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COMMUTATIVE BCK-ALGEBRAS WITH PRODUCT

Abstract. We introduce a product on commutative BCK-algebras with the relative cancellation property, i.e., commutative BCK-algebras $(X;*,0)$ satisfying the condition for $x,y,a \in X$, with $a \leq x$, $a \leq y$ and $x*a = y*a$ we have $x = y$. The product is left and right distributive with respect to the partial operation $+$ derived from the BCK-operation $*$. We show that the category of product BCK-algebras is categorically equivalent to the category of ℓ-rings with special properties. Moreover, we study $\cdot$-ideals and we introduce BCKf-algebras.

1. Introduction

BCK-algebras entered mathematics in 1966 due to Imai and Iséki [ImIs] and they met interest of mathematicians, logicians, algebraists, experts in fuzzy sets as well as in quantum structures [Cor], [CST], [Pal], [RoPa], [DvKi]. Recently Dvurečenskij and Graziano [DvGr] introduced a family of commutative BCK-algebras, commutative BCK-algebras with the relative cancellation property, i.e., for $x,y,a \in X$, with $a \leq x$, $a \leq y$ and $x*a = y*a$ we have $x = y$. MV-algebras introduced by Chang [Cha] form its proper subfamily.

An important representation of commutative BCK-algebras with the relative cancellation property was found in [DvGr 1], where we have shown that they may be represented as BCK-subalgebras of the positive cone of an Abelian ℓ-group. We recall that Bosbach [Bos] has proved an embedding of semiclans into the positive ℓ-group; our commutative BCK-algebra with the relative cancellation property can be converted into a semiclan. However, we have shown that there is a categorical equivalence between the category

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of commutative BCK-algebras with the relative cancellation property and the category of Abelian ℓ-groups with some additional properties, [Dvu 1].

For example, the system $X = [0,1]^{\Omega}$ of all fuzzy sets on $\Omega \neq \emptyset$ has a natural BCK-operation $*$ defined by $(f * g)(\omega) := \max\{0, f(\omega) - g(\omega)\}$, $\omega \in \Omega$. The partial sum $f + g$ is defined in X iff $f \leq 1 - g$, and X admits a natural multiplication $f \cdot g$ which is a total operation and is left and right distributive with respect to the addition $+$.

A product on MV-algebras was introduced in [DvDi]. Other attempts to introduce a product can be found also in [DvRi], [Rie], [BDG], and [DiGe]. In the last two papers, the product is defined only on their radicals.

In the present paper, we introduce a product on commutative BCK-algebras with the relative cancellation property as a (total) binary operation $\cdot$ defined on the BCK-algebra which is left and right distributive with respect to a partial operation $+$ derived from the BCK-operation $*$. We show that product BCK-algebras became from Abelian ℓ-rings, and the category of product BCK-algebras is categorically equivalent to the category of Abelian ℓ-rings with special properties. In addition, we introduce and study BCKf-algebras

2. Commutative BCK-algebras

A BCK-algebra is a non-empty set X with a binary operation $*$ and with a constant element 0 such that the following axioms are satisfied: for all x, y, $z \in X$,

(BCK-1) $((x * y) * (x * z)) * (z * y) = 0$;
(BCK-2) $(x * (x * y)) * y = 0$;
(BCK-3) $x * x = 0$;
(BCK-4) $x * y = 0$ and $y * x = 0$ imply $x = y$;
(BCK-5) $0 * x = 0$.

In every BCK-algebra $X = (X; *, 0)$ we can define a partial order $\leq$ via $x \leq y$ iff $x * y = 0$; then X is a poset with the least element 0. A BCK-algebra $(X; *, 0)$ is said to be *commutative* if

$$x * (x * y) = y * (y * x), \quad x, y \in X,$$

and in this case, $x \wedge y = x * (x * y)$, where $\wedge$ is a g.l.b.

A BCK-algebra $(X; *, 0)$ is *bounded* if there is a greatest element 1 in X. The class of bounded commutative BCK-algebras is categorically equivalent to the class of MV-algebras, [FRT].

Let $(X; *, 0)$ and $(X_1; *_1, 0_1)$ be two BCK-algebras. A mapping $f : X \to X_1$ such that $f(x * y) = f(x) *_1 f(y)$, $x, y \in X$, is said to be a *BCK-homomorphism*; it is evident that $f(0) = 0$. If f is injective, f is said to be a *BCK-embedding*; if f is injective and surjective, f is said to be a *BCK-isomorphism.* It is evident that if X and X_1 are commutative BCK-algebras, then any BCK-homomorphism from X into X_1 preserves meets from X.

