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THE JENSEN INEQUALITY FOR s-BRECKNER CONVEX FUNCTIONS IN LINEAR SPACES

Abstract. We derive some inequalities of Jensen's type for s-convex functions in the sense of Breckner on subsets of linear spaces and give some applications connected with special means.

1. Introduction

Let X be a real linear space and C a convex subset of X. If f is a real valued convex function on C, then *Jensen's inequality* states that

$$f\Big(\sum_{i=1}^{n} p_i x_i\Big) \le \sum_{i=1}^{n} p_i f(x_i), \tag{1.1}$$

whenever $p_i \ge 0$, $x_i \in C$ and $\sum_{i=1}^{n} p_i = 1$. For some classical results connected with Jensen's inequality see [12], Chapter 1. New inequalities related to Jensen's inequality are derived in [2]–[9] and [13].

A function f on C is said to be of *Q-type* provided

$$f(tx + (1-t)y) \le f(x)/t + f(y)/(1-t) \tag{1.2}$$

for all $t \in (0,1)$ and x, $y \in C$ (see [12], p. 410). In particular, every nonnegative convex function and every nonnegative monotonic function on an interval are of Q-type.

The following inequality for functions of Q-type was derived by Mitrinovic and Pečarić [11] and is analogous to Jensen's inequality

$$f\Big(\sum_{i=1}^{n} p_i x_i\Big) \le \sum_{i=1}^{n} f(x_i)/p_i, \tag{1.3}$$

whenever $p_i > 0$, $x_i \in C$ and $\sum_{i=1}^{n} p_i = 1$.

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In this paper we consider for $s \in (0, \infty)$ the class of s-Breckner convex functions (which for $0 < s < 1$ were called in [1], [10]) s-convex in the second sense and we will prove some inequalities similar to Jensen's inequality and its refinements.

2. s-Breckner convex functions in linear spaces

DEFINITION 2.1. If f is a real valued convex function on C and $s > 0$, then we say that f is *s-Breckner convex function* provided

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y), \tag{sB}$$

whenever $0 < t < 1$ and $x, y \in C$. For $s = 1$ it reduces to the usual notion of convexity.

THEOREM 2.2. *Let f be a real valued s-Breckner convex function on C and $s > 0$. Then we have the inequality (generalising* (1.1)*)*

$$f\Big(\sum_{i=1}^{n} p_i x_i\Big) \le \sum_{i=1}^{n} p_i^s f(x_i), \tag{2.1}$$

whenever $p_i \ge 0$, $x_i \in C$ and $\sum_{i=1}^{n} p_i = 1$.

Proof. We proceed by induction on n. For $n = 2$ it is just (sB). Now, if (2.1) holds for $n = k-1$, then given $p_i \ge 0$, $x_i \in C$ and $\sum_{i=1}^{k} p_i = 1$ we may and do assume that all $p_i > 0$. We put $\beta_j := p_j/(p_1 + \ldots + p_{k-1})$ for $j < k$. Then $\beta_1 + \ldots + \beta_{k-1} = 1$ and thus we have

$$f(\beta_1 x_1 + \ldots + \beta_{k-1} x_{k-1}) \le \beta_1^s f(x_1) + \ldots + \beta_{k-1}^s f(x_{k-1}).$$

Now for $P := p_1 + \ldots + p_{k-1}$ we have

$$\begin{aligned} f(p_1 x_1 + \ldots + p_k x_k) &= f(P(p_1 x_1 + \ldots + p_{k-1} x_{k-1})/P + p_k x_k) \\ &\le P^s f((p_1 x_1 + \ldots + p_{k-1} x_{k-1})/P) + p_k^s f(x_k) \\ &\le P^s((p_1^s f(x_1) + \ldots + p_{k-1}^s f(x_{k-1}))/P^s) + p_k^s f(x_k) \\ &= \sum_{i=1}^{k} p_i^s f(x_i) \end{aligned}$$

which establishes (2.1) for $n = k$.

The following corollaries follow trivially.

