DEMONSTRATIO MATHEMATICA
Vol. XXXIII No 1 2000

Serhiy Lavrenyuk, Lech Zaręba

THE INITIAL-BOUNDARY VALUE PROBLEM FOR THE FIRST ORDER DEGENERATED HYPERBOLIC SYSTEM

Abstract. In this paper we consider the linear hyperbolic system of the first order with degeneracy at $x \to 0$ and $x \to l$. For such system we assume that initial data are unbounded on the interval $(0, l)$. Some conditions of the uniqueness, existence and stability of solution for the initial-boundary value problem are obtained.

The initial and initial-boundary value problem for the first order linear hyperbolic systems were investigated by many authors. In some of these papers [1]–[6] hyperbolic systems degenerated at $t \to 0$ were considered. In this paper we shall study the linear hyperbolic system of the first order with degeneracy at $x \to 0$ and $x \to l$. For such system we assume that initial data are unbounded on the interval $(0, l)$. We obtain some conditions of the uniqueness, existence and stability of solutions for the initial-boundary value problem. Notice that the stability of solutions of some initial and initial-boundary value problems for linear hyperbolic systems and equations (without degeneracy) were studied in papers [7]–[18].

Let $Q = \{(x,t) : 0 < x < l,\ 0 < t < \infty\}$. We shall consider in this domain the hyperbolic system of the form

$$u_{it}(x,t) + \lambda_i(x)u_{ix}(x,t) + \sum_{j=1}^{n} a_{ij}(x,t)u_j(x,t) = f_i(x,t) \quad i = 1, \dots, n. \tag{1}$$

Suppose that the following conditions for the functions λ_i hold:

$$\begin{aligned} &\lambda_1(x) \geq \lambda_1(x) \geq \dots \geq \lambda_{k_1}(x) > 0,\ x \in (0, l];\quad \lambda_1(0) = \dots = \lambda_{k_1}(0) = 0; \\ &0 > \lambda_{k_1+1}(x) \geq \dots \geq \lambda_{k_2}(x),\ x \in [0, l);\quad \lambda_{k_1+1}(l) = \dots = \lambda_{k_2}(l) = 0; \\ &0 > \lambda_{k_2+1}(x) \geq \dots \geq \lambda_n(x),\ x \in [0, l]. \end{aligned} \tag{2}$$

For the system (1) we consider initial data

$$u_i(x, 0) = \psi_i(x), \quad i = 1, \dots, n \tag{3}$$

and boundary data

$$u_i(l,t) = 0, \quad i = k_2 + 1, \dots, n. \tag{4}$$

Denote by $H^1_{\alpha,\beta,\lambda}(0,l)$, $\alpha > 0$, $\beta > 0$, a closed set of functions belonging to $C_0^\infty[0,l]$ such that

$$\int_0^l [g^2(x) + \lambda^2(x) g_x^2(x)] x^\alpha (l-x)^\beta dx < \infty$$

with the norm

$$\|g\|_{\alpha,\beta,\lambda} = \left(\int_0^l [g^2(x) + \lambda^2(x) g_x^2(x)] x^\alpha (l-x)^\beta dx \right)^{1/2}.$$

Moreover, let $L^2_{\alpha,\beta}(0,l)$ denote a closed set of functions belonging to $C_0^\infty[0,l]$ such that

$$\int_0^l g^2(x) x^\alpha (l-x)^\beta dx < \infty$$

with the norm

$$\|g\|_{\alpha,\beta} = \left(\int_0^l g^2(x) x^\alpha (l-x)^\beta dx \right)^{1/2}.$$

Rewrite the matrix $A = (a_{ij})$ of the system (1) in the form

$$A(x,t) = \begin{pmatrix} A_{11}(x,t) & A_{12}(x,t) \\ A_{21}(x,t) & A_{22}(x,t) \end{pmatrix},$$

where A_{11} is the square matrix of the size $k_2 \times k_2$, A_{22} – the square matrix of the size $(n-k_2) \times (n-k_2)$, A_{12} – the matrix of the size $k_2 \times (n-k_2)$, and A_{21} – the matrix of the size $(n-k_2) \times k_2$.

