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SOME REMARKS ON ENDOMORPHISMS OF UNIVERSAL n-ALGEBRAS

0. Let S be a semigroup of transformation on a set X. In [1] Grätzer has raised the problem of characterizing those S which are the endomorphism semigroup of some finitary algebra over X. There are many papers in which the problem has been solved for special momoids S, e.g. [2], [3], [8], [9] and [10]. In 1977 Sauer and Stone [7] found a characterization of endomorphism semigroups in terms of solutions of some systems of functional equations with coefficiens from S. This description seems to be rather unpleasent because of existential character. However the authors gave many application of their result. One can of course ask [5] about a characterization of endomorphism semigroups of n-algebras (= n-ary algebras). In this note we give some observations on this question. In particular we show that in special cases the endomorphism semigroups of n-algebras can be characterized by the conditions closely related to the condition on the group of automorphisms of n-algebras.

I. Let S be a monoid of trasformations on a non-empty set X. For $x = (x_1, x_2, \ldots, x_r) \in X^r$, $s \in S$, $1 \leq r$, $A \subseteq X$ and $U \subseteq S$ we use the natural convention $sx = (sx_1, sx_2, \ldots, sx_r)$, $UA = \{ua : u \in U,\ a \in A\}$. We will denote by $\mathcal{F}_{S,r}$ the set of all functions $f : X^r \longrightarrow X$, which commute with all $s \in S$, i.e. such that the equality

$$sf(x_1, x_2, \ldots x_r) = f(sx_1, sx_2, \ldots sx_r)$$

holds for all $s \in S$ and all $(x_1, x_2, \ldots, x_r) \in X^r$. The set of all fixed points of the monoid S will be denoted by $\mathcal{F}_{S,0} = \{x \in X : sx = x$ for all $s \in S\}$. If all operations of an algebra $\mathfrak{A}$ are of rank $< n+1$, where $n \in \{\aleph_0, 0, 1, 2, \ldots\} = \{\aleph_0, 0\} \bigcup \mathcal{N}$, we say that $\mathfrak{A}$ is an n-algebra, but a $\aleph_0$-algebra is called simply an algebra. Let End $\mathfrak{A}$ denotes the endomorphism semigroup of an algebra $\mathfrak{A}$.

PROPOSITION 1. *We have*

1. Let $n \in \{\aleph_0, 0, 1, 2, \ldots\}$. *There exists an* n*-algebra* $(X;\mathcal{F})$ *such that* $S = \mathrm{End}(X;\mathcal{F})$ *if and only if* $S = \mathrm{End}(X;\bigcup_{r<n+1}\mathcal{F}_{S,r})$.

2. For a finite n *we have* $\mathrm{End}(X;\bigcup_{n<n+1}\mathcal{F}_{S,r}) = \mathrm{End}(X;\mathcal{F}_{S,n})$.

Proof. It follows immediately from the implication

$$\mathcal{F} \subseteq \mathcal{F}_{S,n} \Longrightarrow \mathrm{End}(X;\mathcal{F}_{S,n}) \subseteq \mathrm{End}(X,\mathcal{F})$$

and the fact that the injection $i : X^{X^r} \longrightarrow X^{X^{r+1}}$, $r > 0$, defined by the formula

$$(if)(x_1, x_2, \ldots, x_{r+1}) = f(x_1, x_2, \ldots, x_r)$$

preserves the commutation relation i.e., the equivalence

$$\varphi i(f) = i(f)\varphi \Leftrightarrow \varphi f = f\varphi$$

holds for each transformation φ of X.

In particular, for $n = 0$, we get

COROLLARY 1. *A monoid* S *of transformations of* X *is the semigroup of endomorphisms of some* 0*-algebra if and only if* S *contains each transformation of* X *which moves no element of* $\mathcal{F}_{S,0}$.

In the next proposition we use a simple property of the automorphism groups of n-algebras, which was introduced by B. Jónsson [4].

PROPOSITION 2. *For every* $n \in \mathcal{N}$ *the endomorphism semigroup of an* n*-algebra, satisfies*

(β_{n+2}): *Each transformation* φ *of* X *such that for every sequence* $\boldsymbol{x} \in X^{n+1}$ *there exists* $s \in S$ *with* $\varphi(\boldsymbol{x}) = s(\boldsymbol{x})$, *belongs to* S.

