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SPACES OF ANALYTIC FUNCTIONS OF TWO COMPLEX VARIABLES

Abstract. Let $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$ be an analytic function of two complex variables z_1 and z_2 in the polydisc $|z_i| < 1$. The growth parameters such as order etc. were defined by Juneja and Kapoor [1]. Based on these, the spaces of such analytic functions have been considered and a norm defined. The topological properties of these spaces have been obtained. Besides finding the duals of these spaces, the proper bases are characterized in terms of growth parameters.

1. Let

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n,$$

$a_{mn} \in \mathbb{C}$, $(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_i| \le r_i < 1$, $i = 1, 2$, be analytic. Various workers, such as, Stoll [5], Juneja and Kapoor [1] etc. have studied the growth aspect of these functions and defined their order etc. Let

$$M(r_1, r_2) = \max_{|z_i| \le r_i} |f(z_1, z_2)|.$$

Then $f(z_1, z_2)$ is said to be of order ϱ if

$$\limsup_{r_1, r_2 \to 1} \frac{\log^+ \log^+ M(r_1, r_2)}{-\log \log (r_1 r_2)^{-1}} = \varrho, \quad 0 \le \varrho \le \infty. \tag{1.1}$$

Further, $f(z_1, z_2)$ will be of order ϱ if and only if

$$\limsup_{m,n \to \infty} \frac{\log(m+n)}{\log(m+n) - \log^+ \log^+ |a_{mn}|} = \varrho + 1. \tag{1.2}$$

The above formula follows from Theorem 5.2.2 of Juneja and Kapoor [1].

Let $X(\varrho)$ denote the class of all analytic functions $f(z_1, z_2)$ as defined above such that

$$\limsup_{r_1, r_2 \to 1} \frac{\log^+ \log^+ M(r_1, r_2)}{-\log \log (r_1 r_2)^{-1}} \le \varrho, 0 \le \varrho \le \infty.$$

Then, under pointwise addition and scalar multiplication, the set $X(\varrho)$ is a linear space over field $\mathbb{C}$. In view of (1.2), for any given $\varepsilon > 0$, we have

$$(1.3) \qquad |a_{mn}| < \exp[(m+n)^{(\varrho+\varepsilon)/(\varrho+\varepsilon+1)}],$$

for all large values of m and n. In view of (1.3), for any $\delta > 0$, the double series

$$(1.4) \qquad \|f;\varrho+\delta\| = \sum_{m,n=0}^{\infty} |a_{mn}| \exp[-(m+n)^{(\varrho+\delta)/(\varrho+\delta+1)}]$$

converges for $f \in X(\varrho)$, and $\|f;\varrho+\delta\|$ defines a norm on $X(\varrho)$. We denote the corresponding normed space by $X(\varrho,\delta)$. The lattice product of these normed topologies is denoted by $X(\varrho)$. The space $X(\varrho)$ is metrizable and its metric is given by

$$(1.5) \qquad \lambda(f,g) = \sum_{q=1}^{\infty} 2^{-q} \frac{\|f-g\|_q}{1+\|f-g\|_q}, f,g \in X(\varrho),$$

where $\|f\|_q = \sum_{m,n=0}^{\infty} |a_{mn}| \exp[-(m+n)^{(\varrho+q^{-1})/(\varrho+q^{-1}+1)}]$.

We denote by $X_\lambda(\varrho)$ the space $X(\varrho)$ equipped with above metric. J. K. Srivastava and Rajiv K. Srivastava ([3], [4]) have studied the properties of spaces of analytic functions of several complex variables represented by Dirichlet series such as their completeness, linear hemeomorphisms etc. O. P. Juneja and A. Sinha [2] have also obtained properties of proper bases.

In this paper, we study different properties of the space $X_\lambda(\varrho)$ such as completeness, its dual and linear transformations of $X_\lambda(\varrho)$ into itself.