Define the sets $I_1 = \{1, \dots, k_1\}$, $I_2 = \{k_1+1, \dots, k_2\}$, and $I_3 = \{k_2 + 1, \dots, n\}$. Let

$$\beta_i = \begin{cases} \beta, & i \in I_1 \cup I_2, \\ 0, & i \in I_3, \end{cases}$$

$Q_T = \{(x,t) : 0 < x < l,\ 0 < t < T\}$, $T > 0$.

THEOREM 1. *Let the conditions* (2) *hold. Moreover, let*

$$x^{-1}\lambda_i \in L^\infty(0,l),\ i \in I_1;$$
$$(l-x)^{-1}\lambda_i \in L^\infty(0,l),\ i \in I_2;$$
$$\lambda_i \in L^\infty(0,l),\ i \in I_3;$$
$$\lambda_{ix} \in L^\infty(0,l),\ i = 1, \dots, n;$$

$$\frac{\partial^j A_{11}}{\partial t^j},\ \left\|\frac{\partial^j A_{12}}{\partial t^j}\right\|(l-x)^{\beta/2},\ \left\|\frac{\partial^j A_{21}}{\partial t^j}\right\|(l-x)^{-\beta/2},\ \frac{\partial^j A_{22}}{\partial t^j} \in L^\infty(Q),$$

$$j = 0, 1,\ \beta > 0;$$
$$(A_{11}(x,t)\xi, \xi) \geq a_1|\xi|^2\ \forall \xi \in R^{k_2},\ (x,t) \in Q;$$
$$(A_{22}(x,t)\eta, \eta) \geq a_2|\eta|^2\ \forall \eta \in R^{n-k_2},\ (x,t) \in Q;$$
$$\frac{\partial^j f_i}{\partial t^j} x^{\alpha/2}(l-x)^{\beta_i/2} \in L^2(Q), \quad i = 1, \ldots, n,\ j = 0, 1,\ \alpha > 0;$$
$$\psi_i \in H^1_{\alpha,\beta,\lambda_i}(0,l),\ i = 1, \ldots, n; \quad \psi_i(l) = 0,\ i = k_2 + 1, \ldots, n.$$

Then there exists the only solution $u = (u_1, \ldots, u_n)$ *of the problem* (1), (3), (4) *and*

(5) $$\int_0^l [u_i^2(x,t) + u_{it}^2(x,t) + \lambda_i^2(x) u_{ix}^2(x,t)] x^\alpha (l-x)^{\beta_i} dx \leq C(T),$$
$$t \in (0,T), \quad i = 1, \ldots, n.$$

Proof. We shall use the system of functions $\varphi_k = \sin \frac{k\pi}{l} x$, $k = 1, 2, \ldots$. This system is complete in each of the spaces $H^1_{\alpha,\beta_i,\lambda_i}(0,l)$, $i = 1, \ldots, n$. Consider the sequence of functions

$$u_i^N(x,t) = \sum_{k=1}^N c_{ki}^N(t) \varphi_k(x), \quad N = 1, 2, \ldots,$$

where $c_{1i}^N, \ldots, c_{Ni}^N$, $i = 1, \ldots, n$, are solutions of the following Cauchy problems:

(6) $$\int_0^l \left(u_{it}\varphi_k + \lambda_i u_{ix} \varphi_k + \sum_{j=1}^n a_{ij} u_j \varphi_k - f_i \varphi_k \right) x^\alpha (l-x)^{\beta_i} dx = 0,$$

(7) $$c_{ik}^N(0) = \psi_{ik}^N, \quad i = 1, \ldots, n,\ k = 1, \ldots, N,$$

with ψ_{ik}^N defined by the conditions:

$$\psi_i^N(x) = \sum_{k=1}^N \psi_{ik}^N \varphi_k(x), \quad \lim_{N\to\infty} \|\psi_i - \psi_i^N\|_{\alpha,\beta_i,\lambda_i} = 0, \quad i = 1, \ldots, n.$$

