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Zbigniew Walczak

ON CERTAIN MODIFIED SZÁSZ-MIRAKYAN OPERATORS FOR FUNCTIONS OF TWO VARIABLES

Abstract. We introduce certain modified Szász-Mirakyan operators in polynomial weighted spaces of functions of two variables and we study approximation properties of these operators. The similar theorems for function of one variable are given in [2].

1. Preliminaries

1.1. Let as in [1], for $p \in N_0 := \{0, 1, 2, \ldots\}$,

(1) $w_0(x) := 1, \quad w_p(x) := (1 + x^p)^{-1} \quad \text{if} \quad p \geq 1, \quad x \in R_0 := [0, +\infty).$

Next, for fixed $p, q \in N_0$, we define the weighted function

$$w_{p,q}(x, y) := w_p(x)\, w_q(y), \qquad (x, y) \in R_0^2 := R_0 \times R_0, \tag{2}$$

and the weighted space $C_{p,q}$ of all real-valued functions f continuous on R_0^2 for which $w_{p,q} f$ is uniformly continuous and bounded on R_0^2 and the norm is defined by the formula

$$\|f\|_{p,q} \equiv \|f(\cdot, \cdot)\| := \sup_{(x,y) \in R_0^2} w_{p,q}(x, y)\, |f(x, y)|. \tag{3}$$

The modulus of continuity of $f \in C_{p,q}$ we define as usual

$$\omega(f, C_{p,q}; t, s) := \sup_{0 \leq h \leq t,\, 0 \leq \delta \leq s} \|\Delta_{h,\delta} f(\cdot, \cdot)\|_{p,q}, \qquad t, s \geq 0, \tag{4}$$

where $\Delta_{h,\delta} f(x, y) := f(x + h, y + \delta) - f(x, y)$. Moreover, for fixed $p, q \in N_0$ and $m \in N := \{1, 2, \ldots\}$, let $C_{p,q}^m$ denote the set of all functions $f \in C_{p,q}$ with the partial derivatives $f_{x^j, y^{k-j}}^{(k)}$, $k = 1, \ldots, m$, belonging also to $C_{p,q}$.

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1.2. For $f \in C_{p,q}$, $p, q \in N_0$, we define operators $S_{m,n}(f; a_m, b_m, c_n, d_n; x, y) \equiv S_{m,n}(f; x, y)$

$$(5) \quad S_{m,n}(f; x, y) := \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \varphi_j(a_m x)\, \varphi_k(c_n y)\, f\left(\frac{j}{b_m}, \frac{k}{d_n}\right), \quad (x, y) \in R_0^2, \quad m, n \in N,$$

where

$$(6) \qquad \varphi_i(t) := e^{-t} \frac{t^i}{i!} \quad \text{for } t \in R_0, \quad i \in N_0,$$

and $(a_n)_1^\infty$, $(b_n)_1^\infty$, $(c_n)_1^\infty$, $(d_n)_1^\infty$ are given increasing and unbounded sequences of positive numbers and such that

$$(7) \qquad \frac{a_n}{b_n} = 1 + o\left(\frac{1}{b_n}\right), \qquad \frac{c_n}{d_n} = 1 + o\left(\frac{1}{d_n}\right).$$

Write

$$(8) \qquad M := \sup_{n \in N} \frac{a_n}{b_n}, \qquad M^* := \sup_{n \in N} \frac{c_n}{d_n}.$$

If $a_n = b_n = c_n = d_n = n$ for all $n \in N$, then $S_{m,n}$ defined by (5) is classical Szász-Mirakyan operator examined for continuous and bounded functions in [3].

In the paper [2] there were considered modified Szász-Mirakyan operators $S_n(f; a_n, b_n; x) \equiv S_n(f; x)$ for functions of one variable

$$(9) \qquad S_n(f; x) := \sum_{k=0}^{\infty} \varphi_k(a_n x) f\left(\frac{k}{b_n}\right), \quad x \in R_0, \quad n \in N,$$

with given sequences (a_n) and (b_n) as above. The classical Szász-Mirakyan operators, i.e. S_n with $a_n = b_n = n$ were examined in [1].