According to [DvGr], we say that a commutative BCK-algebra $(X;*,0)$ has the *relative cancellation property* if, for $a,x,y \in X$, $a \leq x,y$ with $x*a = y*a$ imply $x=y$. In this case we can introduce a partial binary operation $+$ on X as follows: $a+b$ is defined in X and equals c iff $c \geq a$ and $b = c*a$. For the basic properties of $+$ see [DvGr, DvGr 1]; we recall only that $+$ is commutative, associative, cancellative and has a neutral element 0.

In [DvGr], we have proved that any upwards directed BCK-algebra (i.e., given $x,y \in X$ there exists $z \in X$ with $x,y \leq z$), and in particular, any bounded commutative BCK-algebra, has the relative cancellation property.

For example, $([0,\infty);*_{\mathbb{R}},0)$, where

$$s *_{\mathbb{R}} t = \max\{0, s-t\},$$

$s,t \in [0,\infty)$, ($\mathbb{R}$ stands for real numbers) is an example of a commutative BCK-algebra with the relative cancellation property.

EXAMPLE 2.1. Suppose that $(G;+,\leq,0)$ is an Abelian ℓ-group with the positive cone $G^+ = \{g \in G : g \geq 0\}$. Then $(G^+;*_G,0)$ is a commutative BCK-algebra with the relative cancellation property, where $*_G$ is defined via

$$u *_G v := (u-v) \vee 0, \tag{2.1}$$

for $u,v \in G^+$. More generally, if G_0 is a non-void subset of G^+ such $u *_G v \in G_0$ for $u,v \in G_0$, then $(G_0;*_G,0)$ is a commutative BCK-subalgebra of $(G^+;*_G,0)$ having the relative cancellation property.

The later example is in some sense an archetype of commutative BCK-algebras with the relative cancellation property because of the following basic representation theorem for commutative BCK-algebras proved in [DvGr 1, Thm 6.4]:

THEOREM 2.2. *Let $(X;*,0)$ be a commutative BCK-algebra with the relative cancellation property. Then there exists an Abelian ℓ-group $(G;+,\leq,0)$ and a non-void subset G_0 of the positive cone G^+, G_0 generates G^+,*[1] *such that, for any $u,v \in G_0$, $u *_G y \in G_0$, and there exists a BCK-isomorphism h from X onto G_0.*

We recall that an ideal of a commutative BCK-algebra $(X;*,0)$ is a non-empty subset I of X such that (i) $0 \in I$, and (ii) $x*y \in I$ and $y \in I$ entail $x \in I$.

An ideal I of X is said to be *maximal* if it is a proper ideal of X and if it is not contained in any other proper ideal of X. We denote by $\mathcal{M}(X)$ the

[1] I.e., for any $g \in G^+$, there exist $g_1,\dots,g_n \in G_0$ with $g = g_1 + \dots + g_n$.

set of all maximal and prime ideals of X. We recall that it can happen that $\mathcal{M}(X) = \emptyset$.

Define recursively, for all $x, y \in X$:

$$x *^0 y = x, \; x *^1 y = x * y, \ldots, \; x *^{n+1} y = (x *^n y) * y, \; n \geq 1.$$

An element u of X is said a quasi strong unit for X if, for any $x \in X$, there exists an integer $n \geq 1$ such that $x *^n u = u$. If X possesses a quasi strong unit, then $\mathcal{M}(X) \neq \emptyset$.

If $\mathcal{M}(X) \neq \emptyset$, then the set $\mathrm{Rad}(X) := \bigcap\{M : M \in \mathcal{M}(X)\}$ is said to be a *radical* of X. The radical carries an important part of the propositional system. For example, if $\mathrm{Rad}(X) = \{0\}$, X is said to be *semisimple*, and in this case X can be represented by functions or even by fuzzy sets, [Dvu].

Let $n \geq 1$ be an integer and $a \in X$. If $a_1 + \ldots + a_n$ is defined in X, where $a_i = a$ for $i = 1, \ldots, n$, then $na := a_1 + \ldots + a_n$. An element x is said to be *infinitesimal* if nx is defined in X for any $n \geq 1$. The set of all infinitesimal elements in X will be denoted by $Infinit(X)$.