2. In this section, we obtain results concerning linear transformations on $X_\lambda(\varrho)$.

First we prove a completeness theorem. In what follows, we shall denote for $m,n \geq 0$,

$$\delta_{mn}(z_1,z_2) = z_1^m z_2^n.$$

We have

THEOREM 1. *The space $X_\lambda(\varrho)$ is a Frechet space.*

Proof. In order to prove the theorem, it is sufficient to show that the space $X_\lambda(\varrho)$ is complete. Thus, let $\{f_\alpha\}$ be a λ-*Cauchy* sequence in $X_\lambda(\varrho)$. Then for any $\eta' > 0$, there exists a positive integer $m_0 = m_0(\eta')$ such that

$$(2.1) \qquad \|f_\alpha - f_\beta\|_q < \eta' \quad \text{for all } \alpha,\ \beta \geq m_0,\ q \geq 1.$$

Let us denote by

$$f_\alpha(z_1,z_2) = \sum_{m,n=0}^{\infty} a_{mn}^{(\alpha)} z_1^m z_2^n,$$

$$f_\beta(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn}^{(\beta)} z_1^m z_2^n.$$

Then we have for $\alpha, \beta \geq m_0$, $q \geq 1$.

$$\sum_{m,n=0}^{\infty} |a_{mn}^{(\alpha)} - a_{mn}^{(\beta)}| \Phi < \eta', \tag{2.2}$$

where $\Phi = \exp[-(m+n)^{(\varrho+q^{-1})/(\varrho+1+q^{-1})}]$ contains m, n but no α, β. Since each term in (2.2) on the left hand side is obviously less than η', we get

$$|a_{mn}^{(\alpha)} - a_{mn}^{(\beta)}| < \eta'/\Phi \forall \alpha, \beta \geq m_0, \text{ and fixed } m, n.$$

Thus for given $\eta > 0$ and fixed m and n, $\exists m_0$ such that

$$|a_{mn}^{(\alpha)} - a_{mn}^{(\beta)}| < \eta \quad \forall \alpha, \beta \geq m_0.$$

The letter implies that $\{a_{mn}^{(\alpha)}\}_{\alpha=1}^{\infty}$, is a Cauchy sequence of complex numbers for each fixed m and n. Therefore, there exists a double sequence $\{a_{mn}\}_{m,n=0}^{\infty}$ such that

$$\lim_{\alpha \to \infty} a_{mn}^{(\alpha)} = a_{mn}, \qquad m, n = 0, 1, 2, \ldots.$$

Now, taking $\beta \to \infty$ in (2.2), we get for $\alpha \geq m_0$.

$$\sum_{m,n=0}^{\infty} |a_{mn}^{(\alpha)} - a_{mn}| \exp[-(m+n)^{(\varrho+q^{-1})/(\varrho+1+q^{-1})} \leq \eta. \tag{2.3}$$

Taking $\alpha = m_0$, we get for any fixed q,

$$\begin{aligned} &|a_{mn}| \exp[-(m+n)^{(\varrho+q^{-1})/(\varrho+1+q^{-1})}] \\ &\qquad \leq |a_{mn}^{(m_0)}| \exp[-(m+n)^{(\varrho+q^{-1})/(\varrho+1+q^{-1})}] + 2\eta. \end{aligned}$$

Now, $f_{m_0}(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn}^{(m_0)} z_1^m z_2^n \in X_\lambda(\varrho)$. Hence the condition (1.3) is satisfied. Therefore, for arbitrary $p > q$, we have

$$\begin{aligned} &|a_{mn}| \exp[-(m+n)^{(\varrho+q^{-1})/(\varrho+1+q^{-1})}] \\ &\quad < 2\eta + |a_{mn}^{(m_0)}| \exp[-(m+n)^{(\varrho+q^{-1})/(\varrho+1+q^{-1})}] \\ &\quad < 2\eta + \exp[-(m+n)^{(\varrho+p^{-1})/(\varrho+1+p^{-1})} - (m+n)^{(\varrho+q^{-1})/(\varrho+1+q^{-1})}]. \end{aligned}$$

Since $p > q$ is arbitrary, the second term on the right hand side approaches zero as $(m+n) \to \infty$. Also, since $\eta > 0$ was arbitrarily chosen, therefore the sequence $\{a_{mn}\}$ satisfies (1.3) for sufficiently large values of $(m+n)$. Consequently,