From (5)–(9) we deduce that $S_{m,n}(f)$ are well-defined in every space $C_{p,q}$, $p, q \in N_0$. Moreover we have

$$(10) \qquad S_{m,n}(1; a_m, b_m, c_n, d_n; x, y) = 1 \quad \text{for} \quad (x, y) \in R_0^2, \quad m, n \in N,$$

and if $f \in C_{p,q}$ and $f(x, y) = f_1(x) f_2(y)$ for all $(x, y) \in R_0^2$, then

$$(11) \qquad S_{m,n}(1; a_m, b_m, c_n, d_n; x, y) = S_m(f_1; a_m, b_m; x) S_n(f_2; c_n, d_n; y)$$

for all $(x, y) \in R_0^2$ and $m, n \in N$.

In Section 2 we give some auxiliary results. In Section 3 we prove main results.

In this paper by $M_k(\alpha, \beta)$ we shall denote suitable positive constants depending only on indicated parameters α, β.

2. Auxiliary results

From (9) and (6) we get for $x \in R_0$ and $n \in N$

$$S_n(1; a_n, b_n; x) = 1, \tag{12}$$

$$\begin{aligned} S_n(t-x; a_n, b_n; x) &= \left(\frac{a_n}{b_n} - 1\right) x, \\ S_n((t-x)^2; a_n, b_n; x) &= \left(\frac{a_n}{b_n} - 1\right)^2 x^2 + \frac{a_n x}{b_n^2}. \end{aligned} \tag{13}$$

In the paper [2] the following two lemmas for S_n defined by (9) were proved.

LEMMA 1. *For every fixed $p \in N_0$ there exist positive constants $M_i \equiv M_i(p, b_1, M)$, $i = 1, 2$, such that for all $x \in R_0$ and $n \in N$*

$$\begin{cases} w_p(x)\, S_n\left(\dfrac{1}{w_p(t)}; x\right) \le M_1, \\ w_p(x)\, S_n\left(\dfrac{(t-x)^2}{w_p(t)}; x\right) \le M_2 \left[\dfrac{x}{b_n} + \left(\dfrac{a_n}{b_n} - 1\right)^2 x^2\right]. \end{cases} \tag{14}$$

LEMMA 2. *For every $x \in R_0$*

$$\lim_{n\to\infty} b_n S_n((t-x)^k; x) = \begin{cases} 0 & \text{if } k = 1, \\ x & \text{if } k = 2, \end{cases}$$

$$\lim_{n\to\infty} b_n^2 S_n((t-x)^k; x) = \begin{cases} x & \text{if } k = 3, \\ 3x^2 & \text{if } k = 4. \end{cases}$$

Applying Lemma 1 we shall prove two lemmas on $S_{m,n}$ defined by (5).

LEMMA 3. *For every $p, q \in N_0$ there exists a positive constant $M_4 \equiv M_4(p, q, b_1, d_1, M, M^*)$ such that*

$$\left\| S_{m,n}\left(\frac{1}{w_{p,q}(t,z)}\right)\right\|_{p,q} \le M_4 \quad \text{for} \quad m, n \in N. \tag{16}$$

Moreover for every $f \in C_{p,q}$ we have

$$\|S_{m,n}(f)\|_{p,q} \le M_4 \|f\|_{p,q} \quad \text{for} \quad m, n \in N. \tag{17}$$

The formulas (5)–(8) and the inequality (17) show that $S_{m,n}$, $m, n \in N$, defined by (5) are linear positive operators from the space $C_{p,q}$ into $C_{p,q}$.

Proof. The inequality (16) follows immediately from (2), (11) and (14).

From (5) and (3) we get for $f \in C_{p,q}$

$$\|S_{m,n}(f)\|_{p,q} \le \|f\|_{p,q} \left\| S_{m,n}\left(\frac{1}{w_{p,q}}\right)\right\|_{p,q}, \quad m, n \in N,$$

which by (16) implies (17).