Proof. Let us suppose that $\varphi \in X^X$, $f \in \mathcal{F}_{S,n}$ and $(x_1, x_2, \ldots x_n) \in X^n$. We put $\boldsymbol{x} = (x_1, x_2, \ldots, x_n, f(x_1, x_2, \ldots, x_n))$. If $\varphi(\boldsymbol{x}) = s(\boldsymbol{x})$ for some $s \in S$, then we have

$$\begin{aligned}\varphi f(x_1, x_2, \ldots, x_n) &= sf(x_1, x_2, \ldots, x_n) = f(sx_1, sx_2, \ldots, sx_n)\\ &= f(\varphi x_1, \varphi x_2, \ldots, \varphi x_n),\end{aligned}$$

which means that φ is an endomorphism of the algebra $(X;\mathcal{F}_{S,n})$ and therefore, by Proposition 1, it has to belong to S, as required.

In the case S being a group, $\beta_{n+1}(S)$ ensure the existence of an n-algebra with S as the automorphism group [4]. In general case it is not true.

For a finite $n > 0$ and a transformation semigroup S we set

$\varepsilon_n(S)$: *There exists a family;* $\{\boldsymbol{x}^t\}_{t\in T}$ *of elements of* X^n *such that* X^n *is the sum of pairwise disjoint sets* $S\boldsymbol{x}^t$, $t \in T$.

EXAMPLES.

1. *If* S *is a group, then* $\varepsilon_n(S)$ *holds for all* $n \in \mathcal{N}$.

2. *If S is the semigroup of all injection of X into itself, then $\varepsilon_n(S)$ holds for every $n \in \mathcal{N}$. In fact, if $x_1, x_2, x_3 \dots$ is a sequence of pairwise different elements of X, then*

$X = Sx_1$,

$X^2 = S(x_1, x_1) \cup S(x_1, x_2)$,

$X^3 = S(x_1, x_1, x_1) \cup S(x_1, x_2, x_2) \cup S(x_2, x_1, x_2) \cup S(x_2, x_2, x_1) \cup S(x_1, x_2, x_3)$, etc.

3. *Let S be a group and let $K = \{x \in X : sx = x \text{ for all } s \in S\}$. Then $S \cup K$ forms a semigroup and $\varepsilon_1(S \cup K)$ holds.*

4. *If $X = Sx_0$ for some $x_0 \in X$, then, of course, S satisfies ε_1.*

5. *Let H be a group of bijection of S such that $sHx \subseteq Hx$ for all $s \in S$ and $x \in X$, then $\varepsilon_1(S)$ holds.*

For $n \in N$, $\boldsymbol{x} \in X^n$ and $S \subseteq X^X$, we define

$$C_S(\boldsymbol{x}) = \{y \in X : s\boldsymbol{x} = s'\boldsymbol{x} \text{ implies } sy = s'y \text{ for all } s, s' \in\}.$$

Clearly, $\{x_1, x_2, \dots, x_n\} \subseteq C_S(\boldsymbol{x})$. Therefore if $C_S(\boldsymbol{x})$ is one element set, then $\boldsymbol{x}$ has to be of the form $\boldsymbol{x} = (x, x, \dots, x)$.

THEOREM 1. 1. *Let S be a monoid on a set X and $n \in \mathcal{N}$. Then $f(\boldsymbol{x}) \in C_S(\boldsymbol{x})$ for every $f \in \mathcal{F}_{S,n}$ and $\boldsymbol{x} \in X^n$.*

2. *If X^n is the sum of disjoint sets $S\boldsymbol{x}^t$, $t \in T$, then for every $y \in C_S(\boldsymbol{x})$ and $1 \le i \le n$, there exists $f_i \in \mathcal{F}_{S,n}$ such that $f_i(\boldsymbol{x}^t) = y$ and $f_i(\boldsymbol{z}) = z_i$ for $\boldsymbol{z} \in \boldsymbol{x}^t$.*

Proof. If $y = f(\boldsymbol{x})$ for some $f \in \mathcal{F}_{S,n}$ and $s\boldsymbol{x} = s'\boldsymbol{x}$ for $s, s' \in S$, then we have $sy = sf(\boldsymbol{x}) = f(s\boldsymbol{x}) = f(s'\boldsymbol{x}) = s'f(\boldsymbol{x}) = s'y$. Thus $y \in C_S(\boldsymbol{x})$.

