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Jacek Dziok

CLASSES OF p-VALENT ANALYTIC FUNCTIONS WITH FIXED ARGUMENT OF COEFFICIENTS

Abstract. Using the integral operator $\Omega^{\lambda} f(z)$ defined by Owa [6] and by Srivastava and Owa [10] we consider some classes of functions with fixed argument of coefficients. In those classes we determine coefficient estimates, distortion theorems and extreme points.

1. Introduction

Let $\mathcal{A}(p,k)$ denote the class of functions f of the form

$$f(z) = z^p + \sum_{n=k}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in $\mathcal{U} = \mathcal{U}(1)$, where $\mathcal{U}(r) = \{z : z \in \mathbf{C} \text{ and } |z| < r\}$. Also let us put $\mathcal{A}(p) = \mathcal{A}(p, p+1)$ and $\mathcal{A} = \mathcal{A}(1)$.

We say that a function $f \in \mathcal{A}$ is subordinate to a function $F \in \mathcal{A}$, and write $f \prec F$, if and only if there exists a Schwarz function $\omega \in \mathcal{A}$, $\omega(0) = 0$, $|\omega(z)| < 1$, $z \in \mathcal{U}$, such that $f(z) = F(\omega(z))$, $z \in \mathcal{U}$.

Let Γ denote the Gamma function. Owa [6] and Srivastava and Owa [10] defined the fractional derivative operator D_z^{λ} as follows:

DEFINITION 1.1. The *fractional integral of order* λ, $\lambda < 0$, is defined, for a function f, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}\, d\zeta,$$

where f is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

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DEFINITION 1.2. The fractional derivative of order λ, $0 \le \lambda < 1$, is defined, for a function f, by

$$D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta,$$

where f is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1.1.

DEFINITION 1.3. Under the hypotheses of Definition 1.2, the *fractional derivative of order* $n+\lambda$, $0 \le \lambda < 1$; $n \in \mathrm{N} \cup \{0\}$, is defined, for a function f, by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^\lambda f(z).$$

By using these definitions of fractional calculus we define the linear operator

$$\Omega_p^\lambda : \mathcal{A}(p) \to \mathcal{A}(p)$$

by

$$\Omega_p^\lambda f(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^\lambda D_z^\lambda f(z).$$

Let $p, k \in \mathrm{N}$, $A, B, \lambda, \theta \in \mathrm{R}$, $\lambda \le p < k$, $0 \le B \le 1$, $-B \le A < B$, ($B \ne 1$ or $\cos\theta < 0$).

DEFINITION 1.4. Let $H^\lambda(p,k;A,B)$ denote the class of functions $f \in \mathcal{A}(p,k)$ satisfying the following condition

$$\frac{\Omega_p^\lambda f(z)}{z^p} \prec \frac{1+Az}{1+Bz}. \tag{1.2}$$

The class $H^\lambda(p,k;A,B)$ for $\lambda = 0$ and $\lambda = 1$ have been investigated by Stankiewicz *et al.* ([13] and [15]) (see also [1], [5] and [8]).

DEFINITION 1.5. Let $H_\theta^\lambda(p,k;A,B)$ denote the subclass of the class $H^\lambda(p,k;A,B)$ of functions f of the form (1.1), such that $\arg a_n = \theta$ for $a_n \ne 0, n = k, k+1, \ldots$

We can write every function f from the class $H_\theta^\lambda(p,k;A,B)$ in the form

$$f(z) = z^p + e^{i\theta} \sum_{n=k}^{\infty} |a_n| z^n, \quad z \in \mathcal{U}. \tag{1.3}$$

Let $Q_\theta(p,k;A,B) = H_\theta^0(p,k;A,B)$, $R_\theta(p,k;A,B) = H_\theta^1(p,k;A,B)$.

These classes $Q_\pi(p,k;A,B)$, $R_\pi(p,k;A,B)$ have been studied by Srivastava *et al.* ([9], [11] and [12]) (see also [2], [3] and [14]).