COROLLARY 2.3. *Let f be a real valued s-Breckner convex function on C and $s > 0$. Then*

$$f\Big(P^{-1} \sum_{i=1}^{n} p_i x_i\Big) \le P^{-s} \sum_{i=1}^{n} p_i^s f(x_i) \tag{2.2}$$

whenever $p_i \ge 0$, $x_i \in C$ and $\sum_{i=1}^{n} p_i = P$.

COROLLARY 2.4. *Let f be a real valued s-Breckner convex function on C and $s > 0$. Then*

$$f\Big(n^{-1}\sum_{i=1}^{n} x_i\Big) \le n^{-s}\sum_{i=1}^{n} f(x_i), \tag{2.3}$$

whenever $x_i \in C$.

In the paper [10] the following class of functions was considered.

DEFINITION 2.5. Let $s \in (0,1]$. A real valued function f on an interval $I \subset [0,\infty)$ is *s-convex in the second sense* provided

$$f(\alpha u + \beta v) \le \alpha^s f(u) + \beta^s f(v)$$

for all u, $v \in I$ and α, $\beta \ge 0$ with $\alpha + \beta = 1$. This is denoted by $f \in K_s^2$.

This definition of s-convexity was considered in [1], where was explored the problem: whether rationally s-convex functions are s-convex.

We record here some of the results from [10] about s-convex functions.

THEOREM 2.6. *Let $0 < s < 1$. If $f \in K_s^2$, then f is nonnegative.*

Recall that a function $f : [0,\infty) \to [0,\infty)$ is a *ϕ-function* if $f(0) = 0$ and f is nondecresing and continuous, on $[0,\infty)$.

THEOREM 2.7 ([10, Corollary 2]). *If Φ is a convex ϕ-function and f is a ϕ-function in K_s^2, then the composition $\Phi \circ f$ belongs to K_s^2. In particular $\Phi^s \in K_s^2$.*

If $0 < s < 1$ then there are, however, ϕ-functions in K_s^2 which are neither of the form $\Phi(u^s)$ nor Φ^s for any convex ϕ-function Φ ([10], Example 3).

3. Functionals related to single sums

Let C be a convex set in a linear space X and $s > 0$. We define

$$B(f,I,p,x) := \sum_{i\in I} p_i^s f(x_i) - P_I^s f\Big(P_I^{-1}\sum_{i\in I} p_i x_i\Big)$$

for each s-Breckner convex function f on C, each finite set I of natural numbers, each sequence p of positive numbers and sequence x of points of C, where $P_I := \sum_{i\in I} p_i$.

It is clear that (2.2) is equivalent to $B \ge 0$. We prove some slightly less obvious facts about this functional.

THEOREM 3.1. *Let f be a s-Breckner convex function on C and take a sequence p of positive numbers and a sequence x of points of C, and set $P_I := \sum_{i\in I} p_i$. Let I and K be disjoint finite sets of natural numbers. Then*

$$B(f, I\cup K, p, x) \ge B(f,I,p,x) + B(f,K,p,x). \tag{3.1}$$

Proof. We have

$$\begin{aligned}
B(f,I\cup K,p,x) &= \sum_{i\in I\cup K} p_i^s f(x_i) - P_{I\cup K}^s f\Big(P_{I\cup K}^{-1}\sum_{i\in I\cup K} p_i x_i\Big)\\
&= \sum_{i\in I} p_i^s f(x_i) + \sum_{k\in K} p_k^s f(x_k) - P_{I\cup K}^s f\\
&\quad\times\Big(\frac{P_I}{P_{I\cup K}}\sum_{i\in I}\frac{p_i x_i}{P_I} + \frac{P_K}{P_{I\cup K}}\sum_{k\in K}\frac{p_k x_k}{P_K}\Big)\\
&\geq \sum_{i\in I} p_i^s f(x_i) + \sum_{k\in K} p_k^s f(x_k) - P_{I\cup K}^s\\
&\quad\times\Big(\frac{P_I^s}{P_{I\cup K}^s} f(\sum_{i\in I}\frac{p_i x_i}{P_I}) + \frac{P_K^s}{P_{I\cup K}^s} f\Big(\sum_{k\in K}\frac{p_k x_k}{P_K}\Big)\Big)\\
&= \sum_{i\in I} p_i^s f(x_i) - P_I^s f(\sum_{i\in I} p_i x_i/P_I)\\
&\quad+ \sum_{k\in K} p_k^s f(x_k) - P_K^s f\Big(\sum_{k\in K} p_k x_k/P_K\Big)\\
&= B(f,I,p,x) + B(f,K,p,x).
\end{aligned}$$

