by Hahn–Banach Theorem, we can choose $x^*\in S(X^*)$ such that $x^*(x)=1$, so $x(x^*)=1$ and $x\in S_{x^*}, x_1,x_2,\ldots,x_k\in S(X^{**})$. Choose $\delta'(x^*,x,\epsilon)=\delta>0$, then $\|x+x_1+\ldots+x_k\|>(k+1)-\delta'$. By the assumption that X^* is $LkUS$, we have $B(x,x_1,\ldots,x_k)<\epsilon$, hence $A(x,x_1,\ldots,x_k)=B(x,x_1,\ldots,x_k)<\epsilon$. This shows that X is $LkUR$ space.

(b) If for any $\epsilon>0$, $x\in S(X)$, $x^*\in S_x$, there is a $\delta=\delta(x,x^*,\epsilon)>0$ such that for $x_1^*,\ldots,x_k^*\in S(X^*)$, if $\|x^*+x_1^*+\ldots+x_k^*\|>(k+1)-\delta$, then we can choose $\delta'(x^*,\epsilon)=\delta>0$, such that $\|x^*+x_1^*+\ldots+x_k^*\|>(k+1)-\delta'$. By the assumption that X^* is $LkUR$, we have $A(x^*,x_1^*,\ldots,x_k^*)<\epsilon$. Since, for any $x_1,\ldots,x_k\in S(X)$, we have $x_1,\ldots,x_k\in S(X^{**})$, hence $B(x^*,x_1^*,\ldots,x_k^*)\le A(x^*,x_1^*,\ldots,x_k^*)<\epsilon$. This shows that X is $LkUS$ space.

Theorem 1 shows that $LkUR$ and $LkUS$ are dual notions.

COROLLARY 1 [1].

(a) *If X^* is $L1US$ space, then X is $L1UR$ space,*

(b) *If X^* is $L1UR$ space, then X is $L1US$ space.*

In fact, the $L1UR$ and $L1US$ space are, clearly, nothing else but the usual LUR[2] and LUS[1] space, resp. Hence, the $LkUS$ space is a generalization of the locally smooth space.

DEFINITION 4 [5]. A Banach space X is said to be a k-strictly convex if and only if for any $(k+1)$ elements $x_1, x_2, \ldots, x_{k+1}$ of $S(X)$ with $\|\sum_{i=1}^{k+1} x_i\| = \sum_{i=1}^{k+1} \|x_i\|$ implies that $x_1, x_2, \ldots, x_{k+1}$ are linearly dependent.

DEFINITION 5 [4]. A Banach space X is said to be a k-smooth space if and only if for any $x \in S(X)$, $\dim S_x \le k$.

DEFINITION 6 [4]. A Banach space X is said to be a k-strongly smooth space if X is a k-smooth space. And for each $x \in S(X)$, if $x_n^* \in S(X^*)$ $x_n^*(x) \to 1$, then (x_n^*) is relatively compact.

LEMMA 1 [6]. *If X^* is k-smooth space, then X is k-strictly convex space; if X^* is k-strictly convex space, then X is k-smooth space.*

LEMMA 2. *We have*

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_{1,2} & a_{1,3} & \cdots & a_{1,k+2} \\ a_{2,2} & a_{2,3} & \cdots & a_{2,k+2} \\ \vdots & \vdots & & \vdots \\ a_{k,2} & a_{k,3} & \cdots & a_{k,k+2} \end{vmatrix}$$

$$= \sum_{j=2}^{k+2} (-1)^j \begin{vmatrix} 1 & 1 & \ldots & 1 & 1 & \ldots & 1 \\ a_{1,1} & a_{1,2} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1,k+2} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2,k+2} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,j-1} & a_{k,j+1} & \cdots & a_{k,k+2} \end{vmatrix}.$$

Proof. Consider the determinant (this determinant is equal to zero)

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ a_{1,1} & a_{1,2} & \cdots & a_{1,k+2} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k+2} \\ \vdots & \vdots & & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k+2} \end{vmatrix}.$$

Expanding in minors along the first row, we obtain the desired result.

In [4], we have proved that k-uniformly smooth space implies k-strongly smooth space (Theorem 4 in [4]). Using the similar method which is used in proof of Theorem 4 in [4], we can prove the following lemma.