In [Dvu 2], we have proved that if X possesses a quasi strong unit, then

$$\mathrm{Rad}(X) = \mathrm{Rad}(X_u) = \mathrm{Infinit}(X_u), \tag{2.2}$$

where $X_u := \{x \in X : x \leq u\}$. In addition, if $x, y \in \mathrm{Rad}(X)$, then $x + y$ is defined in X and $x + y \in \mathrm{Rad}(X)$.

We denote by $\mathcal{BCK}$ the category whose objects are commutative BCK-algebras and morphisms are BCK-homomorphisms.

Let G_1 and G_2 be two Abelian ℓ-groups. A mapping $h : G_1 \to G_2$ is said to be an *ℓ-group homomorphism* iff h is both a group-homomorphism and a lattice-homomorphism. In other words, for each $a, b \in G_1$, $h(a+b) = h(a) + h(b)$, $h(a \wedge b) = h(a) \wedge h(b)$ (as well as for joins).

We denote by $\mathcal{LG}$ the category whose objects are pairs (G, G_0), where G is an Abelian ℓ-group and G_0 is a non-void subset of the positive cone G^+ of G such that G_0 generates G^+ and $(G_0; *_G, 0)$ is a BCK-algebra (in fact a BCK-subalgebra of $(G^+; *_G, 0)$, see Example 2.1). A morphism from (G, G_0) into (G', G'_0) is an ℓ-group homomorphism $h : G \to G'$ such that $h(G_0) \subseteq G'_0$.

Now let (G, G_0) be an object of $\mathcal{LG}$ and define a morphism $\mathcal{X}$ from the category $\mathcal{LG}$ into the category $\mathcal{BCK}$ as follows

$$\mathcal{X}(G, G_0) = (G_0; *_G, 0), \tag{2.3}$$

where $*_G$ is defined via (2.1). Let h be a morphism from (G, G_0) into (G', G'_0). We define $\mathcal{X}(h)$ as a mapping from $\mathcal{X}(G, G_0)$ into $\mathcal{X}(G', G'_0)$ via

$$\mathcal{X}(h)(a) := h(a), \; a \in G_0. \tag{2.4}$$

The following result has been proved in [Dvu 1]:

THEOREM 2.3. *$\mathcal{X}$ is a faithful, full and right-adjoint functor from the category $\mathcal{LG}$ of Abelian ℓ-groups into the category $\mathcal{BCK}$ of commutative BCK-algebras with the relative cancellation property. Moreover, $\mathcal{X}$ is a categorical equivalence.*

Remark 2.4. From Theorem 2.3 we have, in particular, see also [DvGr 1], that any commutative BCK-algebra with the relative cancellation property admits a *universal group*, i.e. a pair $(G(X), h)$, where $G(X)$ is an Abelian ℓ-group and h is a mapping from X into $G(X)$ preserving the order in X and $+$, such that if $g : X \to G_1$ is an order and $+$ preserving mapping into an ℓ-group G_1, then there exists a unique group homomorphism $g' : G \to G_1$ such that $g = g' \circ g$. We recall that h is a BCK-embedding. In this case, $X = \mathcal{X}(G(X), h(X))$.

On the other hand, every commutative BCK-algebra $(X; *, 0)$ with the relative cancellation property can be embedded into a commutative BCK-algebra $(\widehat{X}; *, 0)$, called the *BCK-hull* of X, such that $\widehat{X}$ is a lattice consisting of all finite joins of elements from X. Moreover, every element from $\widehat{X}$ is a finite sum of elements from X, where the sum is taken in $\widehat{X}$, [Dvu]. Then $(G(X), \widehat{h})$ is a universal group for $\widehat{X}$, where $\widehat{h}$ is a unique extension of h.

3. Product on commutative BCK-algebras

In many important commutative BCK-algebras, for example, in semisimple MV-algebras, we are able to introduce besides a total BCK-binary operation $*$ and the derived partial addition $+$ also a multiplication as a total binary operation. For example, if $X = [0,1]^{\Omega}$, we define $(f * g)(\omega) = f(\omega) *_{\mathbb{R}} g(\omega)$, $\omega \in \Omega$. Then we define the product $\cdot$ as a natural multiplication of functions. The derived $+$ is such one that $f + g$ exists in X iff $f(\omega) + g(\omega) \le 1$ for any $\omega \in \Omega$. Then the natural product is left and right distributive with respect to the derived $+$.