Multiplying every equation of the system (6) by the function $c_{ik}^N(t)$, respectively, adding from k=1 to k=N and integrating on the interval $(0,\tau)$, $\tau > 0$, we obtain the equality

(8) $$\int_0^t \int_0^l \left(u_{it}^N u_i^N + \lambda_i u_{ix}^N u_i^N + \sum_{j=1}^n a_{ij} u_j^N u_i^N - f_i u_i^N \right) x^\alpha (l-x)^{\beta_i} dx dt = 0,$$
$$i = 1, \ldots, n.$$

Denote

$$a_{12} = \sup_Q(\|A_{12}(x,t)\|(l-x)^{\beta/2}); \quad a_{21} = \sup_Q(\|A_{21}(x,t)\|(l-x)^{-\beta/2})$$

and put

$$h_i(x;\varepsilon,\delta) = \begin{cases} 2a_1 - \lambda_{ix}(x) - \alpha x^{-1}\lambda_i(x) - \varepsilon(a_{12}+a_{21}) - \delta, & i \in I_1, \\ 2a_1 - \lambda_{ix}(x) + \beta(l-x)^{-1}\lambda_i(x) - \varepsilon(a_{12}+a_{21}) - \delta, & i \in I_2, \\ 2a_2 - \lambda_{ix}(x) - \dfrac{1}{\varepsilon}(a_{12}+a_{21}) - \delta, & i \in I_3, \end{cases}$$

where $\varepsilon > 0$, $\delta > 0$.

Taking into account the assumptions of the theorem it is easy to obtain the inequality

$$\begin{aligned}(9)\quad & \int_0^l \sum_{i=1}^n (u_i^N(x,\tau))^2 x^\alpha (l-x)^{\beta_i} dx \\ & + \int_0^\tau\int_0^l \sum_{i=1}^n h_i(x;\varepsilon,\delta)(u_i^N(x,t))^2 x^\alpha (l-x)^{\beta_i} dx\, dt \\ & \le \int_0^l \sum_{i=1}^n (\psi_i^N(x))^2 x^\alpha (l-x)^{\beta_i} dx + \int_0^\tau\int_0^l \sum_{i=1}^n f_i^2(x,t) x^\alpha (l-x)^{\beta_i} dx\, dt.\end{aligned}$$

Putting in the estimate (9) $\varepsilon = \delta = 1$ and using Gronwall-Bellman inequality we get

$$\begin{aligned}(10)\quad & \int_0^l \sum_{i=1}^n (u_i^N(x,\tau))^2 x^\alpha (l-x)^{\beta_i} dx \le C_1(\tau)\Bigg[\int_0^l \sum_{i=1}^n (\psi_i(x))^2 x^\alpha (l-x)^{\beta_i}\, dx \\ & + \int_0^\tau\int_0^l \sum_{i=1}^n f_i^2(x,t) x^\alpha (l-x)^{\beta_i} dx\, dt\Bigg] \equiv C_2(\tau), \quad 0 \le \tau < \infty.\end{aligned}$$

Differentiating the system (6) with respect to t, then multiplying every equation respectively by the function $c_{ikt}^N(t)$, adding on k from 1 to N and integrating on the interval $[0,\tau]$, we obtain the system

$$\begin{aligned}(11)\quad & \int_0^\tau\int_0^l \Bigg(u_{itt}^N u_{it}^N + \lambda_i u_{ixt}^N u_{it}^N + \sum_{j=1}^n a_{ij} u_{jt}^N u_{it}^N \\ & + \sum_{j=1}^n a_{ijt} u_j^N u_{it}^N - f_{it} u_{it}^N\Bigg) x^\alpha (l-x)^{\beta_i} dx\, dt = 0, \quad i = 1,\dots,n.\end{aligned}$$

