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SEMIRING-VALUED KORCZYŃSKI NETS

Abstract. Korczyński [4]–[6] generalized the notion of a Petri net by introducing the notion of an **e-net**. The purpose of this generalization, based on a nonstandard definition of a directed graph, was to allow a wider class of morphisms, including those which send edges to vertices. Around the same time, Golan [2] introduced the notion of a semiring-valued Petri net in order to present a common framework which includes classical Petri nets, colored Petri nets, fuzzy Petri nets, etc. In this note we use the same techniques for presenting semiring-valued Korczyński nets, which contain semiring-valued Petri nets as a special case.

1. Notation and terminology

For nonempty sets X and Y, we will denote the set of all functions from X to Y by Y^X. Functions in Y^X are always written as acting on the left. Any function $\alpha: X \to X$ can be extended to a function $Y^\alpha: Y^X \to Y^X$ by setting $Y^\alpha: g \mapsto g\alpha$. Note that if $\alpha, \beta: X \to X$ then $Y^\alpha Y^\beta = Y^{\beta\alpha}$. A Y-**valued relation** on X is an element of $Y^{X\times X}$.

There is a bijective correspondence between the family of all subsets of a set X and $\mathbb{B}^X$, where $\mathbb{B} = \{0, 1\}$, through which a subset A of X corresponds to its **characteristic function** h_A, defined by

$$h_A : x \mapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if $\alpha : X \to X$ then $\mathbb{B}^\alpha : h_A \mapsto h_{\alpha^{-1}(A)}$.

A **semiring** R is a nonempty set on which we have operations of addition and multiplication defined such that the following conditions are satisfied:

(1) $(R, +)$ is a commutative monoid with identity element 0;
(2) $(R, \cdot)$ is a monoid with identity element $1 \neq 0$;
(3) Multiplication distributes over addition from either side;
(4) $0r = 0 = r0$ for all $r \in R$.

For information and background on semirings, refer to [1, 3]. A semiring R is **difference ordered** if and only if the relation $\preceq$ on R defined by the condition $r \preceq r'$ if and only if there exists an element $r'' \in R$ satisfying $r+r''+r'$, is a partial order on R. It is **uniquely difference ordered** if the element r'' is always unique. Thus, for example, the semiring $(\mathbb{B}, \max, \min)$ is a difference-ordered semiring in which we take addition to be max and multiplication to be min. It is not uniquely difference ordered. Examples of uniquely difference-ordered semirings include the semiring $(\mathbb{N}, +, \cdot)$, where $\mathbb{N}$ denotes the set of nonnegative integers, and the semiring $(\mathbb{R}^+, +, \cdot)$, where $\mathbb{R}^+$ denotes the set of all nonnegative real numbers. Wu [7] has defined an intermediate notion: a semiring R is **weakly uniquely difference ordered** if and only if $r \preceq r'$ when and only when $r = r'$ or $r \neq r'$ and there exists an element $0 \neq r'' \in R$ satisfying $r + r'' = r'$. The semiring $(\mathbb{B}, \max, \min)$ is in fact weakly uniquely difference ordered. Other weakly uniquely difference-ordered semirings include the semiring $(\mathbb{I}, \max, \sqcap)$, where $\mathbb{I}$ is the unit interval on the real line and $\sqcap$ is any triangular norm defined on $\mathbb{I}$, the semiring $(\mathbb{I}, \min, \sqcup)$, where $\sqcup$ is any triangular conorm defined on $\mathbb{I}$, and the semiring $(\mathbb{R}^+ \cup \{-\infty\}, \max, +)$.

A semiring R is **zerosumfree** if and only if $r+r'=0$ only when $r=r'=0$ and it is **entire** if and only if $rr' \neq 0$ when $r \neq 0$ and $r' \neq 0$. Difference-ordered semirings are zerosumfree. Indeed, if R is difference-ordered and if $r + r' = 0$ then $0 \succeq r \succeq 0$ and so $r = 0$. Similarly $r' = 0$. A sufficient condition for a semiring R to be entire is that it be a **division semiring**, namely that for each $0 \neq r \in R$ there exists an element $r^{-1} \in R$ satisfying $1 = rr^{-1} = r^{-1}r$. Every zerosumfree division semiring is difference ordered; see Proposition 20.30 of [3].