Motivating that example, we introduce in the present Section product BCK-algebras. We show that they are closely connected with ℓ-rings. Other examples of product BCK-algebras are given after Theorem 3.3.

DEFINITION. We say that a commutative BCK-algebra $(X; *, 0)$ with the relative cancellation property admits a *product* if there is a binary operation $\cdot$ on X satisfying for all $a, b, c \in X$ the following

(i) if $a + b$ is defined in X, then $a \cdot c + b \cdot c$ and $c \cdot a + c \cdot b$ exist and

$$(a + b) \cdot c = a \cdot c + b \cdot c,$$
$$c \cdot (a + b) = c \cdot a + c \cdot b,$$

where $+$ is a derived partial operation on X, and we say that X is a *product BCK-algebra.* Sometimes we write $X = (X; *, 0, \cdot)$. An element u of a product BCK-algebra X is said to be a *unity,* if $a \cdot u = u \cdot a = a$ for any $a \in X$.

A product $\cdot$ on X is

(ii) *associative* if $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, $a, b, c \in X$;
(iii) *commutative* if $a \cdot b = b \cdot a$, $a, b \in X$.

It is worth saying that if $\cdot$ is a product on X, then

(iv) $a \cdot 0 = 0 = 0 \cdot a$,
(v) if $a \leq b$, then for any $c \in X$, $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.

Property (iv) follows easily from the following: $a \cdot 0 = a \cdot (0+0) = a \cdot 0 + a \cdot 0$, and the cancellation property gives $a \cdot 0 = 0$. Similarly, $0 \cdot a = 0$.

THEOREM 3.1. *The class of product BCK-algebras is equationally definable.*

Proof. The class of commutative BCK-algebras with the relative cancellation property is equationally definable, [DvGr 1, Thm 5.4]. The condition (i) is equivalent to the identities, for all $c, x, y \in X$,

$$c \cdot (x * (x \wedge y)) = (c \cdot x) * (c \cdot (x \wedge y)),$$
$$(x * (x \wedge y)) \cdot c = (x \cdot c) * ((x \wedge y) \cdot c).$$

Hence the class of product BCK-algebras is equationally definable. ∎

We recall that an *ℓ-ring* is a usual ring $(R; +, \cdot, 0)$ with a partial order $\leq$ such that $(R; +, 0, \leq)$ is an ℓ-group and, if $0 \leq a$ and $0 \leq b$, then $0 \leq a \cdot b$. A *function ring* or an *f-ring* is an ℓ-ring such that $a \wedge b = 0$ and $0 \leq c$ imply $(c \cdot a) \wedge b = (a \cdot c) \wedge b = 0$. An ℓ-ring R is Archimedean if $(R; +, 0, \leq)$ is Archimedean. For an ℓ-ring R we shall write $R = (R; +, \cdot, 0, \leq)$.

We recall that if unity 1 of an ℓ-ring R is a strong unit, then R is an f-ring, [Bir, Lem XVII.5.2], and any Archimedean f-ring is commutative and associative, [Bir, Thm XVII.10]. More about ℓ-rings see, e.g., [Bir] or [Fuc].

THEOREM 3.2. *Let $(R; +, \cdot, 0, \leq)$ be an (associative) ℓ-ring with $\emptyset \neq R_0 \subseteq R^+$ such that $a, b \in R_0$ entails $a *_R b \in R_0$ and $a \cdot b \in R_0$. Then $(X; *, 0) := \mathcal{X}(R, R_0)$ is a product BCK-algebra with an (associative) product $\cdot$, which is the restriction of $\cdot$ to $R_0 \times R_0$.*

Conversely, let $(X; *, 0)$ be a product BCK-algebra with an (associative) product $\cdot$. Then there exists a unique (up to isomorphism) (associative) ℓ-ring $(R; +, \cdot, 0, \leq)$ with a non-empty set $R_0 \subseteq R^+$ generating R^+ as a group-cone such that $a, b \in R_0$ entails $a *_R b \in R_0$, $a \cdot b \in R_0$, and $X \cong \mathcal{X}(R, R_0)$, and $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$, $a, b \in X$, where ϕ is a BCK-isomorphism of X onto $\mathcal{X}(R, R_0)$.