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n \text{ belongs to } X_\lambda(\varrho).$$

Again from (2.3), for arbitrary $\varepsilon > 0$ and $q = 1, 2, \ldots$, we have $\|f_\alpha - f\|_q < \varepsilon$. Hence

$$\lambda(f_\alpha, f) = \sum_{q=0}^{\infty} 2^{-q} \frac{\|f_\alpha - f\|_q}{1 + \|f_\alpha - f\|_q} < \frac{\varepsilon}{1+\varepsilon} \sum_{q=0}^{\infty} 2^{-q} = \frac{\varepsilon}{1+\varepsilon} < \varepsilon.$$

Since the above inequality holds for all $\alpha \geq m_0$, it follows that. $f_\alpha \to f$ as $\alpha \to \infty$. Since we have already proved that $f \in X_\lambda(\varrho)$, this showes that $X_\lambda(\varrho)$ is complete.

The next result characterizes the linear continuous functionals on $X_\lambda(\varrho)$. We have the following

THEOREM 2. *A continuous linear functional F on $X_\lambda(\varrho)$ is of the form $F(f) = \sum_{m,n=0}^{\infty} a_{mn} c_{mn}$ if and only if*

$$\text{(2.4)} \qquad |c_{mn}| \leq L \exp[-(m+n)^{(\varrho+q^{-1})/(\varrho+1+q^{-1})}], m, n \geq 0, q \geq 1,$$

where L is a finite, positive number and $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$.

Proof. Let $F : X_\lambda(\varrho) \to \mathbb{C}$, where $\mathbb{C}$ is the complex field, be a linear, continuous functional. Then for any sequence $\{f_j\} \subseteq X_\lambda(\varrho)$ with $f_j \to f$, we have $F(f_j) \to F(f)$ as $j \to \infty$. Now, let $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$, where $\{a_{mn}\}$ satisfies (1.3). Then $f \in X_\lambda(\varrho)$. Also for $j = 1, 2, \ldots$, let us put $f_j(z_1, z_2) = \sum_{m,n=0}^{j} a_{mn} z_1^m z_2^n$. Then $f_j \in X_\lambda(\varrho)$ for $j = 1, 2, \ldots$. Let q be any fixed positive integer and let $0 < \varepsilon < q^{-1}$. Then from (1.3), we can find a positive integer j such that

$$|a_{mn}| < \exp[(m+n)^{(\varrho+\varepsilon)/(\varrho+1+\varepsilon)}] \forall m, n > j.$$

Now,

$$\begin{aligned}
\|f - f_j\| &= \Big\| \sum_{m,n=j+1}^{\infty} a_{mn} z_1^m z_2^n \Big\|_q \\
&= \sum_{m,n=j+1}^{\infty} |a_{mn}| \exp[-(m+n)^{(\varrho+q^{-1})/(\varrho+1+q^{-1})}] \\
&< \sum_{m,n=j+1}^{\infty} \exp[(m+n)^{(\varrho+\varepsilon)/(\varrho+1+\varepsilon)} - (m+n)^{(\varrho+q^{-1})/(\varrho+1+q^{-1})}] \\
&< \delta
\end{aligned}$$

for sufficiently large values of j, where $\delta > 0$ is arbitrarily small. Hence

$$\lambda(f, f_j) = \sum_{q=1}^{\infty} 2^{-q} \frac{\|f - f_j\|_q}{1 + \|f - f_j\|_q} < \sum_{q=1}^{\infty} 2^{-q} \frac{\delta}{(\delta+1)} < \delta.$$