Let R be a semiring and let X be a set. If $f, g \in R^X$ we can the define the elementwise sum and product of f and g by

$$f + g : x \mapsto f(x) + g(x)$$

and

$$fg : x \mapsto f(x)g(x).$$

Also, we define the function $f \neg g$ by setting

$$f \neg g : x \mapsto \begin{cases} f(x) & \text{if } g(x) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

For each $r \in R$ we have the constant R-valued function $c_r \in R^X$ defined by $c_r : x \mapsto r$.

If $f, g \in R^X$ satisfy $f(x) \preceq g(x)$ for all $x \in X$ then we write $f \preceq g$. If R is a weakly uniquely difference-ordered semiring and X be a nonempty set, and if $f, g \in R^X$ satisfy $f \preceq g$, then we define the function $g - f \in R^X$ by setting $(g - f)(x) = 0$ if $g(x) = f(x)$ and otherwise $(g - f)(x) = r$, where r is the unique element of R satisfying $f(x) + r = g(x)$.

The **support** of an R-valued function $f \in R^X$ is

$$\operatorname{supp}(f) = \{x \in X \mid f(x) \neq 0\}.$$

Let R be a semiring and let X be a nonempty set. If $f \in R^{X \times X}$ is an R-valued relation on X and if $x \in X$, we set

$$f(x, \bullet) = \{x' \in X \mid f(x, x') \neq 0\}$$

and

$$f(\bullet, x) = \{x' \in X \mid f(x', x) \neq 0\}.$$

The **domain** of an R-valued relation f on X is $\operatorname{dom}(f) = \cup_{x \in X} f(\bullet, x)$. In particular, $\operatorname{dom}(f) = X$ if and only if for each $x' \in X$ there exists an $x \in X$ such that $f(x', x) \neq 0$.

For any nonempty subset Y of X we always have the diagonal R-valued relation i_Y defined by

$$i_Y : (x, x') \mapsto \begin{cases} 1 & \text{if } x = x' \in Y \\ 0 & \text{otherwise.} \end{cases}$$

If f and g are R-valued relations on a finite set X we define

$$f \circ g : (x, x') \mapsto \sum_{x'' \in X} f(x, x'')g(x'', x').$$

Note that if R is entire and zerosumfree, if $f, g \in R^{X \times X}$, and if $x \in X$ then

$$\begin{aligned} (f \circ g)(x, \bullet) &= \{x' \in X \mid (f \circ g)(x, x') \neq 0\} \\ &= \left\{x' \in X \,\middle|\, \sum_{x'' \in X} f(x, x'')g(x'', x') \neq 0\right\} \\ &= \{x' \in X \mid f(x, x'') \neq 0 \neq g(x'', x') \text{ for some } x'' \in X\} \\ &= \bigcup_{x'' \in f(x, \bullet)} g(x'', \bullet). \end{aligned}$$

2. Semiring-valued Korczyński nets

We now define a semiring-valued version of the notion of an e-net, studied by Korczyński [4, 5, 6]. Let R be a semiring. An R-**valued Korczyński net** on a finite set X is a pair $\Delta = (f, g)$ of R-valued relations on X satisfying the following conditions:

(K1) $\operatorname{dom}(f) = \operatorname{dom}(g) = X$;
(K2) $f \circ g = f \circ f = f$;
(K3) $g \circ f = g \circ g = g$;
(K4) $f \circ (f \neg i_X) = c_0$;
(K5) $g \circ (g \neg i_X) = c_0$.

The case studied by Korczyński is obtained by taking $R = \mathbb{B}$.

Since R is entire and zerosumfree, these last two conditions can be rephrased as follows:

(K4′) If $x \neq x'$ in X then for each $x'' \in X$ we have $f(x,x'') = 0$ or $f(x'',x') = 0$.