DEFINITION 1.6. Let $\tilde{H}_\theta^\lambda(p,k;A,B)$ denote the class of functions f of the form (1.3) satisfying the following condition

$$\sum_{n=k}^{\infty}\Gamma_n^{-1}|a_n|\le\delta(\theta,A,B), \tag{1.4}$$

where

$$\Gamma_n=\frac{\Gamma(n+1-\lambda)\Gamma(p+1)}{\Gamma(n+1)\Gamma(p+1-\lambda)},\quad \delta(\theta,A,B)=\frac{B-A}{\sqrt{1-B^2\sin^2\theta}-B\cos\theta}. \tag{1.5}$$

These classes of functions defined by Definitions 1.5 and 1.6 are called the classes of functions with fixed argument of coefficients.

In the present paper we obtain coefficient estimates, distortion theorems and extreme points for the classes of functions defined above.

2. Coefficients estimates

LEMMA 2.1 [7]. *Let f be a function of the form* (1.1). *If $f\prec g$ and g is convex function, then $|a_n|\le 1, n=k,k+1,\dots$*

LEMMA 2.2 [4]. *If a function f of the form* (1.1) *belongs to the class $\mathcal{A}(p,k)$, then*

$$\Omega_p^\lambda f(z)=z^p+\frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)}\sum_{n=k}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)}a_nz^n,\ z\in\mathcal{U}.$$

THEOREM 2.1. *If a function f of the form* (1.3) *belongs to the class $H_\theta^\lambda(p,k;A,B)$, then it satisfies the condition* (1.4).

Proof. Let $f\in H_\theta^\lambda(p,k;A,B)$. By Definition 1.3 we obtain

$$\frac{\Omega_p^\lambda f(z)}{z^p}=\frac{1+A\omega(z)}{1+B\omega(z)},$$

where $\omega(z)$ is an analytic function in $\mathcal{U}$, such that $\omega(0)=0$ and $|\omega(z)|<1$ for $z\in$ U. Thus we have

$$\left|\frac{\Omega_p^\lambda f(z)-z^p}{B\Omega_p^\lambda f(z)-Az^p}\right|=|\omega(z)|<1.$$

Using Lemma 2.2 we obtain

$$\left|\sum_{n=k}^{\infty}\Gamma_n^{-1}|a_n|z^{n-p}\right|<\left|B-A+Be^{i\theta}\sum_{n=k}^{\infty}\Gamma_n^{-1}|a_n|z^{n-p}\right|,$$

where Γ_n is defined by (1.5). Thus putting $z=r, 0<r<1$, we have

$$|w|<|B-A+Bwe^{i\theta}|, \tag{2.1}$$

where $w = \sum_{n=k}^{\infty} \Gamma_n^{-1}|a_n|r^{n-p}$. Since w is real number, by (2.1) we have

$$(1 - B^2)w^2 - (2B(B - A)\cos\theta)w - (B - A)^2 < 0.$$

Solving this inequality with respect to w we obtain

$$\sum_{n=k}^{\infty} \Gamma_n^{-1}|a_n|r^{n-p} < \delta(\theta, A, B),$$

where $\delta(\theta, A, B), \Gamma_n$ are defined by (1.5). Thus letting $r \to 1^-$ we obtain (1.4).

From Theorem 2.1 we obtain

COROLLARY 2.1. $H_\theta^\lambda(p, k; A, B) \subset \tilde{H}_\theta^\lambda(p, k; A, B)$.

From Definition 1.4 we have

COROLLARY 2.2. *If a function f of the form* (1.3) *belongs to the class $\tilde{H}_\theta^\lambda(p, k; A, B)$, then*

$$|a_n| \le \delta(\theta, A, B)\Gamma_n, \quad n = k, k + 1, \dots, \tag{2.2}$$

where Γ_n is defined by (1.5). The result is sharp. The extremal functions are functions f_n of the form

$$f_n(z) = z^p + e^{i\theta}\, \delta(\theta, A, B)\Gamma_n z^n, \quad n = k, k + 1, \dots. \tag{2.3}$$

THEOREM 2.2. *If a function f of the form* (1.1) *belongs to the class $H^\lambda(p, k; A, B)$, then*

$$|a_n| \le (B - A)\Gamma_n, \quad n = k, k + 1, \dots, \tag{2.4}$$

where Γ_n is defined by (1.5). *The result is sharp.*

Proof. Let a function f of the form (1.1) belong to the class $H^\lambda(p, k; A, B)$ and let us put

$$g(z) = \left(\frac{\Omega_p^\lambda f(z)}{z^p} - 1\right) \Big/ (A - B), \quad h(z) = \frac{z}{1 + Bz}.$$