LEMMA 4. *Let $f \in C_{p,q}$, $p,q \in N_0$. Then there exists a positive constant $M_5 \equiv M_5(p,q,b_1,d_1,M,M^*)$ such that for all $m,n \in N$*

$$\|(S_{m,n}(f))'_x\|_{p,q} \le M_5 \|f\|_{p,q} a_m, \tag{18}$$

$$\|(S_{m,n}(f))'_y\|_{p,q} \le M_5 \|f\|_{p,q} c_n. \tag{19}$$

Proof. We shall prove only (18) because the proof of (19) is identical. From (5) and (6) we get

$$(S_{m,n}(f))'_x(x,y) = a_m\Big\{ - S_{m,n}(f;x,y) + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \varphi_j(a_m x)\varphi_k(c_n y) f\Big(\frac{j+1}{b_m}, \frac{k}{d_n}\Big)\Big\}$$

for all $(x,y) \in R_0^2$ and $m,n \in N$. Thus

$$\|(S_{m,n}(f))'_x\|_{p,q} \le a_m\Big\{\|S_{m,n}(f)\|_{p,q} + \|f\|_{p,q} \Big\|S_{m,n}\Big(\frac{1}{w_{p,q}(t+1/b_m,z)};\cdot;\cdot\Big)\Big\|_{p,q}\Big\}. \tag{20}$$

By (1), (2) and (5) we have

$$(w_{p,q}(t+1/b_m,z))^{-1} \le 2^p(1+b_1^{-p})(w_{p,q}(t,z))^{-1},$$

which implies the inequality

$$\|S_{m,n}((w_{p,q}(t+1/b_m,z))^{-1}\|_{p,q} \le 2^p(1+b_1^{-p})\|S_{m,n}(1/w_{p,q}(t,z)\|_{p,q}.$$

Now, using (16) and (17), we obtain (18) from (20).

3. Theorems

3.1. First we shall give two theorems on the degree of approximation of functions by $S_{m,n}$ defined by (5).

THEOREM 1. *Suppose that $f \in C^1_{p,q}$ with fixed $p,q \in N_0$. Then there exists a positive constant $M_6 = M_6(p,q,b_1,d_1,M,M^*)$ such that for all $(x,y) \in R_0^2$ and $m,n \in N$*

$$w_{p,q}(x,y)|S_{m,n}(f;x,y) - f(x,y)| \le M_6\Big\{\|f'_x\|_{p,q}\sqrt{\frac{x}{b_m}} + \|f'_y\|_{p,q}\sqrt{\frac{y}{d_n}}\Big\}. \tag{21}$$

Proof. Let $(x,y) \in R_0^2$ be a fixed point. Then for $f \in C^1_{p,q}$ we have the formula

$$f(t,z) - f(x,y) = \int_x^t f'_u(u,z)\,du + \int_y^z f'_v(x,v)\,dv, \quad (t,z) \in R_0^2.$$

Thus, by (10), we obtain

(22) $$S_{m,n}(f(t,z);x,y)-f(x,y) = S_{m,n}\left(\int_x^t f'_u(u,z)du;x,y\right)+S_{m,n}\left(\int_y^z f'_v(x,v)dv;x,y\right).$$

But, by (1)–(3), we have

$$\left|\int_x^t f'_u(u,z)\,du\right| \le \|f'_x\|_{p,q}\left|\int_x^t \frac{du}{w_{p,q}(u,z)}\right| \le \|f'_x\|_{p,q}\left(\frac{1}{w_{p,q}(t,z)}+\frac{1}{w_{p,q}(x,z)}\right)|t-x|,$$

which implies by, (1), (2), (5) and (9)–(12), that

$$\begin{aligned} w_{p,q}(x,y)&\left|S_{m,n}\left(\int_x^t f'_u(u,z)du;x,y\right)\right| \\ &\le w_{p,q}(x,y)S_{m,n}\left(\left|\int_x^t f'_u(u,z)\,du\right|;x,y\right) \\ &\le \|f'_x\|_{p,q}\,w_{p,q}(x,y) \\ &\quad\cdot\left\{S_{m,n}\left(\frac{|t-x|}{w_{p,q}(t,z)};x,y\right)+S_{m,n}\left(\frac{|t-x|}{w_{p,q}(x,z)};x,y\right)\right\} \\ &\le \|f'_x\|_{p,q}\,w_q(y)S_n\left(\frac{1}{w_q(z)};c_n,d_n;y\right) \\ &\quad\cdot\left\{w_p(x)S_m\left(\frac{|t-x|}{w_p(t)};a_m,b_m;x\right)+S_m\left(|t-x|;a_m,b_mx\right)\right\}. \end{aligned}$$