(K5′) If $x \neq x'$ in X then for each $x'' \in X$ we have $g(x,x'') = 0$ or $g(x'',x') = 0$.

PROPOSITION 2.1. *Let R be an entire zerosumfree semiring and let $\Delta = (f,g)$ be an R-valued Korczyński net on a finite set X. If $y \in X$ then:*

(1) $x \in (f \circ f)(y,\bullet) \Rightarrow f(x,\bullet) = g(x,\bullet) = \{x\}$;
(2) $x \in (g \circ g)(y,\bullet) \Rightarrow f(x,\bullet) = g(x,\bullet) = \{x\}$;
(3) $x \in (f \circ g)(y,\bullet) \Rightarrow f(x,\bullet) = g(x,\bullet) = \{x\}$;
(4) $x \in (g \circ f)(y,\bullet) \Rightarrow f(x,\bullet) = g(x,\bullet) = \{x\}$.

Proof. (1) If $x \in (f \circ f)(y,\bullet)$ then there exists an element $z \in X$ satisfying $f(y,z) \neq 0 \neq f(z,x)$. Moreover, $f = f \circ f$ and so $f(y,x) \neq 0$. First let us consider the case $y = x$. Then we know that $f(x,x) \neq 0$ so $x \in f(x,\bullet)$. Conversely, suppose that $x' \in f(x,\bullet)$ for some $x' \neq x$. Then $f(x,x)f(x,x') \neq 0$ and so $0 \neq [f \circ (f \neg i_X)](x,x')$, which is a contradiction. Therefore $f(x,\bullet) = \{x\}$. Now suppose that $y \neq x$. If $x' \in f(x,\bullet)$ for some $x' \neq x$ then $f(y,x)f(x,x') \neq 0$ and so $[f \circ (f \neg i_X)](y,x') \neq 0$, which is a contradiction. Moreover, we know by hypothesis that $f(z,x) \neq 0$. If $z \neq x$ then $f(y,x) = f(y,z)f(z,x) \neq 0$ and so $[f \circ (f \neg i_X)](y,x) \neq 0$, which is a contradiction. Therefore we must have $z = x$ and so $f(x,x) \neq 0$. Thus $f(x,\bullet) = \{x\}$ in this case as well.

Since $f \circ f = f \circ g$ we know that there exists a $z \in X$ such that $f(y,z)g(z,x) \neq 0$. First let us consider the case $z = x$. Then surely $x \in g(x,\bullet)$. If $g(x,x') \neq 0$ for $x \neq x'$ then $0 \neq g(x,x)g(x,x') = [g \circ (g \neg i_X)](x,x')$, which is a contradiction. Thus $g(x,\bullet) = \{x\}$. Now assume that $z \neq x$. If $g(x,x') \neq 0$ for $x' \neq x$ then $0 \neq g(z,x)g(x,x') = [g \circ (g \neg i_X)](z,x')$, which is a contradiction. On the other hand,

$$0 \neq f(x,x) = (f \circ g)(x,x) = \sum_{x' \in X} f(x,x')g(x',x) = f(x,x)g(x,x)$$

and so $g(x,x) \neq 0$. Thus $g(x,\bullet) = \{x\}$.

(2)–(4): These are proven similarly. ∎

PROPOSITION 2.2. *Let R be an entire zerosumfree semiring and let $\Delta = (f,g)$ be an R-valued Korczyński net on a finite set X. For $x, y \in X$ the following conditions are equivalent:*

(1) $y \in f(x,\bullet) \cup g(x,\bullet)$;
(2) $f(y,\bullet) = g(y,\bullet) = \{y\}$.

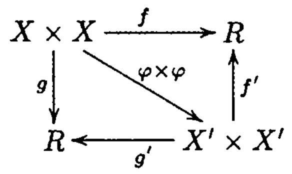
P r o o f. This is a direct consequence of Proposition 2.1 and the fact that $f \circ f = f$ and $g \circ g = g$. ∎

PROPOSITION 2.3. *Let R be an entire zerosumfree semiring and let $\Delta = (f, g)$ be an R-valued Korczyński net on a finite set X. For $x \in X$ the following conditions are equivalent:*

(1) *There exists an element $y \neq x$ in $f(x, \bullet) \cup g(x, \bullet)$;*
(2) $f(x, \bullet) \neq \{x\} \neq g(x, \bullet)$.