THEOREM 3.2. *Let f be a s-Breckner convex function on C and take a sequence p of positive numbers and a sequence x of points of C, and set $P_I := \sum_{i\in I} p_i$. Let I and K be finite sets of natural numbers with $I \subset K$. Then*

$$B(f,K,p,x) \geq B(f,I,p,x). \tag{3.2}$$

Proof. We have

$$B(f,K,p,x) = B(f,I\cup(K\setminus I),p,x) \geq B(f,I),p,x) + B(f,K\setminus I,p,x),$$

by (3.1), so that $B(f,K,p,x) - B(f,I,p,x) \geq B(f,K\setminus I,p,x) \geq 0$.

Now consider the sequence $\{B_n\}$ with

$$B_n := \sum_{i=1}^{n} p_i^s f(x_i) - P_n^s f\Big(P_n^{-1}\sum_{i=1}^{n} p_i x_i\Big),$$

where $P_n := \sum_{i=1}^n p_i$.

COROLLARY 3.3. *The sequence $\{B_n\}$ is monotone nondecreasing and*

$$B_n \geq \max_{1\leq i<j\leq n}[p_i^s f(x_i) + p_j^s f(x_j) - (p_i+p_j)^s f((p_i x_i + p_j x_j)/(p_i+p_j))] \geq 0.$$

Now consider the functional

$$H(f,I,p,x) := P_I^s f(\sum_{i\in I} p_i x_i/P_I),$$

for each s-Breckner convex function f on C, each finite set I of natural numbers, each sequence p of positive numbers and sequence x of points of C, where $P_I := \sum_{i\in I} p_i$.

THEOREM 3.4. *The mapping $H(f,I,\cdot,x)$ is subadditive, that is, if p and $q>0$ are sequences of positive numbers, then*

$$H(f,I,p+q,x) \le H(f,I,p,x) + H(f,I,q,x). \tag{3.3}$$

Proof. Let $Q_I := \sum_{i\in I} q_i$. Then

$$\begin{aligned} H(f,I,p+q,x) &= (P_I+Q_I)^s f\Big(\sum_{i\in I}(p_i+q_i)x_i/(P_I+Q_I)\Big) \\ &= (P_I+Q_I)^s f\left(\frac{P_I}{(P_I+Q_I)}\sum_{i\in I}\frac{p_i x_i}{P_I} + \frac{Q_I}{(P_I+Q_I)}\sum_{i\in I}\frac{q_i x_i}{Q_I}\right) \\ &\le (P_I+Q_I)^s\left(\frac{P_I^s}{(P_I+Q_I)^s} f\Big(\sum_{i\in I}\frac{p_i x_i}{P_I}\Big)\right. \\ &\quad \left.+ \frac{Q_I^s}{(P_I+Q_I)^s} f\Big(\sum_{i\in I}\frac{q_i x_i}{Q_I}\Big)\right) \\ &= H(f,I,p,x) + H(f,I,q,x)). \end{aligned}$$

Similarly one can prove that H is subadditive as an index set mapping.

THEOREM 3.5. *If I and K are disjoint finite sets of natural numbers, then*

$$H(f,I\cup K,p,x) \le H(f,I,p,x) + H(f,K,p,x). \tag{3.4}$$

It is obvious that H has the following homogeneity property.