Applying the Hölder inequality and (12)–(15), we get the inequalities

$$\begin{aligned} S_m\left(|t-x|;a_m,b_m;x\right) &\le \left\{S_m((t-x)^2;a_m,b_m;x)S_m(1;a_m,b_m;x)\right\}^{\frac{1}{2}} \\ &\le M_7(M)\sqrt{\frac{x}{b_m}}, \end{aligned}$$

$$\begin{aligned} &w_p(x)S_m\left(\frac{|t-x|}{w_{p(t)}};a_m,b_m;x\right) \\ &\le \left\{w_p(x)S_m\left(\frac{(t-x)^2}{w_{p(t)}};a_m,b_m;x\right)\right\}^{\frac{1}{2}}\left\{w_p(x)S_m\left(\frac{1}{w_{p(t)}};a_m,b_m;x\right)\right\}^{\frac{1}{2}} \\ &\le M_8(p,b_1,M)\sqrt{\frac{x}{b_m}} \end{aligned}$$

for $x \in R_0$ and $m \in N$. Consequently

$$w_{p,q}(x,y)\left|S_{m,n}\left(\int\limits_x^t f'_u(u,z)du;x,y\right)\right| \le M_9(p,b_1,M)\sqrt{\frac{x}{b_m}}, \quad m \in N.$$

Analogously we obtain

$$w_{p,q}(x,y)\left|S_{m,n}\left(\int\limits_y^z f'_v(x,v)dv;x,y\right)\right| \le M_{10}(q,d_1,M^*)\sqrt{\frac{y}{d_n}}, \quad n \in N.$$

Combining the last two inequalities, we derive from (22)

$$w_{p,q}(x,y)\left|S_{m,n}(f;x,y) - f(x,y)\right| \le M_{11}\left\{\|f'_x\|_{p,q}\sqrt{\frac{x}{b_m}} + \left\|f'_y\right\|_{p,q}\sqrt{\frac{y}{d_n}}\right\},$$

for all $m,n \in N$, $M_{11} = M_{11}(p,q,b_1,d_1,M,M^*) = \text{const.} > 0$. Thus the proof of (21) is completed.

THEOREM 2. *Suppose that $f \in C_{p,q}$, $p,q \in N_0$. Then there exists a positive constant $M_{12} \equiv M_{12}(p,q,b_1,d_1,M,M^*)$ such that*

$$w_{p,q}(x,y)\left|S_{m,n}(f;x,y) - f(x,y)\right| \le M_{12}\,\omega\left(f, C_{p,q}; \sqrt{\frac{x}{b_m}}, \sqrt{\frac{y}{d_n}}\right), \tag{23}$$

for all $(x,y) \in R_0^2$ and $m,n \in N$.

Proof. We shall apply the Stieklov function $f_{h,\delta}$ for $f \in C_{p,q}$

$$f_{h,\delta}(x,y) := \frac{1}{h\delta}\int\limits_0^h du \int\limits_0^\delta f(x+u,y+v)dv, \quad (x,y) \in R_0^2, h,\delta > 0. \tag{24}$$

From (24) it follows that

$$f_{h,\delta}(x,y) - f(x,y) = \frac{1}{h\delta}\int\limits_0^h du \int\limits_0^\delta \Delta_{u,v}f(x,y)\,dv,$$

and therefore

$$(f_{h,\delta})'_x(x,y) = \frac{1}{h\delta}\int\limits_0^\delta \left(\Delta_{h,v}f(x,y) - \Delta_{0,v}f(x,y)\right)dv,$$

$$(f_{h,\delta})'_y(x,y) = \frac{1}{h\delta}\int\limits_0^h \left(\Delta_{u,\delta}f(x,y) - \Delta_{u,0}f(x,y)\right)du.$$

