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OSCILLATION PROPERTIES FOR CERTAIN HYPERBOLIC EQUATIONS WITH DISTRIBUTED ARGUMENTS

1. Introduction

Recently, it has been an increasing interest in oscillation theory of hyperbolic partial functional differential equations. We can refer to [1-6] and their cited references. However, it seems that very little is known about the work of the case with continuous distributed deviating arguments. The purpose of this paper is to extend some of the known in the literature results to the more general equations, with continuous distributed deviating arguments. Namely we obtain some oscillatory criteria for the boundary value problem of the form

$$(E)\quad \frac{\partial^2}{\partial t^2}\Big[u+\sum_{i=1}^{n}\lambda_i(t)u(x,\tau_i(t))\Big]=a(t)\Delta u+\sum_{j=1}^{m}a_j(t)\Delta u(x,\rho_j(t))$$
$$-c(x,t,u,u[x,\eta(t)])-\int_a^b q(x,t,\xi)u[x,g(t,\xi)]d\sigma(\xi)+f(x,t)$$

and

$$(B)\qquad \frac{\partial u}{\partial n}=\psi(x,t)\quad \text{on}\quad (x,t)\in\partial\Omega\times R_+,$$

where Δu is a Laplace operator in R^n, $(x,t)\in\Omega\times R_+=G$, $R_+=[0,+\infty)$, $u=u(x,t)$. Ω is a bounded domain in R^n with a piecewise smooth boundary

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$\partial\Omega$, $\psi(x,t)$ is a continuous function on $\partial\Omega \times R_+$ and n denotes the unit exterior normal vector to $\partial\Omega$.

It is easy to see that $Eq.(E)$ includes the following delay hyperbolic equations

$$(E') \qquad \frac{\partial^2}{\partial t^2}\Big[u + \sum_{i=1}^{n} \lambda_i(t) u(x, t-\tau_i)\Big] = a(t)\Delta u + \sum_{j=1}^{m} a_j(t)\Delta u(x, \rho_j(t))$$
$$-c(x,t,u,u[x,\eta(t)]) - \sum_{j=1}^{m} q_j(x,t)u[x,g_j(t)] + f(x,t).$$

Our results extend some of the known in the literature theorems. For example, Kreith, Kusano and Yoshida in [2]; Chen and Yu in [3] concerned the following equations

$$(E_1) \qquad \frac{\partial^2 u}{\partial t^2} = \Delta u - c(x,t,u) + f(x,t),$$

$$(E_2) \qquad \frac{\partial^2}{\partial t^2}[u + \lambda u(x,t-\tau)] = \Delta u - c(x,t,u) + f(x,t)$$

respectively. Those equations all are special case of $Eq.(E')$.

Suppose that the following conditions (H) hold:

(H_1) $a(t), a_j(t), \lambda_i(t), \tau_i(t), \rho_j(t), \eta(t) \in C(R_+, R_+)$, $q(x,t,\xi) \in C(\overline{\Omega} \times R_+ \times [a,b], R_+)$; $\tau_i(t) \le t, \rho_j(t) \le t$, $i = 1,2\ldots,n$; $j = 1,2,\ldots m$; and $\lim_{t\to+\infty} \tau_i(t) = \lim_{t\to+\infty} \rho_j(t) = \lim_{t\to+\infty} \eta(t) = +\infty$,

(H_2) $g(t,\xi) \in C(R_+ \times [a,b], R); g(t,\xi) \le t, \xi \in [a,b]; g(t,\xi)$ are nondecreasing with to t, ξ; and $\lim_{t\to+\infty} \min_{\xi\in[a,b]} \{g(t,\xi)\} = +\infty$,

(H_3) $c(x,t,u,v) \in C(\overline{\Omega} \times R_+ \times R \times R, R); c(x,t,u,v) \ge p(t)\varphi(v)$, in which $p(t) \in C(R_+, R_+)$, $\varphi(v) \in C([a,b], R)$; $\varphi(v)$ is a positive and convex function in $(0,+\infty)$, and $c(x,t,-u,-v) = -c(x,t,u,v)$,

(H_4) $f(x,t) \in C(\overline{\Omega} \times R_+, R); \sigma(\xi) \in ([a,b], R)$ is nondecreasing, integral of $Eq.(E)$ is Stieltjes integral.