LEMMA 3. *If X is a locally k-uniformly smooth space, then X is a k-strongly smooth space.*

LEMMA 4 [7]. *Let (x_n) be a bounded sequence in X and ϵ be a positive real number. If (x_n) has no finite ϵ-net, then for any positive integers n_0, k and any $x \in X$, there exist $n_1, \ldots, n_k > n_0$, such that $d(x_{n_1}, \operatorname{span}\{x\}) > \frac{\epsilon}{2}$ and $d(x_{n_{i+1}}, \operatorname{span}\{x, x_{n_1}, \ldots, x_{n_i}\}) > \frac{\epsilon}{2}$ for $i = 1, 2, \ldots, k-1$.*

THEOREM 2. *If X is a $LkUS$ space, then X is a $L(k+1)US$ space.*

Proof. If X is a $LkUS$ space, then for any $\epsilon > 0, x \in S(X), x^* \in S_x$, there is a $\delta = \delta(x, x^*, \epsilon) > 0$ such that for $x_1^*, \ldots, x_k^* \in S(X^*)$, if $\|x^* + x_1^* + \ldots + x_k^*\| > (k+1) - \delta$, then

$$(*) \qquad B(x^*, x_1^*, \ldots, x_k^*) < \frac{\epsilon}{2(k+1)}.$$

Suppose $x_1^*, \ldots, x_{k+1}^* \in S(X^*)$ and $\|x^* + x_1^* + \ldots + x_{k+1}^*\| > (k+2) - \delta$, then for each j we have $\|x^* + x_1^* + \ldots + x_{j-1}^* + x_{j+1}^* + \ldots + x_{k+1}^*\| > (k+1) - \delta$.

Using $(*)$, we can deduce

$$(**) \qquad B(x^*, x_1^*, \ldots, x_{j-1}^*, x_{j+1}^*, \ldots, x_{k+1}^*) < \frac{\epsilon}{2(k+1)}.$$

By Lemma 2, we have

$$(***) \qquad B(x_1^*, x_2^*, \ldots, x_{k+1}^*) < \frac{\epsilon}{2}.$$

Combining $(**)$ and $(***)$, we have

$$B(x^*, x_1^*, \ldots, x_{k+1}^*) < \epsilon.$$

This completes the proof that X is a $L(k+1)US$ space.

The converse to Theorem 2 is not true.

EXAMPLE. There exists an infinite-dimensional $LkUS$ space X which is not $L(k-1)US$ space.

Let $k \geq 2$ be an integer, and let $i_1 < i_2 < \ldots < i_k$. For each $x = (a_1, a_2, \ldots) \in l_2$, define

$$\|x\|_{i_1,\ldots,i_k}^2 = \Big(\sum_{j=1}^{k} |a_{i_j}|\Big)^2 + \sum_{i \neq i_1,\ldots,i_k} a_i^2.$$

The space $X_{i_1,\ldots,i_k} = (l_2, \|\bullet\|_{i_1,\ldots,i_k})$ is k-uniformly rotund space[8] and reflexive. Hence, $X_{i_1,\ldots,x_k}^{**}$ is k-uniformly rotund space. Obviously, $X_{i_1,\ldots,i_k}^{**}$

is $LkUR$ space, by Theorem 1, we know that $X^*_{i_1,\dots,i_k}$ is $LkUS$ space. But, $X^*_{i_1,\dots,i_k}$ is not $L(k-1)US$ space. In fact, take $e_{i_j} = (0,\dots,0,\underset{i_j}{1},0,\dots)$, $j = 1,2,\dots,k$, then $\|e_{i_j}\|_{i_1,\dots,i_k} = 1, j = 1,2,\dots,k$ and $\{e_{i_j}\}_{j=1}^k$ is linearly independent. But $\|\sum_{j=1}^k e_{i_j}\|_{i_1,\dots,i_k} = k$, this shows that $X_{i_1,\dots,i_k}$ is not (k-1)-strictly convex space. By Lemma 1, we know that $X^*_{i_1,\dots,x_k}$ is not (k-1)-smooth space. By Definition 6 and Lemma 3, we know that $X^*_{i_1,\dots,i_k}$ is not $L(k-1)US$ space.

DEFINITION 7. A Banach space X is said to be LkS (resp. $CLkS$) if and only if for any $x \in S(X)$, if for any sequence (x_n^*) in $U(X^*)$ and $x^* \in S_x$, $\lim_{n_1,\dots,n_k\to\infty} \|x^* + x^*_{n_1} + \ldots + x^*_{n_k}\| = k+1$, then $\|x_n^* - x^*\| \to 0$ (resp. (x_n^*) is relatively compact). LkS (resp. $CLkS$) and LkR [7] (resp. $CLkR$) [9] are dual notions.