Proof. (1) Let $X = \mathcal{X}(R, R_0)$. For $a, b \in X$ we have $a \cdot b \in X$ which says that the restriction of $\cdot$ onto $X \times X$ defines the (associative) product on the BCK-algebra X.

(2) Let X be a BCK-algebra with a product $\cdot$. According to Theorem 2.3, there is an ℓ-group $(R; +, 0, \leq)$ with a BCK-subalgebra R_0 and a BCK-isomorphism ϕ from X onto $\mathcal{X}(R, R_0)$. We can define the product $\cdot$ on $\mathcal{X}(R, R_0)$ as follows

$$\phi(a) \cdot \phi(b) := \phi(a \cdot b), \; a, b \in X.$$

Because $\mathcal{X}(R, R_0)$ is generating for the positive cone R^+ of R, ϕ preserves all existing $+$ in X, we see that $\cdot$ is a product on $\mathcal{X}(R, R_0)$.

Given $g \in R^+$ there exist $a_1, \ldots, a_n \in X$ such that $g = \sum_{i=1}^n \phi(a_i)$. For any $\phi(c)$, where $c \in X$, we define

$$g \cdot \phi(c) = \phi(a_1 \cdot c) + \ldots + \phi(a_n \cdot c). \tag{3.1}$$

We claim that (3.1) is defined unambiguously. Indeed, if $g = \sum_{j=1}^m \phi(b_j)$, for some $b_1, \ldots, b_m \in X$, due to the Riesz decomposition property holding on ℓ-groups [Goo, Prop 2.2], there exist elements $c_{ij} \in X$ such that $a_i = \sum_{j=1}^m c_{ij}$ and $b_j = \sum_{i=1}^n c_{ij}$ for all i, $1 \leq i \leq n$ and all j, $1 \leq j \leq m$. Then

$$\begin{aligned}
\sum_{i=1}^n \phi(a_i \cdot c) &= \sum_{i=1}^n \phi((\sum_{j=1}^m c_{ij}) \cdot c) = \sum_{i=1}^n \phi(\sum_{j=1}^m (c_{ij} \cdot c)) \\
&= \sum_{i=1}^n \sum_{j=1}^m \phi(c_{ij} \cdot c) = \sum_{j=1}^m \sum_{i=1}^n \phi(c_{ij} \cdot c) \\
&= \sum_{j=1}^m \phi(\sum_{i=1}^n c_{ij} \cdot c) = \sum_{j=1}^m \phi((\sum_{i=1}^n c_{ij}) \cdot c) \\
&= \sum_{j=1}^m \phi(b_j \cdot c),
\end{aligned}$$

which proves that the extension of $\cdot$ on $R^+ \times \mathcal{X}(R, R_0)$ is correct. We now extend $\cdot$ to $R \times \mathcal{X}(R, R_0)$ as follows: If $g = g_1 - g_2$, $g_1, g_2 \in R^+$, then

$$g \cdot \phi(c) := g_1 \cdot \phi(c) - g_2 \cdot \phi(c).$$

Since if $g_1 - g_2 = h_1 - h_2$ for $g_i, h_i \in R^+$, $i = 1, 2$, then $g_1 + h_2 = h_1 + g_2$, so that by (3.1)

$$\begin{aligned}
(g_1 + h_2) \cdot \phi(c) &= (h_1 + g_2) \cdot \phi(c) \\
g_1 \cdot \phi(c) + h_2 \cdot \phi(c) &= h_1 \cdot \phi(c) + g_2 \cdot \phi(c) \\
g_1 \cdot \phi(c) - g_2 \cdot \phi(c) &= h_1 \cdot \phi(c) - h_2 \cdot \phi(c).
\end{aligned}$$

Now let $c \in R^+$. Then $c = \phi(c_1) + \ldots + \phi(c_s)$, where $c_t \in X$, $t = 1, \ldots, s$. We extend $\cdot$ to $R \times R^+$ as follows

$$g \cdot c := \sum_{t=1}^{s} g \cdot \phi(c_t), \; g \in R.$$

If $c = \sum_{w=1}^{v} \phi(d_w)$, using the Riesz decomposition property, we have

$$\sum_{t=1}^{s} g \cdot \phi(c_t) = \sum_{w=1}^{v} g \cdot \phi(d_w).$$

It is clear that the "multiplication" $\cdot$ can be extended to whole $R \times R$: If $c = c_1 - c_2$, where $c_1, c_2 \in R^+$, then

$$g \cdot c := g \cdot c_1 - g \cdot c_2.$$

It is evident that if $g, h \in R^+$, then $g \cdot h \in R^+$, and due to (ii), $\cdot$ is associative on R, so that $(R; +, \cdot, 0, \leq)$ is an ℓ-ring with the BCK-subalgebra R_0, which proves Theorem. ■