Hence $f_j \to f$ in $X_\lambda(\varrho)$ as $j \to \infty$. Therefore,

$$\lim_{j\to\infty} F(f_j) = F(f).$$

Put $c_{mn} = F(z_1^m z_2^n)$. Then

$$F(f) = \lim_{j\to\infty} F(f_j) = \lim_{j\to\infty} \sum_{m,n=0}^{j} a_{mn}c_{mn} = \sum_{m,n=0}^{\infty} a_{mn}c_{mn},$$

where, $|c_{mn}| = |F(z_1^m z_2^n)|$. Since F is continuous on $X_\lambda(\varrho)$, it is continuous on $X_{\|\|_q}(\varrho)$ for each $q = 1, 2, \ldots$. Consequently, there exists a positive constant L independent of q such that

$$|F(z_1^m z_2^n)| = |c_{mn}| \le L\|\delta_{mn}\|_q, \quad q \ge 1,$$

where $\delta_{mn}(z_1, z_2) = z_1^m z_2^n$. Now, using the definition of the norm for $\delta_{mn}(z_1, z_2)$, we get

$$|c_{mn}| \le L\exp[-(m+n)^{(\varrho+q^{-1})/(\varrho+1+q^{-1})}],$$

for all $m, n \ge 0$, $q \ge 1$. Hence we have $F(f) = \sum_{m,n=0}^{\infty} a_{mn}c_{mn}$, where c_{mn}'s satisfy (2.4).

Conversely, suppose that c_{mn}'s satisfy (2.4) and for any sequence of complex numbers $\{a_{mn}\}$, $F(f) = \sum_{m,n=0}^{\infty} a_{mn}c_{mn}$. Then for $q \ge 1$, we have

$$|F(f)| \le L \sum_{m,n=0}^{\infty} |a_{mn}| \exp[-(m+n)^{(\varrho+q^{-1})/(\varrho+1+q^{-1})}].$$

Then $|F(f)| \le L\|f\|_q$, $q \ge 1$. Hence $F \in X'_{\|\|_q}(\varrho)$ for $q = 1, 2, \ldots$ Now since

$$\lambda(f,g) = \sum_{q=1}^{\infty} 2^{-q} \frac{\|f-g\|_q}{1+\|f-g\|_q}$$

therefore $X'_\lambda(\varrho) = \bigcup_{q=1}^{\infty} X'_{\|\|_q}(\varrho)$. Hence $F \in X'_\lambda(\varrho)$. This proves Theorem 2.

3. In this section we shall study continuous linear transformations and proper bases in $X(\varrho)$. The sequence of functions $\{\alpha_{mn}\}$ is said to be linearly independent if $\sum_{m,n=0}^{\infty} a_{mn}\alpha_{mn} = 0$ implies that $a_{mn} = 0$ for all sequences $\{a_{mn}\}$ of complex numbers for which the series $\sum_{m,n=0}^{\infty} a_{mn}\alpha_{mn}$ converges in $X(\varrho)$. $\{\alpha_{mn}\}$ spans a subspace X_0 of $X(\varrho)$ provided X_0 consists of all linear combinations $\sum_{m,n=0}^{\infty} a_{mn}\alpha_{mn}$ which are convergent in $X(\varrho)$. A sequence $\{\alpha_{mn}\} \subseteq X(\varrho)$ which is linearly independent and spans a subspace X_0 of $X(\varrho)$ is said to be a base in X_0. Finally, a sequence $\{\alpha_{mn}\} \subseteq X(\varrho)$ will be called a "proper base" if it is a base and satisfies the condition: "For all sequences $\{a_{mn}\}$ of complex numbers, the convergence of $\sum_{m,n=0}^{\infty} a_{mn}\alpha_{mn}$ in $X(\varrho)$ implies the convergence of $\sum_{m,n=0}^{\infty} a_{mn}\delta_{mn}$ in $X(\varrho)$, and conversely".

Now, we prove

THEOREM 3. *A necessary and sufficient condition that there exists a continuous linear transformation $F : X(\varrho) \to X(\varrho)$ with*

$$F(\delta_{mn}) = \beta_{mn}, m, n, = 0, 1, 2, \ldots; \delta_{mn}(z_1, z_2) = z_1^m z_2^n,$$

is that for each $\delta > 0$,

$$\limsup_{m,n\to\infty} \frac{\log(m+n) - \log^+ \log^+ \|\beta_{mn}; \varrho+\delta\|^{-1}}{\log(m+n)} < \frac{1}{(\varrho+1)}. \tag{3.1}$$