By (1.2) we have $g \prec h$. Since the function g is a function of the form

$$g(z) = \sum_{n=k}^{\infty} [(A - B)\Gamma_n]^{-1} a_n z^{n-p}$$

and the function h is convex in $\mathcal{U}$, by Lemma 2.1 we obtain

$$[(B - A)\Gamma_n]^{-1}|a_n| \le 1, \quad n = k, k + 1, \dots \tag{2.5}$$

Thus we have (2.4). The equality in (2.5) holds for the functions g_n of the form

$$g_n(z) = h(z^{n-p}) = z^{n-p} + \dots, \quad n = k, k + 1, \dots$$

Thus the equality in (2.4) holds for the functions f_n of the form

$$f_n(z) = z^p + (A - B)\Gamma_n z^n + \dots, \quad n = k, k+1, \dots$$

By Corollary 2.1 and Corollary 2.2 we obtain

COROLLARY 2.3. *If a function f of the form* (1.3) *belongs to the class $H_\theta^\lambda(p, k; A, B)$, then*

$$|a_n| \le \delta(\theta, A, B)\Gamma_n, \quad n = k, k+1, \dots,$$

where $\delta(\theta, A, B)$, Γ_n are defined by (1.5). *The result is sharp for $\theta = \pi$. The extremal functions are functions f_n of the form*

$$f_n(z) = z^p - \frac{B - A}{1 + B}\Gamma_n z^n, \quad n = k, k+1, \dots \tag{2.6}$$

Putting $\lambda = 0$ and $\lambda = 1$ in Corollary 2.3 we obtain the following two corollaries:

COROLLARY 2.4. *If a function f of the form* (1.3) *belongs to the class $Q_\theta(p, k; A, B)$, then*

$$|a_n| \le \delta(\theta, A, B), \quad n = k, k+1, \dots,$$

where $\delta(\theta, A, B)$ is defined by (1.5). *The result is sharp for $\theta = \pi$. The extremal functions are functions f_n of the form*

$$f_n(z) = z^p - \frac{B - A}{1 + B} z^n, \quad n = k, k+1, \dots \tag{2.7}$$

COROLLARY 2.5. *If a function f of the form* (1.3) *belongs to the class $R_\theta(p, k; A, B)$, then*

$$|a_n| \le \delta(\theta, A, B)\frac{p}{n}, \quad n = k, k+1, \dots,$$

where $\delta(\theta, A, B)$ is defined by (1.5). *The result is sharp for $\theta = \pi$. The extremal functions are functions f_n of the form*

$$f_n(z) = z^p - \frac{B - A}{1 + B}\frac{p}{n} z^n, \quad n = k, k+1, \dots \tag{2.8}$$

3. Distortion theorems and extreme points

THEOREM 3.1. *If $f \in H_\theta^\lambda(p, k; A, B)$, $|z| = r < 1$, then for $0 \le \lambda \le p$*

$$r^p - \delta(\theta, A, B)\Gamma_k r^k \le |f(z)| \le r^p + \delta(\theta, A, B)\Gamma_k r^k \tag{3.1}$$

and for $1 \le \lambda \le p$

$$pr^{p-1} - k\delta(\theta, A, B)\Gamma_k r^{k-1} \le |f'(z)| \le pr^{p-1} + k\delta(\theta, A, B)\Gamma_k r^{k-1}, \tag{3.2}$$

where $\delta(\theta, A, B)$, Γ_k are defined by (1.5). *The result is sharp for $\theta = \pi$. The extremal function is function f_k of the form* (2.6).