A solution $u(x,t)$ of $Eq.(E)$ is called oscillatory in the domain G if for each positive number μ there exists a point $(x_0, t_0) \in \Omega \times [\mu, +\infty)$ such that the condition $u(x_0, t_0) = 0$ holds.

2. Oscillation criteria

LEMMA 1. *Suppose that (H_1)-(H_4) hold. If u is a positive solution of the Eq.(E), (B) in $\Omega \times [\mu, +\infty)$, $\mu \geq 0$, then the function*

$$U(t) = \frac{\int_\Omega u(x,t)dx}{\int_\Omega dx} \tag{2.1}$$

satisfies the following differential inequality

$$\frac{d^2}{dt^2}\left[U(t) + \sum_{i=1}^{n} \lambda_i(t) U(\tau_i(t))\right] + p(t)\varphi(U(\eta(t))) + \int_a^b Q(t,\xi) U[g(t,\xi)] d\sigma(\xi) \leq H(t) \tag{2.2}$$

in which $Q(t,\xi) = \min_{x\in\Omega}\{q(x,t,\xi)\}$,

$$H(t) = \left[\int_\Omega dx\right]^{-1} \left\{ \int_{\partial\Omega} \left[a(t)\psi(x,t) + \sum_{j=1}^{m} a_i(t)\psi(x,\rho_j(t))\right] d\omega + \int_\Omega f(x,t)\, dx\right\}.$$

P r o o f. Let $u(x,t)$ be a positive solution of the $Eq.(E)$, (B) in $\Omega \times [\mu, +\infty)$, for $\mu \geq 0$. Note that by (H_2), there exist a $t_1 \geq \mu$ such that $u(x, g(t,\xi)) > 0$, $(x,t,\xi) \in \Omega \times [t_1, +\infty) \times [a,b]$, and $u(x,\tau_i(t)) > 0, u(x,\rho_j(t)) > 0, u(x,\eta(t)) > 0$, $(x,t) \in \Omega \times [t_1, +\infty)$

Using Green's formula, we have

$$\int_\Omega \triangle u\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} d\omega = \int_\Omega \psi\, d\omega, \quad t \geq t_1 \tag{2.3}$$

and

$$\int_\Omega \triangle u(x,\rho_j(t))\, dx = \int_{\partial\Omega} \frac{\partial u(x,\rho_j(t))}{\partial n}\, d\omega = \int_\Omega \psi(x,\rho_j(t))\, d\omega, \quad t \geq t_1. \tag{2.4}$$

Integrating $Eq.(E)$ with respect to x over the domain Ω, we have

$$\begin{aligned} \frac{d^2}{dt^2}\left[\int_\Omega u\, dx + \sum_{i=1}^{n} \lambda_i(t) \int_\Omega u(x,\tau_i(t))\, dx\right] &= a(t) \int_{\partial\Omega} \psi\, d\omega \\ &+ \sum_{j=1}^{m} a_j(t) \int_{\partial\Omega} \psi(x,\rho_j(t))\, d\omega - \int_\Omega c(x,t,u,u[x,\eta(t)])\, dx \\ &- \int_\Omega \int_a^b q(x,t,\xi) u[x,g(t,\xi)] d\sigma(\xi) dx + \int_\Omega f(x,t)\, dx, \quad t \geq t_1. \end{aligned} \tag{2.5}$$

Using the condition (H_3) and Jensen's inequality, we have

$$\int_\Omega c(x,t,u,u[x,\eta(t)])\,dx \geq p(t)\int_\Omega \varphi(u(x,\eta(t)))\,dx \geq p(t)\varphi\left(\frac{\int_\Omega u(x,\eta(t))dx}{\int_\Omega dx}\right)\int_\Omega dx, \quad t \geq t_1. \tag{2.6}$$

Notice that

$$\int_\Omega q(x,t,\xi)u[x,g(t,\xi)]\,dx \geq Q(t,\xi)\int_\Omega u[x,g(t,\xi)]\,dx \tag{2.7}$$

and

$$\int_\Omega\int_a^b q(x,t,\xi)u[x,g(t,\xi)]\,d\sigma(\xi)\,dx = \int_a^b\int_\Omega q(x,t,\xi)u[x,g(t,\xi)]\,dx\,d\sigma(\xi). \tag{2.8}$$