Thus we have

$$\|f_{h,\delta} - f\|_{p,q} \le \omega(f, C_{p,q}; h,\delta), \tag{25}$$

$$\|(f_{h,\delta})'_x\|_{p,q} \le 2h^{-1}\omega(f, C_{p,q}; h,\delta), \tag{26}$$

$$\|(f_{h,\delta})'_y\|_{p,q} \le 2\delta^{-1}\omega(f, C_{p,q}; h,\delta), \tag{27}$$

for all $h, \delta > 0$, which show that $f_{h,\delta} \in C^1_{p,q}$ if $f \in C_{p,q}$ and $h, \delta > 0$. Now, for $S_{m,n}$ defined by (5), we can write

$$\begin{aligned} w_{p,q}(x,y)\,|S_{m,n}(f;x,y) - f(x,y)| \\ \le w_{p,q}(x,y)\,\{|S_{m,n}\,(f(t,z) - f_{h,\delta}(t,z);x,y)| \\ + |S_{m,n}\,(f_{h,\delta}(t,z);x,y) - f_{h,\delta}(x,y)| \\ + |f_{h,\delta}(x,y) - f(x,y)|\} := T_1 + T_2 + T_3. \end{aligned}$$

By (3), (17) and (25) we get

$$\begin{aligned} T_1 &\le \|S_{m,n}(f - f_{h,\delta};\cdot,\cdot)\|_{p,q} \le M_4\|f - f_{h,\delta}\|_{p,q} \\ &\le M_4\,\omega(f, C_{p,q}; h, \delta), \\ T_3 &\le \omega(f, C_{p,q}; h, \delta). \end{aligned}$$

Applying Theorem 1 and (26) and (27), we get

$$\begin{aligned} T_2 &\le M_6\left\{\left\|(f_{h,\delta})'_x\right\|_{p,q}\sqrt{\frac{x}{b_m}} + \left\|(f_{h,\delta})'_y\right\|_{p,q}\sqrt{\frac{y}{d_n}}\right\} \\ &\le 2M_6\omega\,(f, C_{p,q}; h, \delta)\left\{h^{-1}\sqrt{\frac{x}{b_m}} + \delta^{-1}\sqrt{\frac{y}{d_n}}\right\}. \end{aligned}$$

Consequently there exists $M_{13} \equiv M_{13}(p, q, b_1, d_1, M, M^*)$ such that

$$\begin{aligned} \text{(28)} \quad & w_{p,q}(x,y)\,|S_{m,n}(f;x,y) - f(x,y)| \\ &\le M_{13}\omega\,(f, C_{p,q}; h, \delta)\left\{1 + h^{-1}\sqrt{\frac{x}{b_m}} + \delta^{-1}\sqrt{\frac{y}{d_n}}\right\}, \end{aligned}$$

for $(x,y) \in R^2$, $m,n \in N$ and $h, \delta > 0$. Now, for fixed $x, y > 0$ and $m, n \in N$ setting $h = \sqrt{\frac{x}{b_m}}$ and $\delta = \sqrt{\frac{y}{d_n}}$ to (28), we obtain (23).

If $x = 0 = y$, then by (5) we get $S_{m,n}(f;0,0) = f(0,0)$, $m,n \in N$ which implies (23). If $x = 0$, $y > 0$ or $x > 0$, $y = 0$, we obtain (23) similarly as in [2].

From Theorem 2 follows

COROLLARY. *Let $f \in C_{p,q}$, $p, q \in N_0$. Then*

$$\text{(29)} \qquad \lim_{m,n\to\infty} S_{m,n}(f;x,y) = f(x,y) \quad \textit{for all } (x,y) \in R_0^2.$$

Moreover (29) *holds uniformly on every rectangle* $0 \le x \le x_0$, $0 \le y \le y_0$.

3.2. In this part we give the Voronovskaya type theorem for the operators

$$\text{(30)} \qquad S_{n,n}(f(t,z);x,y) := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\varphi_j(a_n x)\varphi_k(c_n y) f\left(\frac{j}{b_n}, \frac{k}{b_n}\right),$$

$(x,y) \in R_0^2$, $n \in N$, where $(a_n)_1^\infty$, $(b_n)_1^\infty$ and $(c_n)_1^\infty$ are given sequences of positive numbers such that

$$\frac{a_n}{b_n} = 1 + o\left(\frac{1}{b_n}\right) \quad \text{and} \quad \frac{c_n}{b_n} = 1 + o\left(\frac{1}{b_n}\right). \tag{31}$$