THEOREM 3. *If X is $LkUS$ space, then X is $CLkS$ space.*

Proof. Let $x \in S(X)$, $x^* \in S_x, (x_n^*)$ be a sequence in $U(X^*)$ and suppose

$$\lim_{n_1,\dots,n_k\to\infty} \|x^* + x^*_{n_1} + \ldots + x^*_{n_k}\| = k+1.$$

We prove that (x_n^*) is relatively compact.

If (x_n^*) is not relatively compact in X^*, then (x_n^*) has no finite ϵ_0-net, for some $\epsilon_0 > 0$. By Lemma 4, for any positive integer n_0 and any $x^* \in X^*$, there exist $n_1,\dots,n_k > n_0$, such that $d(x^*_{n_1}, \operatorname{span}\{x^*\}) > \frac{\epsilon_0}{2}$ and $d(x^*_{n_{i+1}}, \operatorname{span}\{x^*, x^*_{n_1},\dots,x^*_{n_i}\}) > \frac{\epsilon_0}{2}$, for $i = 1,2,\dots,k-1$.

Therefore, there exist $n_1^{(m)}, n_2^{(m)},\dots,n_k^{(m)}$ such that $n_i^{(m)} \to \infty$ and such that for any $n_1^{(m)},\dots,n_k^{(m)}$ we have $d(x^*_{n_1^{(m)}}, \operatorname{span}\{x^*\}) > \frac{\epsilon_0}{2}$ and $d(x^*_{n_{i+1}^{(m)}}, \operatorname{span}\{x^*, x^*_{n_1^{(m)}},\dots,x^*_{n_k^{(m)}}\}) > \frac{\epsilon_0}{2}$, for $i = 1,2,\dots,k-1$. By Hahn-Banach Theorem, we choose $x^{**}_{n_1^{(m)}}, x^{**}_{n_2^{(m)}},\dots,x^{**}_{n_k^{(m)}} \in S(X^{**})$ such that $x^{**}_{n_1^{(m)}}(x^*_{n_1^{(m)}}) > \frac{\epsilon_0}{2}, x^{**}_{n_1^{(m)}}(y^*) = 0$ for all $y^* \in \operatorname{span}\{x^*\}$ and $x^{**}_{n_i^{(m)}}(x^*_{n_i^{(m)}}) > \frac{\epsilon_0}{2}, x^{**}_{n_i^{(m)}}(y^*) = 0$ for all $y^* \in \operatorname{span}\{x^*, x^*_{n_1^{(m)}},\dots,x^*_{n_{i-1}^{(m)}}\}$ for $i = 2,\dots,k$. Then

$$A(x^*, x^*_{n_1^{(m)}},\dots,x^*_{n_k^{(m)}}) > \left(\frac{\epsilon_0}{2}\right)^k > 0.$$

From the definition of $A(x^*, x^*_{n_1^{(m)}},\dots,x^*_{n_k^{(m)}})$, we know that there are $G_1,\dots,G_k \in S(X^{**})$ such that

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ G_1(x^*) & G_1(x^*_{n_1^{(m)}}) & \dots & G_1(x^*_{n_k^{(m)}}) \\ \vdots & \vdots & & \vdots \\ G_k(x^*) & G_k(x^*_{n_1^{(m)}}) & \dots & G_k(x^*_{n_k^{(m)}}) \end{vmatrix} \geq \left(\frac{\epsilon_0}{2}\right)^k.$$

By Goldstine–Weston Theorem, there are $(x_1^\alpha), (x_2^\alpha), \dots, (x_k^\alpha) \subset S(X)$ such that $x_1^\alpha \xrightarrow{w^*} G_1, \dots, x_k^\alpha \xrightarrow{w^*} G_k$. So we have

$$\sup\left\{\begin{vmatrix} 1 & 1 & \dots & 1 \\ x^*(x_1) & x^*_{n_1^{(m)}}(x_1) & \dots & x^*_{n_k^{(m)}}(x_1) \\ \vdots & \vdots & & \vdots \\ x^*(x_k) & x^*_{n_1^{(m)}}(x_k) & \dots & x^*_{n_k^{(m)}}(x_k) \end{vmatrix} : x_1, x_2, \dots, x_k \in S(X)\right\}$$

$$\geq \lim_\alpha \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1^\alpha(x^*) & x_1^\alpha(x^*_{n_1^{(m)}}) & \dots & x_1^\alpha(x^*_{n_k^{(m)}}) \\ \vdots & \vdots & & \vdots \\ x_k^\alpha(x^*) & x_k^\alpha(x^*_{n_1^{(m)}}) & \dots & x_k^\alpha(x^*_{n_k^{(m)}}) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & \dots & 1 \\ G_1(x^*) & G_1(x^*_{n_1^{(m)}}) & \dots & G_1(x^*_{n_k^{(m)}}) \\ \vdots & \vdots & & \vdots \\ G_k(x^*) & G_k(x^*_{n_1^{(m)}}) & \dots & G_k(x^*_{n_k^{(m)}}) \end{vmatrix}$$