A commutative BCK-algebra $(X; *, 0)$ with the relative cancellation property is said to be *Archimedean* if the statement "na is defined and $na \leq b$ for any $n \geq 1$ and for some $b \in X$" implies $a = 0$. Due to [Dvu], a BCK-algebra X is Archimedean iff its universal group (G, h) is an Archimedean ℓ-group. In addition, X is Archimedean if X is semisimple [Dvu]. The converse statement holds for example, if X has a quasi strong unit.

THEOREM 3.3. *Let $(X; *, 0)$ be a product BCK-algebra with a product $\cdot$.*

(1) *If $a \cdot b = b \cdot a$, for all $a, b \in X$, then $(R; +, \cdot, 0, \leq)$ from Theorem 3.2 is a commutative ℓ-ring.*

(2) *If a quasi strong unit u for X is the unity for X, then the ℓ-ring $(R; +, \cdot, 0, \leq)$ from Theorem 3.2 is an f-ring with a strong unit $\phi(u)$*[2] *which is unity in R.*

(3) *If X is semisimple with unity u which is a quasi strong unit for a product $\cdot$, then the ℓ-ring $(R; +, \cdot, 0, \leq)$ from Theorem 3.2 is a commutative and associative f-ring.*

(4) *If there is an element $e \in X$ which is unity for X, then $\phi(e)$ is unity in the ℓ-ring $(R; +, \cdot, 0, \leq)$ from Theorem 3.2.*

Proof. (1) It follows from the construction of $(R; +, \cdot, 0)$ from the proof of Theorem 3.2.

[2] A positive element of an ℓ-ring R is a *strong unit* if, for any $g \in R$, there exists an integer $n \geq 1$ such that $-nu \leq g \leq nu$.

(2) Since $a \cdot u = u \cdot a = a$ for any $a \in X$, it is possible to show that $\phi(1)$ from the proof of Theorem 3.2 is unity in R. From [Bir, Lem XVII.5.2] we conclude that R is an f-ring.

(3) According to (2), R is an f-ring. Due to the semisimplicity of X we infer that $(R; +, 0)$ is an Archimedean ℓ-group, and by [Bir, Thm XVII.10], the multiplication $\cdot$ on R is commutative and associative.

(4) It is evident. ∎

To illustrate our notions, we investigate the possibility of the existence of product BCK-algebras. It is evident that every BCK-algebra under the trivial product, i.e., $a \cdot b = 0$ for all $a, b \in X$, is a product BCK-algebra, called a zero-BCK-algebra. Other product BCK-algebras with non trivial product we can obtain as follows.

Let a be an element of X. If there exists a greatest integer n such that $na := a + \ldots + a$ is defined in X, $ord(a) := n$; if na is defined in X for any integer $n \geq 1$, we put $ord(a) = \infty$.

A non-zero element a of X is said to be an *atom* of X if $b \leq a$, $b \in X$ imply $b = 0$ or $b = a$. A BCK-algebra X is said to be *atomic* if given a non-zero element b of X there is an atom a in X such that $a \leq b$.

A BCK-algebra $(X; *, 0)$ is said to be *implicative* if $x * y = (x * y) * y$, $x, y \in X$. Such BCK-algebras have the relative cancellation property and they can be embedded onto a BCK-algebra $(\mathcal{S}; \setminus, \emptyset)$, where $\mathcal{S}$ is a usual ring of subsets of a non-void set Ω, and $\setminus$ is the set-theoretical difference, [MeJu, Thm VII.2.7].

THEOREM 3.4. *A finite commutative BCK-algebra $(X; *, 0)$ with the relative cancellation property admits a product with unity u if and only if X is implicative. If it is the case, then $a \cdot b = a \wedge b$, $a, b \in X$.*

Proof. Suppose that X is implicative and define $a \cdot b := a \wedge b$ for all $a, b \in X$. Then $\cdot$ is a product in question.

Conversely, assume that the product $\cdot$ satisfies $a \cdot u = a = u \cdot a$ for any $a \in X$.