P r o o f. Clearly (2) implies (1). Assume (1) and let $y \in f(x, \bullet)$. Then surely $f(x, \bullet) \neq \{x\}$. If $g(x, \bullet) = \{x\}$ then $g(x, x)f(x, y) \neq 0$ so $g(x, y) = (g \circ f)(x, y) \neq 0$, which is a contradiction. Thus we have (2). If $y \in g(x, \bullet)$ the proof is similar. ∎

Let $\Delta = (f, g)$ be an R-valued Korczyński net on a finite set X and let $\Delta' = (f', g')$ be an R-valued Korczyński net on a finite set X'. A **morphism** from Δ to Δ' is a function $\varphi : X \to X'$ such that the diagram



commutes.

3. Places and transitions

Let R be a semiring and let Δ be an R-valued Korczyński net on a finite set X. An element $x \in X$ is a **Δ-place** if and only if $f(x, \bullet) = \{x\}$. By Proposition 2.3, this is equivalent to the condition that $g(x, \bullet) = \{x\}$. Note that if x is a place then

$$f(x, x) = \sum_{x' \in X} f(x, x')f(x', x) = f(x, x)^2$$

and

$$f(x, x) = \sum_{x' \in X} f(x, x')g(x', x) = f(x, x)g(x, x).$$

similarly $g(x, x) = g(x, x)^2 = g(x, x)f(x, x)$.

Also note, that as a consequence of (3) and (4), a sufficient condition for $x \in X$ to be a place is that there exist an element $x' \neq x$ in X satisfying $f(x', x) \neq 0$ or $g(x', x) \neq 0$.

Denote the set of all places of Δ by $\mathrm{pl}(\Delta)$. If $x \in X$ is not a Δ-place then it is a **Δ-transition**. We denote the set of all transitions of Δ by $\mathrm{tr}(\Delta)$. We also set $\bar{f} = f \neg id_{\mathrm{pl}(\Delta)}$ and $\bar{g} = g \neg id_{\mathrm{pl}(\Delta)}$.

PROPOSITION 3.1. *Let R be an entire zerosumfree semiring and let $\Delta = (f, g)$ be an R-valued Korczyński net on a finite set X. Then*

(1) $\mathrm{supp}(f) \cup \mathrm{supp}(g) \subseteq X \times \mathrm{pl}(\Delta)$.
(2) $\mathrm{dom}(\bar{f}) = \mathrm{tr}(\Delta) = \mathrm{dom}(\bar{g})$.

Proof. (1) Note that $(x, y) \in \mathrm{supp}(f) \Leftrightarrow y \in f(x, \bullet) \Leftrightarrow f(y, \bullet) = \{y\} \Rightarrow y \in \mathrm{pl}(\Delta)$. The case of $\mathrm{supp}(g)$ is similar.

(2) If $x \in \mathrm{tr}(\Delta)$ then $x \in \mathrm{dom}(f)$ by definition and $x \notin \mathrm{pl}(\Delta)$ so $x \in \mathrm{dom}(\bar{f})$. Conversely, assume that $x \in \mathrm{dom}(\bar{f})$. Then there exists an element y of X such that $(x, y) \in \mathrm{supp}(f)$, and either $x \neq y$ or $x \notin \mathrm{pl}(\Delta)$. In the second case, $x \in \mathrm{tr}(\Delta)$ and we are done. Assume therefore that $x \neq y$ and $x \in \mathrm{pl}(\Delta)$. Then $x \neq y \in f(x, \bullet)$ and $f(x, \bullet) = \{x\}$, which is a contradiction. Thus $x \in \mathrm{tr}(\Delta)$. The proof of the second inequality is similar. ∎

In particular, we see that if $\Delta = (f, g)$ is an R-valued Korczyński net on a finite set X then $\mathrm{supp}(\bar{f}) \cup \mathrm{supp}(\bar{g}) \subseteq \mathrm{tr}(\Delta) \times \mathrm{pl}(\Delta)$.