THEOREM 3. *Suppose that $f \in C_{p,q}^2$, $p,q \in N_0$. Then for every $(x,y) \in R_+^2 := \{(x,y) : x > 0, y > 0\}$*

$$\lim_{n\to\infty} b_n \{S_{n,n}(f;x,y) - f(x,y)\} = \frac{x}{2} f''_{xx}(x,y) + \frac{y}{2} f''_{yy}(x,y). \tag{32}$$

Proof. Choosing $(x,y) \in R_+^2$, by the Taylor formula for $f \in C_{p,q}^2$, we have

$$\begin{aligned} f(t,z) &= f(x,y) + f'_x(x,y)(t-x) + f'_y(x,y)(z-y) \\ &\quad + \frac{1}{2}\{f''_{xx}(x,y)(t-x)^2 + 2f''_{xy}(x,y)(t-x)(z-y) + f''_{yy}(x,y)(z-y)^2\} \\ &\quad + \varepsilon_1(t,z;x,y)\sqrt{(t-x)^4 + (z-y)^4}, \quad (t,z) \in R_0^2, \end{aligned}$$

where $\varepsilon_1(t,z) = \varepsilon_1(t,z;x,y)$ is a function form $C_{p,q}$ and $\varepsilon_1(x,y) = 0$. From this and by (30), (5),(9)–(12) we get

$$\begin{aligned} S_{n,n}(f(t,z);x,y) = f(x,y) &+ f'_x(x,y)S_n(t-x;a_n,b_n;x) \\ &+ f'_y(x,y)S_n(z-y;c_n,b_n;y) \\ &+ \frac{1}{2}\{f''_{xx}(x,y)S_n((t-x)^2;a_n,b_n;x) \\ &+ 2f''_{xy}(x,y)S_n(t-x;a_n,b_n;x)S_n(z-y;c_n,b_n;y) \\ &+ f''_{yy}(x,y)S_n((z-y)^2;c_n,b_n;y)\} \\ &+ S_{n,n}\left(\varepsilon_1(t,z)\sqrt{(t-x)^4+(z-y)^4};x,y\right) \quad \text{for } n \in N, \end{aligned}$$

which, by (13), (31) and Lemma 2, implies that

$$\begin{aligned} \lim_{n\to\infty} b_n \{S_{n,n}(f;x,y) - f(x,y)\} &= \frac{x}{2} f''_{xx}(x,y) + \frac{y}{2} f''_{yy}(x,y) \\ &\quad + \lim_{n\to\infty} b_n S_{n,n}\left(\varepsilon_1(t,z)\sqrt{(t-x)^4+(z-y)^4};x,y\right). \end{aligned} \tag{33}$$

By the Hölder inequality and by (9)–(12) we have

$$\begin{aligned} &\left|S_{n,n}\left(\varepsilon_1(t,z)\sqrt{(t-x)^4+(z-y)^4};x,y\right)\right| \\ &\le \{S_{n,n}(\varepsilon_1^2(t,z);x,y)\}^{\frac{1}{2}}\{S_n((t-x)^4;a_n,b_n;x) + S_n((z-y)^4;c_n,b_n;y)\}^{\frac{1}{2}}. \end{aligned} \tag{34}$$

The properties of ε_1 and Corollary imply that

$$\lim_{n\to\infty} S_{n,n}\left(\varepsilon_1^2(t,z);x,y\right) = \varepsilon_1^2(x,y) = 0. \tag{35}$$

Using (35) and Lemma 2, we obtain from (34)

$$\lim_{n\to\infty} b_n\, S_{n,n}\left(\varepsilon_1(t,z)\,\sqrt{(t-x)^4+(z-y)^4};x,y\right)=0. \tag{36}$$

From (36) and (33) follows (32).