From the assumptions and the estimate (10) it is easy from (11) to get the inequality

$$(12)\quad \int_0^l \sum_{i=1}^n (u_{it}^N(x,\tau))^2 x^\alpha (l-x)^{\beta_i}\,dx \le \int_0^l \sum_{i=1}^n (u_{it}^N(x,0))^2 x^\alpha (l-x)^{\beta_i}\,dx + C_2(\tau)\int_0^\tau\int_0^l \sum_{i=1}^n f_{it}^2(x,t)x^\alpha(l-x)^{\beta_i}\,dx\,dt.$$

To estimate the first term in the right hand side of (12) we shall use the system (6) for $t=0$. Multiplying every equation of this system respectively by $c_{ikt}^N(0)$, adding on k from 1 to N we obtain the estimate

$$(13)\quad \int_0^l \sum_{i=1}^n (u_{it}^N(x,\tau))^2 x^\alpha (l-x)^{\beta_i}\,dx \le C_3(\tau)\int_0^l \left[\sum_{i=1}^n \lambda_i^2 (u_{ix}^N(x,0))^2 + \sum_{i=1}^n f_i^2(x,0)^2\right] x^\alpha (l-x)^{\beta_i}\,dx \le C_4(\tau)\left[\sum_{i=1}^n \|\psi_i\|^2_{\alpha,\beta_i,\lambda_i} + \int_0^l \sum_{i=1}^n f_i^2(x,0)x^\alpha(l-x)^{\beta_i}\,dx\right].$$

Thus, due to the estimates (10), (12), (13) we have

$$(14)\quad \|u_i^N\|_{L^\infty((0,T);\,L^2_{\alpha,\beta_i}(0,l))} + \|u_{it}^N\|_{L^\infty((0,T);\,L^2_{\alpha,\beta_i}(0,l))} \le C_5(T)$$

for arbitrary $T>0$, $i=1,\dots,n$. Since the sequences $\{u_i^N(x,t)\}$, $\{u_{it}^N(x,t)\}$ are bounded we can choose subsequences $\{u_i^m(x,t)\}$, $\{u_{it}^m(x,t)\}$ such that

$$u_i^m(x,t) \xrightarrow[m\to\infty]{} u_i(x,t) \quad *-\text{weak in} \quad L^\infty((0,T);\,L^2_{\alpha,\beta_i}(0,l)),$$

and

$$u_{it}^m(x,t) \xrightarrow[m\to\infty]{} u_{it}(x,t) \quad *-\text{weak in} \quad L^\infty((0,T);\,L^2_{\alpha,\beta_i}(0,l))$$

for arbitrary $T>0$, $i=1,\dots,n$.

Now using equations of the system (6) and data (7) it is easy to show that the functions $u_1,\dots,u_n$ satisfy the system

$$(15)\quad \int_0^\tau\int_0^l \Big[u_{it}v_i - \lambda_{ix}u_iv_i - \lambda_iu_iv_{ix} - \alpha x^{-1}\lambda_iu_iv_i + \beta(l-x)^{-1}\lambda_iu_iv_i + \sum_{j=1}^n a_{ij}u_jv_i - f_iv_i\Big]x^\alpha(l-x)^{\beta_i}\,dx\,dt = 0$$

for arbitrary functions $v_i \in C_0^\infty(Q)$, $i=1,\dots,n$, and the initial data (3) hold.