So we have

$$\frac{d^2}{dt^2}\left[\int_\Omega u\,dx + \sum_{i=1}^n \lambda_i(t)\int_\Omega u(x,\tau_i(t))\,dx\right] + p(t)\varphi(U(\eta(t)))\int_\Omega u\,dx + \int_a^b Q(t,\xi)\int_\Omega u[x,g(t,\xi)]\,dx$$
$$\leq a(t)\int_{\partial\Omega}\psi\,d\omega + \sum_{j=1}^m a_j(t)\int_{\partial\Omega}\psi(x,\rho_j(t))\,d\omega + \int_\Omega f(x,t)\,dx, \quad t \geq t_1 \tag{2.9}$$

and therefore $U(t)$ is a solution of the inequality (2.2).

THEOREM 1. *Suppose that (H_1)-(H_4) hold. If both the differential inequalities*

$$\frac{d^2}{dt^2}\left[U(t) + \sum_{i=1}^n \lambda_i(t)U(\tau_i(t))\right] + p(t)\varphi(U(\eta(t)) + \int_a^b Q(t,\xi)U[g(t,\xi)]d\sigma(\xi) \leq H(t). \tag{2.10}$$

$$\frac{d^2}{dt^2}\left[U(t) + \sum_{i=1}^n \lambda_i(t)U(\tau_i(t))\right] + p(t)\varphi(U(\eta(t))) + \int_a^b Q(t,\xi)U[g(t,\xi)]\,d\sigma(\xi) \leq -H(t) \tag{2.11}$$

have no eventually positive solution, then every solution of Eq.(E) with (B) is oscillatory in G.

Proof. Assume to a contrary that there exists a nonoscillatory solution $u(x,t)$ of the *Eq.*(E), (B). If $u(x,t) > 0$, $(x,t) \in \Omega \times [\mu, +\infty)$, $\mu \geq 0$, then from Lemma 1 it follows that $U(t)$ defined by (2.1) is an eventually positive solution of the inequality (2.10), which is a contradiction.

If $u(x,t) < 0$, $(x,t) \in \Omega \times [\mu, +\infty)$, $\mu \geq 0$, then $v(x,t) = -u(x,t)$ is a positive solution of the problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\Big[u + \sum\limits_{i=1}^{n} \lambda_i(t) u(x, \tau_i(t))\Big] \\ = a(t)\Delta u + \sum\limits_{j=1}^{m} a_j(t)\Delta u(x, \rho_j(t)) - c(x,t,u,u[x,\eta(t)]) \\ - \int\limits_a^b q(x,t,\xi) u[x, g(t,\xi)]\, d\sigma(\xi) + f(x,t), \\ \dfrac{\partial u}{\partial n} = -\psi(x,t), \qquad \text{on } (x,t) \in \partial\Omega \times R_+. \end{cases}$$

Using the above-mentioned method, we can conclude that

$$V(t) = \frac{\int_\Omega v(x,t)dx}{\int_\Omega dx}$$

is an eventually positive solution of the inequality (2.11), which is a contradiction. This completes the proof of Theorem 1.

THEOREM 2. *Suppose that* (H_1)*–*(H_4) *hold and moreover*

$$\liminf_{t \to +\infty} \int\limits_{t_1}^{t} \left(1 - \frac{s}{t}\right) H(s)\, ds = -\infty \tag{2.12}$$

and

$$\limsup_{t \to +\infty} \int\limits_{t_1}^{t} \left(1 - \frac{s}{t}\right) H(s) ds = +\infty \tag{2.13}$$

for sufficiently large t_1. *Then every solution of the Eq.*(E) *with* (B) *is oscillatory in* G.