$$\geq \left(\frac{\epsilon_0}{2}\right)^k.$$

Hence,

$$(\star) \qquad B(x^*, x^*_{n_1^{(m)}}, \dots, x^*_{n_k^{(m)}}) \geq (\frac{\epsilon_0}{2})^k > 0.$$

On the other hand

$$\|x^* + x^*_{n_1^{(m)}} + \dots + x^*_{n_k^{(m)}}\| \to k+1, m \to \infty.$$

By the assumption that X is $LkUS$ space, we have $B(x^*, x^*_{n_1^{(m)}}, \dots, x^*_{n_k^{(m)}}) \to 0$, $(m \to \infty)$. This contridicts inequality $(\star)$, so (x_n^*) is relatively compact.

THEOREM 4. *If X is $LkUS$ space and X^* is strictly convex space, then X is LkS space.*

Proof. Let $x \in S(X), x^* \in S_x, (x_n^*)$ be a sequence in $U(X^*)$ and suppose

$$\lim_{n_1,\ldots,n_k \to \infty} \|x^* + x_{n_1}^* + \ldots + x_{n_k}^*\| = k+1.$$

Since X is $LkUS$ space, it follows from Theorem 3 that (x^*) is relatively compact in X^*.Consequently there exist $y^* \in X^*$ and a subsequence $(x_{n_j}^*)$ of (x_n^*) such that $x_{n_j}^* \to y^*$, obviously $y^* \in S(X^*)$.

On the other hand, we have

$$\begin{aligned} k+1 &= \lim_{n_1,\ldots,n_k \to \infty} \|x^* + x_{n_1}^* + \ldots + x_{n_k}^*\| \\ &= \lim_{n_1,\ldots,n_k \to \infty} \Big\| x^* + ky^* + \sum_{j=1}^{k} (x_{n_j}^* - y^*) \Big\| \\ &\le \|x^* + ky^*\| + \lim_{n_1,\ldots,n_k \to \infty} \sum_{j=1}^{k} \|x_{n_j}^* - y^*\| \\ &\le \|x^* + ky^*\| \le k+1. \end{aligned}$$

So $\|x^* + ky^*\| = k+1$, hence $\|x^* + y^*\| = 2$. By the assumption that X^* is strictly convex space, we have $x^* = y^*$. Hence $x_{n_j}^* \to x^*$. This shows that X is LkS space.

Acknowledgements. I wish to thank Professor Wu Congxin for his kind instruction and the referee for valuable suggestions which helped to improve the paper.

References

[1] C. X. Wu and Y. J. Li, *Strong convexity in Banach spaces*, Chinese J. Math., (13) (1993), 105–108.

[2] A. R. Lovaglia, *Locally uniformly convex Banach spaces*, Trans. Amer. Soc., 78 (1955), 225–238.

[3] F. Sullivan, *A generalization of uniformly rotund Banach spaces*, Canad. J. Math., 31 (1979), 628–636.

[4] Suyalatu and C. X. Wu, *k-uniformly rotund spaces and k-uniformly smooth spaces*, Chinese Sci. Bull., 2 (43) (1998), 92–95.

[5] I. Singer, *On the set of best approximation of an element in a normed linear space*, Rev. Roumaine Math. Pures Appl., 5 (1960), 383–402.

[6] C. X. Nan and J. H. Wang, *k-strict convexity and k-smoothness*, Chinese Ann., (in Chinese). (Series A), 3 (1990), 321–324.

[7] C. X. Nan and J. H. Wang, *On the LkUR and LkR spaces*, Math. Proc. Camb. Phil. Soc., 104 (1988), 521–526.

[8] B. L. Lin and X. T. Yu, *On the k-uniform rotund and the fully convex Banach spaces*, J. Math. Anal. Appl., 2 (1985), 407–410.

[9] J. H. Wang, *Some results on the continuity of matrix projection*, Chinese Math. Appl., 8 (1)(1995), 80–84.

DEPARTMENT OF MATHEMATICS
INNER MONGOLIA NORMAL UNIVERSITY
010022 HUHHOT, INNER MONGOLIA, P.R. CHINA

Received December 15, 1998; revised version September 7, 1999.