Proof. Let $f \in H_\theta^\lambda(p,k;A,B)$, $|z| = r < 1$. Since the sequences $\{\Gamma_n\}$ for $0 \le \lambda \le p$ and $\{\frac{\Gamma_n}{n}\}$ for $1 \le \lambda \le p$ are decreasing and positive, by Theorem 2.1 we obtain

$$\sum_{n=k}^{\infty} |a_n| \le \delta(\theta, A, B)\Gamma_k \quad \text{for } 0 \le \lambda \le p, \tag{3.3}$$

$$\sum_{n=k}^{\infty} n|a_n| \le k\delta(\theta, A, B)\Gamma_k \quad \text{for } 1 \le \lambda \le p. \tag{3.4}$$

Since

$$\begin{aligned} |f(z)| &= \left| z^p + e^{i\theta} \sum_{n=k}^{\infty} |a_n| z^n \right| \le r^p + \sum_{n=k}^{\infty} |a_n| r^n \\ &= r^p + r^k \sum_{n=k}^{\infty} |a_n| r^{n-k} \le r^p + r^k \sum_{n=k}^{\infty} |a_n|, \end{aligned}$$

and

$$\begin{aligned} |f(z)| &= \left| z^p + e^{i\theta} \sum_{n=k}^{\infty} |a_n| z^n \right| \ge r^p - \sum_{n=k}^{\infty} |a_n| r^n \\ &= r^p - r^k \sum_{n=k}^{\infty} |a_n| r^{n-k} \ge r^p - r^k \sum_{n=k}^{\infty} |a_n|, \end{aligned}$$

then by (3.3) we obtain (3.1). Using (3.4) we prove analogously the estimations (3.2).

Putting $\lambda = 0$ and $\lambda = 1$ in Corollary 2.3 we obtain the following two corollaries:

COROLLARY 3.1. *If $f \in Q_\theta(p,k;A,B)$, $|z| = r < 1$, then for $0 \le \lambda \le p$*

$$r^p - \delta(\theta, A, B)r^k \le |f(z)| \le r^p + \delta(\theta, A, B)r^k$$

and for $1 \le \lambda \le p$

$$pr^{p-1} - k\delta(\theta, A, B)r^{k-1} \le |f'(z)| \le pr^{p-1} + k\delta(\theta, A, B)r^{k-1},$$

where $\delta(\theta, A, B)$ is defined by (1.5). The result is sharp for $\theta = \pi$. The extremal function is function f_k of the form (2.7).

COROLLARY 3.2. *If $f \in R_\theta(p,k;A,B)$, $|z| = r < 1$, then for $0 \le \lambda \le p$*

$$r^p - \frac{p}{k}\delta(\theta, A, B)r^k \le |f(z)| \le r^p + \frac{p}{k}\delta(\theta, A, B)r^k$$

and for $1 \le \lambda \le p$

$$pr^{p-1} - p\delta(\theta, A, B)r^{k-1} \le |f'(z)| \le pr^{p-1} + p\delta(\theta, A, B)r^{k-1},$$

where $\delta(\theta, A, B)$ *is defined by* (1.5). *The result is sharp for* $\theta = \pi$. *The extremal function is function* f_k *of the form* (2.8).

THEOREM 3.2. *Let* $\delta(\theta, A, B)$, Γ_n *be defined by* (1.5) *and let us put:*

$$f_n(z) = z^p + e^{i\theta}\delta(\theta, A, B)\Gamma_n z^n,\ n = k, k+1, \dots, \quad f_{k-1}(z) = z^p. \tag{3.5}$$

A function f *belongs to the class* $\tilde{H}_\theta^\lambda(p, k; A, B)$ *if and only if it is of the form*

$$f(z) = \sum_{n=k-1}^{\infty} \gamma_n f_n(z), \quad z \in \mathcal{U}, \tag{3.6}$$

where $\sum_{n=k-1}^{\infty} \gamma_n = 1$, $\gamma_n \ge 0$ *for* $n = k-1, k, k+1 \dots$