First, let X be a bounded BCK-algebra with the greatest element 1. Then $1 \cdot 1 \leq 1 = 1 \cdot u \leq 1 \cdot 1$ which proves $1 \cdot 1 = 1$. For any $a \in X$, we denote by $a^* = 1 * a$. Then $a \cdot 1 \geq a \cdot u = a$ and $1 \cdot a \geq u \cdot a = a$. In addition, $a + a^* = 1 = 1 \cdot 1 = (a + a^*) \cdot 1 = a \cdot 1 + a^* \cdot 1 \geq a + a^*$, which entails $a \cdot 1 = a$. By symmetry, we have $1 \cdot a = a$ for any $a \in X$, and in addition, $1 = 1 \cdot u = u$.

The finiteness of X yields that X is atomic. If a and b are two different atoms, then $(na) \wedge (mb) = 0$ whenever na and mb are defined in X, [Dvu, Thm 7.2] Therefore $((na) \vee (mb)) - (na) = (mb) - ((na) \wedge (mb)) = (mb)$ so that $na + mb$ is defined in X and $na + mb = (na) \vee (mb)$.

Given an atom a, let $j_a(x)$ denote the greatest integer n such that na is defined in X and $na \leq x$. Since X is finite, $j_a(x)$ is finite for any atom a and for any element x, and the element x of X can be uniquely expressed in the form

$$x = j_{a_1}(x)a_1 + \ldots + j_{a_k}(x)a_k,$$

where $a_1, \ldots, a_k$ denote the set of all atoms in X.

Assume again that a and b are two different atoms. Hence $a \cdot b \leq a \cdot 1 = a$ and $a \cdot b \leq 1 \cdot b = b$ which gives $a \cdot b \leq a \wedge b = 0$. Similarly $a \cdot a \leq a$ which entails $a \cdot a \in \{0, a\}$.

If X is not implicative, then there is an atom a of X such that $n := ord(a) \geq 2$. If $a \cdot a = a$, then $(na) \cdot (na) = n^2(a \cdot a) = n^2 a$ which yields $n = ord(a) \geq n^2$. If $a \cdot a = 0$, then $0 \neq a = a \cdot 1 = a \cdot (ord(a_1)a_1 + \ldots + ord(a_k)a_k) = 0$ which is again absurd. Consequently, on X, which is not implicative, there is no product in question.

Assume now that X is a finite BCK-algebra which is not necessarily bounded. In any rate, the BCK-hull $\widehat{X}$ of X is due to finiteness of X bounded and finite. If $(G(X), h)$ is a universal group for X, then by Theorem 3.2, $G(X)$ is an ℓ-ring. By Remark 2.4, $\widehat{h}(\widehat{X})$ is closed under the product in $G(X)$, because if $x, y \in \widehat{X}$, then $x = x_1 + \ldots + x_n$, $y = y_1 + \ldots + y_m$, where $x_i, y_j \in X$; then $x \cdot y = \sum_{i,j} x_i \cdot y_j \in \widehat{X}$.

Therefore, $\widehat{X}$ is a finite product BCK-algebra. It is clear that u is unity in $\widehat{X}$, too. Due to the first part of the present proof, $\widehat{X}$ has to be implicative, so that, X is implicative.

Assume now that X is implicative, and without loss of generality, let X be bounded, and let $a \cdot a = 0$ for some atom a of X. Then $a = a \cdot 1 = a \cdot (a + a^*) = a \cdot a^* = a \cdot (\sum_{a_i \neq a} a_i) = 0$, which is absurd. Hence $a \cdot a = a$ for any atom a. Therefore $x \cdot y = (j_{a_1}(x)a_1 + \ldots + j_{a_k}(x)a_k) \cdot (j_{a_1}(y)a_1 + \ldots + j_{a_k}(y)a_k) = j_{a_1}(x \wedge y)a_1 + \ldots + j_{a_k}(x \wedge y)a_k = x \wedge y$ for any $x, y \in X$. ∎

COROLLARY 3.5. (i) *A product $\cdot$ with unity on a non-implicative BCK-algebra X can exists only if X is an infinite BCK-algebra. For example, on $([0, \infty); *_{\mathbb{R}}, 0)$ we can define a usual product of real numbers.*

(ii) *From the proof of the previous theorem we conclude that if u is a unity for a product $\cdot$ on a bounded product BCK-algebra X, then $1 = u$.*

4. Product BCK-algebras and categorical equivalence

Denote by $\mathcal{PBCK}$ the category of product BCK-algebras, i.e., its objects are BCK-algebras with product, and morphisms are BCK-homomorphisms of product BCK-algebras preserving also $\cdot$.