PROPOSITION 3.6. *For $\alpha>0$ we have $H(f,I,\alpha p,x) = \alpha^s H(f,I,p,x)$.*

Similarly, H is s-Breckner convex over positive sequences.

PROPOSITION 3.7. *If p and q are sequences of positive numbers, then*

$$H(f,I,\alpha p+(1-\alpha)q,x) \le \alpha^s H(f,I,p,x) + (1-\alpha)^s H(f,I,q,x)$$

for all $0<\alpha<1$.

4. Some applications

Suppose that f is a concave function on an interval $[a,b]$ which is also s-Breckner convex. Then we have

$$\sum_{i=1}^n p_i f(x_i)/P_n \le f\Big(P_n^{-1}\sum_{i=1}^n p_i x_i\Big) \le \sum_{i=1}^n p_i^s f(x_i)/P_n^s \tag{4.1}$$

which suggests that we need conditions which guarantee that f has those properties.

THEOREM 4.1. *Let Φ be a ϕ-function on* $[0,\infty)$ *which is twice differentiable on* $(0,\infty)$. *If* $0 < s < 1$ *and*

$$0 \le \Phi(t)\Phi''(t) \le (1-s)[\Phi'(t)]^2$$

for all t, *then* Φ^s *is a concave function which is also* s*-convex in the second sense.*

Proof. Note that, if $\Phi(t) \neq 0$, then $\Phi''(t) \ge 0$, so Φ is convex. By Theorem 2.7, the function $g := \Phi^s$ is s-convex in the second sense. Then

$$g''(t) = s[\Phi(t)]^{s-2}[\Phi(t)\Phi''(t) - (1-s)(\phi'(t))^2] \le 0$$

which shows that g is also concave.

COROLLARY 4.2. *Let* $s \in (0,1)$. *Then for* $1 \le q \le 1/s$ *the function* $g(x) := x^{qs}$ *is concave and* s*-convex in the second sense on* $[0,\infty)$.

Applying (4.1) to the function $g(x) := x^{qs}$, we get

$$\sum_{i=1}^{n} p_i x_i^{qs}/P_n \le \left(P_n^{-1}\sum_{i=1}^{n} p_i x_i\right)^{qs} \le \sum_{i=1}^{n} p_i^s x_i^{qs}/P_n^s, \tag{4.2}$$

with $P_n := \sum_{i=1}^{n} p_i$ if x and p are positive sequences, $s \in (0,1)$ and $1 \le q \le 1/s$. In particular we have

$$\sum_{i=1}^{n} p_i x_i^s/P_n \le \left(P_n^{-1}\sum_{i=1}^{n} p_i x_i\right)^{s} \le \sum_{i=1}^{n} p_i^s x_i^s/P_n^s \tag{4.3}$$

and

$$\sum_{i=1}^{n} p_i x_i/P_n \le \sum_{i=1}^{n} p_i^s x_i/P_n^s. \tag{4.4}$$

Also, letting $q \ge 1$ and $s = 1/q$, we get

$$\sum_{i=1}^{n} p_i x_i^{1/q}/P_n \le \left(P_n^{-1}\sum_{i=1}^{n} p_i x_i\right)^{1/q} \le \sum_{i=1}^{n} p_i^{1/q} x_i^{1/q}/P_n^{1/q}$$

which gives us

$$\left(\sum_{i=1}^{n} p_i x_i^{1/q}/P_n\right)^{q} \le P_n^{-1}\sum_{i=1}^{n} p_i x_i \le \left(\sum_{i=1}^{n} p_i^{1/q} x_i^{1/q}\right)^{q}/P_n. \tag{4.5}$$

Finally, let $x_i := qy_i$ in (4.5) to get

$$\left(\sum_{i=1}^{n} p_i y_i/P_n\right)^{q} \le P_n^{-1}\sum_{i=1}^{n} p_i y_i^q \le \left(\sum_{i=1}^{n} p_i^{1/q} y_i\right)^{q}/P_n$$

which gives an upper bound corressponding to the convexity of the qth power function for $q \ge 1$.

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