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NEW CRITERIA FOR UNIVALENCE OF CERTAIN INTEGRAL OPERATORS

Abstract. In this work there is considered the class of univalent functions defined by the condition $\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1$, $|z| < 1$, where $f(z) = z + a_2 z^2 + \ldots$ is analytic in the unit disc $U = \{z : |z| < 1\}$. The author determines conditions for the univalence of certain integral operators.

1. Introduction

We denote by S the class of regular and univalent functions $f(z) = z + a_2 z^2 + \ldots$ in the unit disc U. Let A be the class of functions f which are analytic in the unit disc U with $f(0) = f'(0) - 1 = 0$.

In their paper [2] Ozaki and Nunokawa proved the following

THEOREM A. *Assume that $f \in A$ satisfies the condition*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1, \quad z \in U, \tag{1}$$

then f is univalent in U.

2. Preliminary results

We shall use the following results.

THEOREM B [3]. *Let α be a complex number, $Re\alpha > 0$ and $f \in A$. If*

$$\frac{1 - |z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{2}$$

for all $z \in U$, then the function

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u)\, du\right]^{\frac{1}{\alpha}} \tag{3}$$

is in the class S.

THEOREM C [4]. *Let α be a complex number, $Re\alpha > 0$ and $f(z) = z + a_2 z^2 + \dots$ is a regular function in U. If*

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{4}$$

for all $z \in U$, then for any complex number $\beta, Re\beta \ge Re\alpha$ the function

$$F_\beta(z) = \left[\beta\int_0^z u^{\beta-1} f'(u)\, du\right]^{\frac{1}{\beta}} = z + \dots \tag{5}$$

is regular and univalent in U.

THE SCHWARZ LEMMA [1]. *Let the analytic function $f(z)$ be regular in the unit circle $|z| < 1$ and let $f(0) = 0$. If, in $|z| < 1$, $|f(z)| \le 1$, then*

$$|f(z)| \le |z|, \quad |z| < 1 \tag{6}$$

where equality can hold only if $f(z) \equiv Kz$ and $|K| = 1$.

3. Main results

THEOREM 1. *Let $g \in A$ satisfy (1) and α be a complex number such that $Re\alpha \ge \frac{3}{|\alpha|}$. If*

$$|g(z)| \le 1 \tag{7}$$

$z \in U$, then for every complex number β, $Re\beta \ge Re\alpha$ the function

$$H_{\alpha,\beta}(z) = \left[\beta\int_0^z u^{\beta-1}\left(\frac{g(u)}{u}\right)^{\frac{1}{\alpha}} du\right]^{\frac{1}{\beta}} \tag{8}$$

is in the class S.

Proof. Let us consider the function

$$h(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\frac{1}{\alpha}} du. \tag{9}$$

The function h is regular in U. From (9) we have

$h'(z) = \left(\frac{g(z)}{z}\right)^{\frac{1}{\alpha}}$,

$h''(z) = \frac{1}{\alpha}\left(\frac{g(z)}{z}\right)^{\frac{1}{\alpha}-1}\frac{zg'(z)-g(z)}{z^2}$ and

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zh''(z)}{h'(z)}\right| = \frac{1-|z|^{2Re\alpha}}{Re\alpha}\frac{1}{|\alpha|}\left|\frac{zg'(z)}{g(z)} - 1\right| \tag{10}$$

for all $z \in U$. From (10) we get

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{1-|z|^{2Re\alpha}}{|\alpha|Re\alpha}\left|\frac{zg'(z)}{g(z)}\right| + \frac{1-|z|^{2Re\alpha}}{|\alpha|Re\alpha} \tag{11}$$

for all $z \in U$. Hence, we have

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{1-|z|^{2Re\alpha}}{|\alpha|Re\alpha}\left(\left|\frac{z^2g'(z)}{g^2(z)}\right|\frac{|g(z)|}{|z|} + 1\right) \tag{12}$$

for all $z \in U$.

By the Schwarz Lemma also $|g(z)| \leq |z|, z \in U$ and using (12) we obtain

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{1-|z|^{2Re\alpha}}{|\alpha|Re\alpha}\left(\left|\frac{z^2g'(z)}{g^2(z)} - 1\right| + 2\right) \tag{13}$$

for all $z \in U$.

Since g satisfies the condition (1) then from (13) we have

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{3}{|\alpha|Re\alpha}(1-|z|^{2Re\alpha}) \leq \frac{3}{|\alpha|Re\alpha} \tag{14}$$

for all $z \in U$.

Since $\frac{3}{|\alpha|Re\alpha} \leq 1$ we conclude that

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \tag{15}$$

for all $z \in U$.

Now (15) and Theorem C imply that the function $H_{\alpha,\beta}$ is in the class S.

THEOREM 2. *Let $g \in A$ satisfy (1) and α be a complex number, with $|\alpha-1| \leq \frac{Re\alpha}{3}$. If*

$$|g(z)| \leq 1 \tag{16}$$

for all $z \in U$, then the function

$$G_\alpha(z) = \left[\alpha\int_0^z g^{\alpha-1}(u)\,du\right]^{\frac{1}{\alpha}} \tag{17}$$

is in the class S.

Proof. From (17) we have

$$G_\alpha(z) = \left[\alpha\int_0^z u^{\alpha-1}\left(\frac{g(u)}{u}\right)^{\alpha-1} du\right]^{\frac{1}{\alpha}}. \tag{18}$$

Let us consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\alpha-1} du. \tag{19}$$

The function f is regular in U. From (19) we obtain $f'(z) = \left(\frac{g(z)}{z}\right)^{\alpha-1}$, $f''(z) = (\alpha-1)\left(\frac{g(z)}{z}\right)^{\alpha-2}\frac{zg'(z)-g(z)}{z^2}$ and

$$(20) \qquad \frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{1-|z|^{2Re\alpha}}{Re\alpha}|\alpha-1|\left(\left|\frac{zg'(z)}{g(z)}\right|+1\right),$$

for all $z \in U$. Hence, we have

$$(21) \qquad \frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le |\alpha-1|\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left(\left|\frac{z^2g'(z)}{g^2(z)}\right|\frac{|g(z)|}{|z|}+1\right).$$

Applying Schwarz Lemma and using (21) we obtain

$$(22) \qquad \frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le |\alpha-1|\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left(\left|\frac{z^2g'(z)}{g^2(z)}-1\right|+2\right).$$

Since g satisfies the condition (1), then from (22) we have

$$(23) \qquad \frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 3|\alpha-1|\frac{1-|z|^{2Re\alpha}}{Re\alpha} \le 3\frac{|\alpha-1|}{Re\alpha}$$

for all $z \in U$.

But $|\alpha-1| \le \frac{Re\alpha}{3}$ so from (23) we get

$$(24) \qquad \frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1$$

for all $z \in U$.

Now Theorem B and (24) imply that the function G_α is in the class S.

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