To prove second part let us suppose that $t \in T$ and $1 \le i \le n$. We put

$$f_i(\boldsymbol{z}) = \begin{cases} s_0 y, & \text{if } \boldsymbol{z} = s_0\boldsymbol{x}^t, s_0 \in S; \\ z_i & \text{otherwise.} \end{cases}$$

It follows from the definition of $C_S(\boldsymbol{x}^x)$ and the hypothesis that f_i is well defined function of X^n into X. We show that f_i commutes with all $s \in S$. Indeed, if $\boldsymbol{z} = s_0\boldsymbol{x}^t$, then $s\boldsymbol{z} = ss_0\boldsymbol{x}^t_x$. Thus $f_i(s\boldsymbol{z}) = ss_0y = sf_i(\boldsymbol{z})$. If $\boldsymbol{z} \notin S\boldsymbol{x}^t$, then an element $s\boldsymbol{z}$ is also off the set $S\boldsymbol{x}^t$ and therefore $f_i(s\boldsymbol{z}) = sz_i = sf_i(\boldsymbol{z})$, as required.

THEOREM 2. *Let n be a natural number and let S be a semigroup of transformations of a set X which satisfies the condition $\varepsilon_n : X^n$ is the sum of pairwise disjoint sets $S\boldsymbol{x}^t$, $t \in T$, where $\boldsymbol{x}^t = (x_1^t, x_2^t, \dots, x_n^t) \in X^n$. Then there exists an n-algebra $\mathfrak{A}$ such that $S = \operatorname{End}\mathfrak{A}$ if and only if each transformation $\varphi \in X^X$ belongs to S provided it satisfies the following condition*

(∗) *for all* $s \in S$ *and* $t \in T$ *either* φs *agrees with some* $s_0 \in S$ *on the set* $C_S(\boldsymbol{x}^t)$, *or* $\varphi s\boldsymbol{x}^t = s_0\boldsymbol{x}^{t'}$ *for some* $t' \in T$, $t \neq t'$ *and* $\varphi s C_S(\boldsymbol{x}^t) = \{\varphi s x_1^t\} = s_0 C_S(\boldsymbol{x}^{t'})$.

Proof. By Proposition 1, $\phi \in X^X$ is an endomorphism of an n-algebra $\mathfrak{A}$ with $S = \operatorname{End}\mathfrak{A}$ if and only if ϕ is an endomorphism of the algebra $(X;\mathcal{F}_{S,n})$, and therefore, it suffices to show that $\phi \in \operatorname{End}(X;\mathcal{F}_{S,n})$ if and only if ϕ satisfies (∗). Let $\phi \in \operatorname{End}(X;\mathcal{F}_{S,n})$, $t \in T$, $s \in S$. It follows from our hypothesis that $\varphi s\boldsymbol{x}^t = s_0\boldsymbol{x}^{t'}$ for some $s_0 \in S$ and $t' \in T$.

If $t = t'$ we claim that φs agrees with some s_0 on the set $C_S(\boldsymbol{x}^t)$. If $y \in C_S(\boldsymbol{x}^t)$, then, by Theorem 1, for every i, $1 \leq i \leq n$, there is a function $f_i \in \mathcal{F}_{S,n}$, such that $f_i(\boldsymbol{x}^t) = y$. Hence we get

$$\varphi s y = \varphi s f_i(\boldsymbol{x}^t) = f_i(\varphi s\boldsymbol{x}^t) = f_i(s_0\boldsymbol{x}^t) = s_0 f_i(\boldsymbol{x}^t) = s_0 y.$$

If $\varphi s\boldsymbol{x}^t = s_0\boldsymbol{x}^{t'}$, where $t \neq t'$ then, by Theorem 1 for each $y \in C_S(\boldsymbol{x}^t)$ and $1 \leq i \leq n$ there is a function $f_i \in \mathrm{F}_{S,n}$ with $f(\boldsymbol{x}^t) = y$ and $f(\varphi s\boldsymbol{x}^t) = \varphi s x_i^t$. This yields

$$\varphi s y = \varphi s f_i(\boldsymbol{x}^t) = f_i(\varphi s\boldsymbol{x}^t) = \varphi s x_i^t.$$