Proof. Let F be a continuous linear transformation from $X(\varrho)$ into $X(\varrho)$ with $F(\delta_{mn}) = \beta_{mn}$, $m, n = 0, 1, 2, \ldots$. Then, for any given $\delta > 0$ one can find $\delta_1 > 0$ and a constant $K = K(\delta)$ such that

$$\|F(\delta_{mn}); \varrho+\delta\| \le K \|\delta_{mn}; \varrho+\delta_1\|$$

i.e.

$$\|\beta_{mn}; \varrho+\delta\| \le K \exp[-(m+n)^{(\varrho+\delta_1)/(\varrho+1+\delta_1)}]$$

i.e.

$$(m+n)^{(\varrho+\delta_1)/(\varrho+1+\delta_1)} \le \log K + \log \|\beta_{mn}; \varrho+\delta\|^{-1}$$

i.e.

$$\frac{\log(m+n) - \log^+ \log^+ \|\beta_{mn}; \varrho+\delta\|^{-1}}{\log(m+n)} \le \frac{1}{\varrho+1+\delta_1} + o(1)$$

i.e.

$$\limsup_{m,n\to\infty} \frac{\log(m+n) - \log^+ \log^+ \|\beta_{mn}; \varrho+\delta\|^{-1}}{\log(m+n)} \le \frac{1}{\varrho+1+\delta_1} < \frac{1}{(\varrho+1)}.$$

Conversely, suppose that the double sequence $\{\beta_{mn}\}$ satisfies (3.1). Then, for any $\eta' > 0$, there exists $N_0 = N_0(\eta')$ such that for all $\delta > 0$ and all $m, n > N_0$,

$$\frac{\log(m+n) - \log^+ \log^+ \|\beta_{mn}; \varrho+\delta\|^{-1}}{\log(m+n)} < \frac{1}{\varrho+1+\eta'}$$

or

$$\|\beta_{mn}; \varrho+\delta\| < \exp[-(m+n)^{(\varrho+\eta')_1/(\varrho+1+\eta')}]. \tag{3.2}$$

Let

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n \in X(\varrho)$$

and choose $0 < \eta < \eta'$. Then from (1.3), there exists $N_1(\eta) = N_1$ such that for all $m, n \ge N_1$

$$|a_{mn}| < \exp[(m+n)^{(\varrho+\eta)/(\varrho+1+\eta)}]. \tag{3.3}$$

Let $n_0 = \max(N_o, N_1)$. Then from (3.2) and (3.3), we have for all $m, n \ge n_0$,

$$|a_{mn}| \|\beta_{mn}; \varrho+\delta\| < \exp[(m+n)^{(\varrho+\eta)/(\varrho+1+\eta)} - (m+n)^{(\varrho+\eta')/(\varrho+1+\eta')}].$$

Since $\eta < \eta'$, this inequality implies that the series $\sum_{m,n=0}^{\infty} a_{mn}\beta_{mn}$ converges absolutely in $X(\varrho)$ and since $X(\varrho)$ is complete, we infer that this series converges to an element of $X(\varrho)$. Let us define a transformation $F : X(\varrho) \to X(\varrho)$ by putting $F(\alpha) = \sum_{m,n=0}^{\infty} a_{mn}\beta_{mn}$ for $\alpha \in X(\varrho)$. We note that F is linear, $F(\delta_{mn}) = \beta_{mn}$ and for $\delta > 0$, $\exists\delta' > 0$, such that

$$\frac{\log(m+n) - \log^+\log^+\|\beta_{mn};\varrho+\delta\|^{-1}}{\log(m+n)} < \frac{1}{\varrho+1+\delta'}$$

for $m,n > N(\delta,\delta')$ i.e $\|\beta_{mn};\varrho+\delta\| \le h\exp[-(m+n)^{(\varrho+\delta')/(\varrho+1+\delta')}]$ for all $m,n \ge 0$, $h = h(\delta)$ being a positive constant. Hence

$$\begin{aligned}\|F(\alpha);\varrho+\delta\| &\le \sum_{m,n=0}^{\infty} |a_{mn}|\|\beta_{mn};\varrho+\delta\| \\ &\le h\sum_{m,n=0}^{\infty} |a_{mn}|\exp[-(m+n)^{(\varrho+\delta')/(\varrho+1+\delta')}] \\ &\le h'\|\alpha;\varrho+\delta'\|,\end{aligned}$$

where $h' = \max(h^{-1}, 1)$. Thus F is continuous and Theorem 3 is proved.