Consider the distribution $(\lambda_iu_i)_x \in (C_0^\infty[0,l])'$. By (15) we have

$$\begin{aligned}
&\langle(\lambda_i u_i)_x, \varphi x^\alpha (l-x)^{\beta_i}\rangle \\
&\quad = -\langle \lambda_i u_i, (\varphi x^\alpha (l-x)^{\beta_i})_x\rangle = -\int_0^l \lambda_i u_i (\varphi x^\alpha (l-x)^{\beta_i})_x \, dx \\
&\quad = \int_0^\tau\int_0^l \Big(\lambda_{ix} u_i - u_{it} - \sum_{j=1}^n a_{ij} u_j + f_i\Big) \varphi x^\alpha (l-x)^{\beta_i} \, dx\, dt \\
&\quad = \Big\langle \lambda_{ix} u_i - u_{it} - \sum_{j=1}^n a_{ij} u_j + f_i, \varphi x^\alpha (l-x)^{\beta_i}\Big\rangle, \quad i = 1, \dots, n
\end{aligned}$$

for all functions $\varphi \in C_0^\infty[0,l]$. From the latter we get

$$(\lambda_i u_i)_x = \lambda_{ix} u_i - u_{it} - \sum_{j=1}^n a_{ij} u_j + f_i, \quad i = 1, \dots, n.$$

In other words the functions $u_1, \dots, u_n$ are the solutions of the system (1).

Moreover, these functions satisfy the initial data (3), boundary data (4) and $u_i \in H^1_{loc}(Q)$, $i = 1, \dots, n$. Thus, $u_i \in C(Q)$, $i = 1, \dots, n$.

Using the method of characteristic [19], [20] it is easy to show that the problem (1), (3), (4) can have no more than one continuous solution in Q. The theorem is proved.

Denote

$$\begin{aligned}
b_1 &= \frac{a_{12} + a_{21}}{\min\limits_{i=k_2+1,\dots,n} \min\limits_{x\in[0,l]} [2a_2 - \lambda_{ix}(x)]}, \\
b_1 &= \min_{i=1,\dots,k_1} \min_{x\in[0,l]} [2a_1 - \alpha x^{-1}\lambda_i(x) - \lambda_{ix}(x)] - b_1(a_{12} + a_{21}), \\
b_3 &= \min_{i=k_1+1,\dots,k_2} \min_{x\in[0,l]} [2a_1 + \beta(l-x)^{-1}\lambda_i(x) - \lambda_{ix}(x)] - b_1(a_{12} + a_{21}).
\end{aligned}$$

THEOREM 2. *Let all conditions of Theorem* 1 *be satisfied and* $b_j > 0$, $j = 1,2,3$; $f_i \in L^2((0,\infty); L^2_{\alpha,\beta_i}(0,l))$, $i = 1, \dots, n$. *Then the solution of the problem* (1), (3), (4) *is asymptotically stable with respect to data* (3) *and the right hand side functions* f_i.

Proof. Since $b_j > 0$, $j = 1,2,3$, we can choose such numbers $\varepsilon > 0$, $\delta > 0$ that $h_i(x;\varepsilon,\delta) \geq \delta_0 > 0$, $i = 1, \dots, n$ for all $x \in [0,l]$. Then from the estimate (9) we have

$$\int_0^l \sum_{i=1}^n (u_i^N(x,\tau))^2 x^\alpha (l-x)^{\beta_i} dx + \delta_0 \int_0^\tau\int_0^l \sum_{i=1}^n (u_i^N(x,t))^2 x^\alpha (l-x)^{\beta_i} dx\, dt$$

$$\leq C_6\left(\int_0^l \sum_{i=1}^n (\psi_i(x))^2 x^\alpha (l-x)^{\beta_i} dx + \int_0^T \int_0^l \sum_{i=1}^n f_i^2(x,t) x^\alpha (l-x)^{\beta_i} dx\, dt\right) \equiv C_7,$$

for $N = 1, 2, \ldots$, where C_7 does not depend on t. In particular from the above inequality we obtain the esimates

$$\|u_i(\cdot,t)\|_{L^\infty((0,\infty);\, L^2_{\alpha,\beta_i}(0,l))} \leq C_8,$$

$$(16) \qquad \int_0^\infty \|u_i(\cdot,t)\|_{L^\infty((0,\infty);\, L^2_{\alpha,\beta_i}(0,l))} dt \leq C_8, \quad i = 1, \ldots, n.$$

The estimates (16) imply the conclusion of the theorem.