3.3. Now, we shall prove certain analogue of (29) for derivatives of operators $S_{n,n}$ defined in (30).

THEOREM 4. *Let $f\in C^1_{p,q}$ with some $p,q\in N_0$. Then for every $(x,y)\in R^2_+$* we have

$$\lim_{n\to\infty}(S_{n,n}(f))'_x=f'_x(x,y), \tag{37}$$

$$\lim_{n\to\infty}(S_{n,n}(f))'_y=f'_y(x,y), \tag{38}$$

Proof. We shall prove only (37) because the proof of (38) is identical. Similarly as in the proof of Lemma 4, we get, for $S_{n,n}$ defined by (30), the relations

$$\begin{aligned}(S_{n,n})'_x(x,y)=&\ -a_nS_{n,n}(f(t,z);x,y)+\frac{b_n}{x}S_{n,n}(tf(t,z);x,y)\\ =&\ (b_n-a_n)S_{n,n}(f(t,z);x,y)\\ &+\frac{b_n}{x}S_{n,n}((t-x)f(t,z):x,y),\quad n\in N.\end{aligned}$$

For fixed $(x,y)\in R^2_+$, we have, by the Taylor formula for $f\in C^1_{p,q}$,

$$\begin{aligned}f(t,z)=&\ f(x,y)+f'_x(x,y)(t-x)+f'_y(x,y)(z-y)\\ &+\varepsilon_2(t,z;x,y)\,\sqrt{(t-x)^2+(z-y)^2},\quad (t,z)\in R^2_0,\end{aligned}$$

where $\varepsilon_2(t,z)=\varepsilon_2(t,z;x,y)$ is a function with $C_{p,q}$ and $\varepsilon_2(x,y)=0$. From this and by (9)–(12) it follows that

$$\begin{aligned}&(S_{n,n}(f))'_x(x,y)\\ &=(b_n-a_n)\Big\{f(x,y)+f'_x(x,y)\,S_n(t-x;a_n,b_n;x)\\ &\quad+f'_y(x,y)\,S_n(z-y;c_n,b_n;y)\\ &\quad+S_{n,n}\left(\varepsilon_2(t,z)\,\sqrt{(t-x)^2+(z-y)^2};x,y\right)\Big\}\\ &\quad+\frac{b_n}{x}\Big\{f(x,y)S_n(t;a_n,b_n;x)+f'_x(x,y)\,S_n((t-x)^2;a_n,b_n;x)\\ &\quad+f'_y(x,y)\,S_n(t-x;a_n,b_n;x)S_n(z-y;c_n,b_n;y)\\ &\quad+S_{n,n}\left(\varepsilon_2(t,z)(t-x)\sqrt{(t-x)^2+(z-y)^2};x,y\right)\Big\},\quad n\in N.\end{aligned} \tag{39}$$

The properties of ε_2 and Corollary imply that

$$\lim_{n\to\infty} S_{n,n}\left(\varepsilon_2(t,z)\sqrt{(t-x)^2+(z-y)^2};x,y\right)=0 \tag{40}$$

and

$$\lim_{n\to\infty} S_{n,n}\left(\varepsilon_2(t,z);x,y)\right)=\varepsilon_2^2(x,y)=0. \tag{41}$$

By the Hölder inequality and by (9)–(12) we get the inequality

$$\begin{aligned}&\left|S_{n,n}\left(\varepsilon_2(t,z)(t-x)\sqrt{(t-x)^2+(z-y)^2};x,y\right)\right|\\ &\qquad\le \left\{S_{n,n}\left(\varepsilon_2^2(t,z);x,y\right)\right\}^{\frac{1}{2}}\left\{S_n\left((t-x)^4;a_n,b_n;x\right)\right.\\ &\qquad\quad \left.+S_n\left((t-x)^2;a_n,b_n;x\right)S_n\left((z-y)^2;c_n,b_n;y\right)\right\}^{\frac{1}{2}},\end{aligned}$$

which, by Lemma 2 and (41), implies that

$$\lim_{n\to\infty} b_n S_{n,n}\left(\varepsilon_2(t,z)(t-x)\sqrt{(t-x)^2+(z-y)^2};x,y\right)=0. \tag{42}$$

Using (40), (42) and Lemma 2 to (39), we obtain the desired assertion (37).

The above theorems extend some results obtained for classical Szász-Mirakyan operators, i.e. $S_{m,n}$ defined by (5) with $a_n=b_n=c_n=d_n=n$ for all $n\in N$.

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INSTITUTE OF MATHEMATICS
POZNAŃ UNIVERSITY OF THECHNOLOGY
Piotrowo 3A,
60-965 POZNAŃ, POLAND

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