Proof. Assume to a contrary that there exists a nonoscillatory solution. If $u(x,t) > 0$, $(x,t) \in \Omega \times [\mu, +\infty)$, for $\mu \geq 0$, then from Lemma 1 it follows that the function $U(t)$ defined in (2.1) is an eventually positive solution of the inequality (2.1). Then

$$\frac{d^2}{dt^2}\Big[U(t) + \sum_{i=1}^{n} \lambda_i(t) U(\tau_i(t))\Big] \leq H(t), \quad t \geq t_1 \geq \mu. \tag{2.14}$$

Integrating the above inequality twice in the segment $[t_1, t]$, we get

$$(2.15)\quad U(t) + \sum_{i=1}^{n} \lambda_i(t) U(\tau_i(t)) \leq c_1 + c_2(t - t_1) + \int_{t_1}^{t} \int_{t_1}^{\eta} H(s)\, ds\, d\omega, \quad t \geq t_1 \geq \mu$$

in which c_1, c_2 are constants. Notice that

$$\int_{t_1}^{t} \int_{t_1}^{\eta} H(s)\, ds\, d\eta = \int_{t_1}^{t} (t - s) H(s)\, ds.$$

Therefore from (2.15) we have

$$(2.16)\quad \frac{1}{t}\Big[U(t) + \sum_{i=1}^{n} \lambda_i(t) U(\tau_i(t))\Big] \leq \frac{c_1}{t} + c_2\Big(1 - \frac{t_1}{t}\Big) + \int_{t_1}^{t} \Big(1 - \frac{s}{t}\Big) H(s)\, ds.$$

Now we get

$$\liminf_{t \to +\infty} \frac{1}{t} \left[U(t) + \sum_{i=1}^{n} \lambda_i(t) U(\tau_i(t))\right] = -\infty$$

which contradicts with the assumption that $U(t) > 0$.

If $u(x,t) < 0$, $(x,t) \in \Omega \times [\mu, +\infty)$, $\mu \geq 0$, let $v(x,t) = -u(x,t)$. Then

$$V(t) = \frac{\int_\Omega v(x,t)dx}{\int_\Omega ds}$$

is an eventually positive solution of the inequality (2.11).

Using (2.13), we get

$$\liminf_{t \to +\infty} \int_{t_1}^{t} \Big(1 - \frac{s}{t}\Big)(-H(s))ds = -\limsup_{t \to +\infty} \int_{t_1}^{t} \Big(1 - \frac{s}{t}\Big) H(s)\, ds = -\infty$$

thus, by using the above-mentioned similar arguments we have a contradiction. This completes the proof of Theorem 2.

EXAMPLE. Consider the following equation

$$(E^*)\quad \frac{\partial^2}{\partial t^2}[u + u(x, t - \pi)] = \frac{1}{2}\Delta u - \frac{1}{2}u - 4 \int_{-\pi}^{0} u(x, t + \xi)\, d\xi + e^t \cos x \sin t + e^{t-\pi} \cos x \cos t$$

with the boundary condition

$$(B^*) \qquad \frac{\partial u}{\partial n}(0,t) = 0, \quad \frac{\partial u}{\partial n}\left(\frac{\pi}{2}, t\right) = -e^{-t}\sin t,$$

where $n = 1, \Omega = (0, \frac{\pi}{2})$, $a(t) = \frac{1}{2}$, $q(x,t,\xi) = 4$, $c(x,t,u) = \frac{1}{2}u$, $g(t,\xi) = t + \xi$, $g(t,\xi) = t + \xi$, $f(x,t) = e^t \cos x \sin t + e^{t-\pi}\cos x \cos t$, $\psi(0,t) = 0$, $\psi(\frac{\pi}{2}, t) = -e^t \sin t$.

It is to easy see that the condition (H) holds, and

$$\int_{t_1}^{t}\left(1 - \frac{s}{t}\right)\left\{\int_{\partial\Omega}\left[a(s)\psi(x,s) + \sum_{j=1}^{m} a_j(s)\psi(x,\rho_j(s))\right]d\omega + \int_{\Omega} f(x,s)\,dx\right\}ds$$
$$= e^t\left\{\left(\frac{1}{2} - \frac{e^{-\pi}}{2} + \frac{1}{2}t\right) - \left(\frac{1}{4} + \frac{e^{-\pi}}{2} + \frac{e^{-\pi}}{t}\right)\cos t\right\} + \frac{c}{t},$$

where c is a constant. Hence all the conditions of Theorem 2 hold. It follows from Theorem 2 that every solution of the problem $(E^*), (B^*)$ is oscillatory in $(0, \frac{\pi}{2}) \times [0, +\infty)$. For example $u(x,t) = e^t \cos x \sin t$ is oscillatory.

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