References

[1] A. V. Deryabina, *On the grew solutions of a strong degenerated hyperbolic system*, Mat. Sb. 181(1990), 447–463 (in Russian).

[2] S. A. Tersenov, *On the theory of hyperbolic equations with data in the line of type degeneracy*, Sibirsk. Mat. Zh. 2(1961), 913–935 (in Russian).

[3] S. A. Tersenov, *On a problem with data in the degeneracy line for the system of hyperbolic type*, Doklady Akad. Nauk SSSR, 155(1964), 285–288 (in Russian).

[4] S. A. Tersenov, *On the singular Cauchy problem for some system of hyperbolic type*, Doklady Akad. Nauk SSSR, 205 (1972), 1046–1049 (in Russian).

[5] Tahara Hidetoshi, *Singular hyperbolic systems, I, Existence, uniqueness and differentiability*, J. Fac. Sci. Univ. Tokyo, Sec. 1A, 26 (1979), 213–238.

[6] Tahara Hidetoshi, *Singular hyperbolic systems. Pseudo- differential operators with a parameter and their applications to singular hyperbolic systems*, J. Fac. Sci. Univ. Tokyo, Sec. 1A, 26 (1979), 391–412.

[7] S. V. Zhestkov, *On the Lyapunov stability of the Cauchy problem for linear system of partial derivatives*, Differentsial'nye Uravneniya, 26 (1990), 706-709 (in Russian).

[8] Knut S. Eckhoff, *Stability problems for linear hyperbolic systems*, Rept. Dep. Appl. Math. Univ. Bergen, 54 (1975), 87pp.

[9] Knut S. Eckhoff, *On stability for symmetric hyperbolic systems. I*, J. Differential Equations 40 (1981), 94–115.

[10] Knut S. Eckhoff, *On stability for symmetric hyperbolic systems. II*, J. Differential Equations 43 (1982), 281–304.

[11] Naito Toshiki, *On the absolute stability of a class of hyperbolic system*, Lect. Notes Math., 243 (1971), 284–288.

[12] M. O. Oliskevych, *The solution behaviour of an unlocalic problem for hyperbolic system of the first order*, Proc. International Conf. on Nonlinear Partial Differential Equations, Kiev, (1997), 138–139.

[13] J. Radzikowski, *Untersuchung der Beschrankheit der Losungen eines unhomogen hyperbolischen Systems*, Demonstratio Math., 12 (1979), 981–994.

[14] J. Radzikowski, W. Sadkowski, *On the stability of solution of some hyperbolic systems*, Nonlinear Vib. Probl., 22 (1984), 179–192.

[15] J. Radzikowski, *Qualitative investigations of a homogeneous linear symmetric t-hyperbolic system*, Demonstratio Math., 18 (1985), 787–805.

[16] J. Radzikowski, *Qualitative study of a linear homogeneous hyperbolic system*, Demonstratio Math., 18 (1985), 945–962.

[17] J. Rauch, M. Taylor, *Exponential decay of solutions to hyperbolic equations in bounded domains*, Indiana Univ. Math. J., 24 (1974), 79–86.

[18] W. Sadkowski, *Boundedness and stability of solutions of some hyperbolic equations*, Demonstratio Math. 11 (1978), 775–785.

[19] V. E. Abolinya, A. D. Myshkis, *Mixed problem for semilinear hyperbolic system on the plane*, Matem. Sb., 50 (1960), 423–442.

[20] R. Courant, *Partial Differential Equations*, New York – London, 1962.

Serhiy Lavrenyuk
DEPARTAMENT OF MECHANICS AND MATHEMATICS
LVIV STATE UNIVERSITY
1, Universiteska str.
LVIV, 290602, UKRAINE

Lech Zaręba
PEDAGOGICAL UNIVERSITY IN RZESZÓW
DEPARTAMENT OF MATHEMATICS
Rejtana 16A
35-310 RZESZOW, POLAND

Received January 14, 1999.