Thus $\varphi s C_S(\mathrm{x}^t) = \{\varphi s x_1^t\}$. It follows again from the previous theorem that for every $y \in C_S(\boldsymbol{x}^{t'})$, there is $f_1 \in \mathrm{F}_{S,n}$ with $f_1(\boldsymbol{x}^{t'}) = y$ and $f_1(\boldsymbol{z}) = z_1$ for $\boldsymbol{z}$ off $S\boldsymbol{x}^{t'}$. Hence

$$s_0 y = s_0 f_1(\boldsymbol{x}^{t'}) = f_1(s_0\boldsymbol{x}^{t'}) = f_1(\varphi s\boldsymbol{x}^t) = \varphi s f_1(\boldsymbol{x}^t) = \varphi s x_1^t.$$

In order to prove the converse implication let us suppose that φ is a transformation of X which satisfies the condition. We show that φ is an endomorphism of the algebra $(X;\mathcal{F}_{S,n})$. In fact, let f be an element of $\mathcal{F}_{S,n}$ and $\boldsymbol{z} = s\boldsymbol{x}^t$ for some $s \in S$, $t \in T$. Suppose that φs agrees with some $s_0 \in S$ on the set $C_S(\boldsymbol{x}^t)$. Since $f(\boldsymbol{x}^t) \in C_S(\boldsymbol{x}^t)$, we infer

$$\varphi f(\boldsymbol{z}) = \varphi f(s\boldsymbol{x}^t) = \varphi s f(\boldsymbol{x}^t) = s_0 f(\boldsymbol{x}^t) = f(s_0\boldsymbol{x}^t) = f(\varphi s\boldsymbol{x}^t) = f(\varphi\boldsymbol{z}).$$

Finally, if $\varphi s\boldsymbol{x}^t = s_0\boldsymbol{x}^{t'}$ for some $t' \in T$, $t \neq t'$, then we get $\varphi f(\boldsymbol{z}) = \varphi f(s\boldsymbol{x}^t) = \varphi s f(\boldsymbol{x}^t) = \varphi s x_1^t$, and also $f(\varphi\boldsymbol{z}) = f(\varphi s\boldsymbol{x}^t) = f(s_0\boldsymbol{x}^{t'}) = s_0 f(\boldsymbol{x}^{t'}) = \varphi s x_1^t$, which completes the proof.

This together with Proposition 1 yields

THEOREM 3. *Let* S *be a monoid of transformation of a set* X, *which satisfies the condition* (β_n) *for all* $n \in \mathcal{N}$. *Then* $S = \operatorname{End}(X;\mathcal{F})$ *for some algebra* $(X;\mathcal{F})$ *if and only if each element of* $\varphi \in X^X$ *which satisfies the condition*

(**) *for each natural* n, $s \in S$ *and* $t_n \in T_n$ *either* φs *agrees with some* $s_0 \in S$ *on the set* $C_S(xx^{t_n})$, *or*

$\varphi s x^{t_n} = s_0 x^{t'_n}$ *for some* $t'_n \in T_n$, $t_n \neq t'_n$ *and* $\varphi s C_S(x^{t_n}) = \{\varphi s x_1^{t_n}\} = s_0 C_S(\varphi x^{t'_n})$ *belongs to* S.

EXAMPLE 6. Consider the semigroup S of all injection of a infinite set X into itself. If a transformation φ is not in S, then $\varphi(x) = \varphi(x')$ for some $x \neq x'$. Thus $\varphi(x, x')$ is not in $S(x, x')$ and consequently $\beta_3(S)$ holds. Clearly, $C_S(y_1, y_2, \ldots, y_n) = \{y_1, y_2, \ldots y_n\}$ for all $(y_1, y_2, \ldots, y_n) \in X^n$. Let $\varphi(x) = x_1$ for all $x \in X$. We show that the mapping φ satisfies (**). Indeed, let $n \in \mathcal{N}, s \in S$. Then we have $\varphi s = Id_X$ on the set $\{x_1\} = C_S(x_1, x_1, \ldots, x_1)$. Now if $y_1, y_2, \ldots, y_n \in \{x_1, x_2, \ldots\}$, and $y_i \neq y_j$ for some $i \neq j$, then for all $s \in S$ we have $\varphi s(y_1, y_2, \ldots y_n) = (x_1, x_1, \ldots x_1) = Id_X(x_1, x_1, \ldots x_1)$, and also $\varphi s C_S(y_1, y_2, \ldots, y_n) = \{\varphi s x_1\} = \{x_1\} = Id_X(C_S(\varphi x_1, \varphi x_1, \ldots, \varphi x_1))$. Since $\varphi \notin S$, the condition of Theorem 2 is not satisfy and therefore there is no algebra for which S is the endomorphism semigroup.

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