4. Markings

When one describes the action of systems using Petri nets, the underlying net describes the environment, or hardware, of the system, while the dynamics of the system is described by markings on the net, which are altered by firing various transitions when allowable. A similar situation holds for Korczyński nets.

Let R be a semiring and let $\Delta = (f, g)$ be an R-valued Korczyński net on a finite set X. A **marking** on Δ with values in R is a function $d \in R^{\mathrm{pl}(\Delta)}$. Every transition $t \in \mathrm{tr}(\Delta)$ defines two such markings, namely

$$\mu_t : p \mapsto \bar{f}(t, p) \text{ and } \nu_t : p \mapsto \bar{g}(p, t).$$

Similarly, a **guard** on Δ with values in R is a function $e \in R^{\mathrm{tr}(\Delta)}$. Every place $p \in \mathrm{pl}(\Delta)$ defines two such guards, namely

$$\mu_p : t \mapsto \bar{f}(t, p) \text{ and } \nu_p : t \mapsto \bar{g}(t, p).$$

If $d \in R^{\mathrm{pl}(\Delta)}$ is a marking on Δ with values in R and if $t \in \mathrm{tr}(\Delta)$ then we say that Δ can be **fired** at t if and only if $f \succeq \mu_t$. In this case, there exists a marking $d'' \in R^{\mathrm{pl}(\Delta)}$ satisfying $d = d'' + \mu_t$ and so the marking $d' = d'' + \nu_t$ satisfies the condition

$$(*) \qquad d + \nu_t = d' + \mu_t.$$

In this case we write $d\ [R|t\rangle\ d'$ and say that d' is a marking **obtained from** d as a result of firing the net at t. One problem which we may encounter while working over an arbitrary zerosumfree semring is that the marking d'' above, and hence the marking d', need not be unique. This cannot happen if R is a weakly uniquely difference-ordered semiring, for in that case we must have $d'' = d - \mu_t$. Otherwise, we may need some outside criterion to allow us to decide which of possibly many values to prefer.

Similarly, if $e \in R^{\mathrm{tr}(\Delta)}$ is a guard with values in R and if $p \in \mathrm{pl}(\Delta)$ then we say that Δ can be **activated** at p to obtain a new guard e' if and only if $e \succeq \nu_p$, an we write $e\ [R|p\rangle\ e'$, where e' is aguard satisfying $e' + \nu_p = e + \mu_p$. Again, if R is a weakly uniquely difference-ordered semiring we select $e'' = e - \mu_p$, but otherwise we may need some outside criterion to allow us to decide which of possibly many values of e' to prefer.

If a marking d' is obtained from a marking d on Δ with values in R by successive firings of a sequence $w = t_1, \ldots, t_n$ of (not necessarily distinct) transitions in $\mathrm{tr}(\Delta)$, we write $d\ [R|w\rangle\ d'$. Thus every marking $d \in R^{\mathrm{pl}(\Delta)}$ defines a subset $L(d)$ of the free monoid $\mathrm{tr}(\Delta)^*$ of all finite sequences of elements of $\mathrm{tr}(\Delta)$, given by the condition that $w \in L(d)$ if and only if there exists a marking $d' \in R^{\mathrm{pl}(\Delta)}$ satisfying $d\ [R|t\rangle\ d'$. The set $L(d)$ is called the **formal language** defined by d. Similarly, if a guard e' is obtained from a guard e with values in R by successive firings of a sequence $y = p_1, \ldots, p_n$ of (not necessarily distinct) places in $\mathrm{pl}(\Delta)$, we write $e\ [R|y\rangle\ e'$. Thus every guard $e \in R^{\mathrm{tr}(\Delta)}$ defines a subset $M(e)$ of the free monoid $\mathrm{pl}(\Delta)^*$ given by the condition that $y \in M(e)$ if and only if there exists a guard e' such that $e\ [R|p\rangle\ e'$. The set $M(e)$ is the **formal language** defined by e.

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