From (1.3), we know that $\sum c_{mn}a_{mn}$ converges in $X(\varrho)$ if and only if

$$\limsup_{m,n\to\infty} \frac{\log(m+n)}{\log(m+n) - \log^+\log^+|c_{mn}|} \le \varrho + 1. \tag{3.4}$$

Now, we prove the following

LEMMA 1. *In $X(\varrho)$, the following three statements are equivalent:*

$$\limsup_{m,n\to\infty} \frac{\log(m+n) - \log^+\log^+\|\beta_{mn};\varrho+\delta\|^{-1}}{\log(m+n)} < \frac{1}{(\varrho+1)}, \delta > 0, \tag{3.5}$$

(3.6) *For all sequences $\{a_{mn}\}$ of complex numbers, the convergence of $\sum_{m,n=0}^{\infty} a_{mn}\delta_{mn}$ in $X(\varrho)$ implies the convergence of $\sum_{m,n=0}^{\infty} a_{mn}\beta_{mn}$ in $X(\varrho)$.*

(3.7) *For all sequences $\{a_{mn}\}$ of complex numbers, the convergence of $\sum_{m,n=0}^{\infty} a_{mn}\delta_{mn}$ in $X(\varrho)$ implies that $\lim_{m,n\to\infty} a_{mn}\beta_{mn} = 0$ in $X(\varrho)$.*

Proof. In proving the sufficiency part of Theorem 3, we have already shown that (3.5)$\Rightarrow$(3.6). The implication (3.6)$\Rightarrow$(3.7) is evident. Hence we have to prove only that (3.7)$\Rightarrow$(3.5). To a contrary let (3.7) be true but, for some $\delta > 0$, (3.5) be not satisfied. Then, say for $\delta = \delta'$, one can pick sequences $\{m_k\}$, $\{n_l\}$ of positive integers such that

$$\frac{\log(m_k+n_l) - \log^+\log^+\|\beta_{m_k n_l};\varrho+\delta'\|^{-1}}{\log(m_k n_l)} > \frac{1}{\varrho+1+(kl)^{-1}}$$

for all $k, l = 1, 2, \ldots$. We define

$$a_{mn} = \begin{cases} \|\beta_{mn}; \varrho + \delta'\|^{-1}, & m = m_k,\ n = n_l \\ 0 & \text{otherwise.} \end{cases}$$

Then, for all large values of k and l and $m = m_k$, $n = n_l$,

$$\frac{\log(m+n)}{\log(m+n) - \log^+ \log^+ |a_{mn}|} = \frac{\log(m+n)}{\log(m+n) - \log^+ \log^+ \|\beta_{mn}; \varrho + \delta'\|^{-1}} < \varrho + 1 + (kl)^{-1}.$$

Hence

$$\limsup_{m,n\to\infty} \frac{\log(m+n)}{\log(m+n) - \log^+ \log^+ |a_{mn}|} \le \varrho + 1.$$

Thus, the sequence $\{a_{mn}\}$ defined above satisfies (3.1) and hence $\sum_{m,n=0}^{\infty} a_{mn}\delta_{mn}$ converges in $X(\varrho)$. So by (3.7), we have $\lim_{m,n\to\infty} a_{mn}\beta_{mn} = 0$. However

$$\|a_{m_k n_l}\beta_{m_k n_l}; \varrho + \delta'\| = |a_{m_k n_l}| \|\beta_{m_k n_l}; \varrho + \delta'\| = 1.$$

Therefore $\{a_{m_k n_l}\}$ does not converge to zero in $X(\varrho)$. This is a contradiction. Hence (3.5) must hold for every $\delta > 0$ and proof of Lemma 1 is complete.