Proof. ($\Rightarrow$) Let a function f of the form (1.3) belong to the class $\tilde{H}_\theta^\lambda(p, k; A, B)$. Let us put

$$\gamma_n = [\delta(\theta, A, B)\Gamma_n]^{-1}|a_n|, \quad n = k, k+1, \dots$$

and

$$\gamma_{k-1} = 1 - \sum_{n=k}^{\infty} \gamma_n.$$

By the assumption we have $\gamma_n \ge 0$, $n = k, k+1, \dots$ By Definition 1.4 we have $\gamma_{k-1} \ge 0$. Thus

$$\sum_{n=k-1}^{\infty} \gamma_n f_n(z) = \gamma_{k-1} f_{k-1}(z) + \sum_{n=k}^{\infty} \gamma_n f_n(z) = \Big(1 - \sum_{n=k}^{\infty} \gamma_n\Big) z^p$$
$$+ \sum_{n=k}^{\infty} [\delta(\theta, A, B)\Gamma_n]^{-1}|a_n|(z^p + e^{i\theta}\delta(\theta, A, B)\Gamma_n z^n)$$
$$= z^p - \sum_{n=k}^{\infty} [\delta(\theta, A, B)\Gamma_n]^{-1}|a_n| z^p + \sum_{n=k}^{\infty} [\delta(\theta, A, B)\Gamma_n]^{-1}|a_n| z^p$$
$$+ e^{i\theta} \sum_{n=k}^{\infty} |a_n| z^p = f(z)$$

and the condition (3.6) follows.

($\Leftarrow$) Let the function f satisfy (3.6). Since

$$\begin{aligned} f(z) &= \sum_{n=k-1}^{\infty} \gamma_n f_n(z) = \gamma_{k-1} f_{k-1} + \sum_{n=k}^{\infty} \gamma_n f_n(z) \\ &= \Big(1 - \sum_{n=k}^{\infty} \gamma_n\Big) z^p + \sum_{n=k}^{\infty} (z^p + e^{i\theta} \delta(\theta, A, B) \Gamma_n z^n) \gamma_n \\ &= z^p + e^{i\theta} \sum_{n=k}^{\infty} \delta(\theta, A, B) \gamma_n \Gamma_n z^n \end{aligned}$$

we can write the function f in the form (1.3), where

$$|a_n| = \delta(\theta, A, B) \gamma_n \Gamma_n.$$

Moreover

$$\sum_{n=k}^{\infty} \Gamma_n^{-1} |a_n| = \sum_{n=k}^{\infty} \gamma_n \delta(\theta, A, B) = \delta(\theta, A, B)(1 - \gamma_{1-k}) \leq \delta(\theta, A, B).$$

Thus we have $f \in H_\theta^\lambda(p, k; A, B)$, which ends the proof.

We prove analogously the following:

THEOREM 3.3. *Let $f_{k-1}(z) = z^p$ and let f_n, $n = k, k+1, \ldots$ be defined by* (2.6). *A function f belongs to the class $H_\pi^\lambda(p, k; A, B)$ if and only if it is of the form* (3.6).

From Theorem 3.3 we obtain the following two corollaries:

COROLLARY 3.3. *Let $f_{k-1}(z) = z^p$ and let f_n, $n = k, k+1, \ldots$ be defined by* (2.7). *A function f belongs to the class $Q_\pi(p, k; A, B)$ if and only if it is of the form* (3.6).

COROLLARY 3.4. *Let $f_{k-1}(z) = z^p$ and let f_n, $n = k, k+1, \ldots$ be defined by* (2.8). *A function f belongs to the class $R_\pi(p, k; A, B)$ if and only if it is of the form* (3.6).

From Theorems 3.2, 3.3 and Corollaries 3.1, 3.2 we obtain the following two corollaries:

COROLLARY 3.5. $H_\pi^\lambda(p, k; A, B) = \tilde{H}_\pi^\lambda(p, k; A, B)$.

COROLLARY 3.6. *These classes $\tilde{H}_\theta^\lambda(p, k; A, B)$, $H_\pi^\lambda(p, k; A, B)$, $Q_\pi(p, k; A, B)$, $R_\pi(p, k; A, B)$ are convex. The extremal points are functions of the form* (3.5), (2.6) *and $f_{k-1}(z) = z^p$,* (2.7) *and $f_{k-1}(z) = z^p$,* (2.8) *and $f_{k-1}(z) = z^p$, respectively.*

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INSTITUTE OF MATHEMATICS,
PEDAGOGICAL UNIVERSITY OF RZESZÓW
ul. Rejtana 16A,
35-310 RZESZÓW, POLAND
E-mail: jdziok@univ.rzeszow.pl

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