LEMMA 2. *The following three conditions are equivalent:*

(3.8) *For all double sequences* $\{a_{mn}\}_{m,n=0}^{\infty}$ *of complex numbers,* $\lim_{m,n\to\infty} a_{mn}\beta_{mn} = 0$ *in* $X(\varrho)$ *implies that* $\sum_{m,n=0}^{\infty} a_{mn}\delta_{mn}$ *converges in* $X(\varrho)$,

(3.9) *For all sequences* $\{a_{mn}\}$ *of complex numbers, convergence of* $\sum_{m,n=0}^{\infty} a_{mn}\beta_{mn}$ *in* $X(\varrho)$ *implies that* $\sum_{m,n=0}^{\infty} a_{mn}\delta_{mn}$ *converges in* $X(\varrho)$,

(3.10) $$\lim_{\delta\to 0}\left[\liminf_{m,n\to\infty} \frac{\log(m+n) - \log^+ \log^+ \|\beta_{mn}; \varrho + \delta\|^{-1}}{\log(m+n)}\right] \ge \frac{1}{\varrho + 1}.$$

Proof. Clearly (3.8)⇒(3.9). Hence we first prove that (3.9)⇒(3.10). Suppose, to a contrary, that (3.9) holds but (3.10) is not true. Then we have

$$\lim_{\delta\to 0}\left[\liminf_{m,n\to\infty} \frac{\log(m+n) - \log^+ \log^+ \|\beta_{mn}; \varrho + \delta\|^{-1}}{\log(m+n)}\right] < \frac{1}{\varrho + 1}.$$

Hence for any $\delta > 0$,

$$\liminf_{m,n\to\infty} \frac{\log(m+n) - \log^+ \log^+ \|\beta_{mn}; \varrho + \delta\|^{-1}}{\log(m+n)} < \frac{1}{\varrho + 1}$$

Let $\eta > 0$ be a fixed number. From (3.9), we can find increasing sequences $\{m_k\}$, $\{n_l\}$ of positive integers such that

$$\frac{\log(m_k + n_l) - \log^+ \log^+ \|\beta_{m_k n_l}; \varrho + \delta\|^{-1}}{\log(m_k + n_l)} < \frac{1}{\varrho + 1 + \eta}.$$

For η_1, $0 < \eta_1 < \eta$, we define a sequence $\{a_{mn}\}$ as

$$a_{mn} = \begin{cases} \exp[(m+n)^{(\varrho+\eta_1)/(\varrho+1+\eta_1)}, & m = m_k,\ n = n_l \\ 0 \quad \text{otherwise}. \end{cases}$$

Then, for any $\delta > 0$, we have

$$\sum_{m,n=0}^{\infty} |a_{mn}| \|\beta_{mn}; \varrho + \delta\| = \sum_{k,l=1}^{\infty} |a_{m_k n_l}| \|\beta_{m_k n_l}; \varrho + \delta\|. \tag{3.11}$$

Now, for any $\delta > 0$, we omit those terms on the right hand side series for which $\delta < (kl)^{-1}$. Then the remainder of the series (3.11) is dominated by

$$\sum_{k,l=0}^{\infty} |a_{m_k n_l}| \|\beta_{m_k n_l}; \varrho + (kl)^{-1}\|.$$

Consequently, by (3.7), we obtain

$$\sum_{k,l=0}^{\infty} |a_{m_k n_l}| \|\beta_{m_k n_l}; \varrho + (kl)^{-1}\|$$
$$\leq \sum_{k,l=0}^{\infty} \exp[(m_k + n_l)^{(\varrho+\eta_1)/(\varrho+1+\eta_1)} - (m_k + n_l)^{(\varrho+\eta)/(\varrho+1+\eta)}].$$

Since $\eta_1 < \eta$, the series on the right hand side is convergent. Since $a_{mn} = 0$ for $m \neq m_k$, $n \neq n_l$, the series $\sum_{m,n=0}^{\infty} a_{mn}\beta_{mn}$ converges for the above choice of $\{a_{mn}\}$. Since this is true for any $\delta > 0$, $\sum_{m,n=0}^{\infty} a_{mn}\beta_{mn}$ converges in $X(\varrho)$. On the other hand, for this sequence $\{a_{mn}\}$, we also have

$$\limsup_{m,n\to\infty} \frac{\log(m+n)}{\log(m+n) - \log^+ \log^+ |a_{mn}|} = \varrho + 1 + \eta_1 > \varrho + 1,$$

which is a contradiction. This proves that (3.9)$\Rightarrow$(3.10). Lastly, we prove that (3.10)$\Rightarrow$(3.8). Suppose to a contrary that (3.10) holds but (3.8) does not. Then, there is a sequence $\{a_{mn}\}$ of complex numbers for which $a_{mn}\beta_{mn} \to 0$ in $X(\varrho)$ but $\sum_{m,n=0}^{\infty} a_{mn}\delta_{mn}$ does not converge in $X(\varrho)$. Hence from the equivalent condition (3.4), we have

$$\limsup_{m,n\to\infty} \frac{\log(m+n)}{\log(m+n) - \log^+ \log^+ |a_{mn}|} > \varrho + 1.$$

Thus, we can pick a positive number ε and sequences $\{m_k\}$, $\{n_l\}$ of positive integers such that

$$\frac{\log(m_k + n_l)}{\log(m_k + n_l) - \log^+ \log^+ |a_{m_k n_l}|} > \varrho + 1 + \varepsilon.$$

Let $0 < \eta < \varepsilon/2$. From (3.10), we can find a positive number δ such that

$$\liminf_{m,n\to\infty} \frac{\log(m+n) - \log^+\log^+\|\beta_{mn};\varrho+\delta\|^{-1}}{\log(m+n)} \geq \frac{1}{\varrho+1+\eta}.$$

Choose an integer $N = N(\eta)$ such that for $m, n \geq N$,

$$\frac{\log(m+n) - \log^+\log^+\|\|\|\beta_{mn};\varrho+\delta\|^{-1}}{\log(m+n)} \geq \frac{1}{\varrho+1+2\eta}.$$

Therefore,

$$\begin{aligned}\|a_{mn}\beta_{mn};\varrho+\delta\| &= \max\{|a_{mn}|\|\beta_{mn};\varrho+\delta\|\}\\ &\geq \max\{|a_{m_k n_l}|\|\beta_{m_k n_l};\varrho+\delta\|\}\\ &\geq \exp[(m_k+n_1)^{(\varrho+\varepsilon)/(\varrho+1+\varepsilon)} - (m_k+n_l)^{(\varrho+2\eta)/(\varrho+1+2\eta)}]\\ &> 1,\end{aligned}$$

since $\varepsilon > 2\eta$. Hence the sequence $\{a_{mn}\beta_{mn}\}$ does not converge to zero for the δ chosen above. Hence, $\{a_{mn}\beta_{mn}\}$ does not converge to zero in $X(\varrho)$. This is clearly contradictory to (3.10) and hence we obtain that (3.10)⇒(3.8). This proves Lemma 2.

The following result, which gives a characterization of a proper base in $X(\varrho)$, follows from Lemma 1 and Lemma 2. Thus, we state

THEOREM 4. *A base $\{\beta_{mn}\}$ in a closed subspace $X_0(\varrho)$ of $X(\varrho)$ is proper if and only if the conditions* (3.5) *and* (3.10) *stated above are satisfied.*

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References

[1] O. P. Juneja and G. P. Kapoor, *Analytic Functions-Growth Aspect*, Pitman, London 1985.

[2] O. P. Juneja and A. Sinha, *Proper bases, continuous linear functionals, continuous linear transformations and simultaneous automorphisms in a complete linear metric space*, Far East J. Math. Sci. 3, no. 1, (1995), 63–95.

[3] J. K. Srivastava and R. K. Srivastava, *Automorphisms and proper bases in space of analytic functions of several complex variables represented by Dirichlet series*, Kyungpook Math. J. 21, no. 2, (1981), 251–262.

[4] J. K. Srivastava and R. K. Srivastava, *On the space of analytic functions of several complex variables represented by Dirichlet series*, Ganita 32 (1981), 133–144.

[5] W. Stoll, *Holomorphic functions of finite order in several complex variables*, Amer. Math. Soc. 1974.

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