We denote by $\mathcal{PR}$ the category whose objects are pairs (R, R_0), where R is an Abelian ℓ-ring and R_0 is a non-void subset of the positive cone R^+

of R such that R_0 generates R^+ and $(R_0; *_R, 0)$ is a product BCK-algebra under $\cdot$. A morphism from (R, R_0) into (R', R'_0) is an ℓ-group homomorphism $h : R \to R'$, preserving $\cdot, \wedge, \vee$, such that $h(R_0) \subseteq R'_0$.

We denote by $\mathcal{X}_{\mathcal{R}}$ a morphism from $\mathcal{PR}$ into $\mathcal{PBCK}$ defined by $\mathcal{X}_{\mathcal{R}}(R, R_0) = (R_0, *_R, 0, \cdot)^3$, and $\mathcal{X}_{\mathcal{R}}(f) := f|R_0$.

THEOREM 4.1. *$\mathcal{X}_{\mathcal{R}}$ is a faithful and full functor from $\mathcal{PR}$ to $\mathcal{PBCK}$.*

Proof. Let h_1 and h_2 be two morphism from (R, R_0) into (R', R'_0) such that $\mathcal{X}_{\mathcal{R}}(h_1) = \mathcal{X}_{\mathcal{R}}(h_2)$. Then $h_1(a) = h_2(a)$ for any $a \in \mathcal{X}_{\mathcal{R}}(R, R_0)$. Since $\mathcal{X}_{\mathcal{R}}(R, R_0)$ generates R^+ and R, we have that $h_1(g) = h_2(g)$ for any $g \in R$ which proves that $\mathcal{X}_{\mathcal{R}}$ is faithful.

To prove that $\mathcal{X}_{\mathcal{R}}$ is a full functor, suppose that f is a morphism from $\mathcal{X}_{\mathcal{R}}(R, R_0)$ into $\mathcal{X}_{\mathcal{R}}(R', R'_0)$. Since $\mathcal{X}_{\mathcal{R}}(R, R_0)$ generates R, due to the Riesz decomposition property, f can be uniquely extended to a group-homomorphism $\widehat{f}$ from R into R'.

CLAIM 1. $\widehat{f}$ is a lattice-homomorphism. The proof will proceed in several steps.

Step 1. f preserves meets in $\mathcal{X}_{\mathcal{R}}(R, R_0)$. This follows from the observation that $a \wedge b = a * b$ which entails f preserves meets in $\mathcal{X}_{\mathcal{R}}(R, R_0)$.

Step 2. Let $a, b, u_0 \in R^+$. If $\widehat{f}(a \wedge b) = \widehat{f}(a) \wedge \widehat{f}(b)$ and if $\widehat{f}(u_0 \wedge (b - (a \wedge b))) = \widehat{f}(u_0) \wedge \widehat{f}(b - (a \wedge b))$, then

$$\widehat{f}((a + u_0) \wedge b) = \widehat{f}(a + u_0) \wedge \widehat{f}(b).$$

Indeed, we have $(a \wedge b) + [u_0 \wedge (b - (a \wedge b))] = ((a \wedge b) + u_0) \wedge b = (a + u_0) \wedge (b + u_0) \wedge b = (a + u_0) \wedge b$.

Therefore

$$\begin{aligned}
\widehat{f}((a + u_0) \wedge b) &= \widehat{f}(a \wedge b) + \widehat{f}(u_0 \wedge (b - (a \wedge b))) \\
&= (\widehat{f}(a) \wedge \widehat{f}(b)) + [\widehat{f}(u_0) \wedge (\widehat{f}(b) - (\widehat{f}(a) \wedge \widehat{f}(b)))] \\
&= ((\widehat{f}(a) \wedge \widehat{f}(b)) + \widehat{f}(u_0)) \wedge \widehat{f}(b) \\
&= (\widehat{f}(a) + \widehat{f}(u_0)) \wedge (\widehat{f}(b) + \widehat{f}(u_0)) \wedge \widehat{f}(b) = \widehat{f}(a + u_0) \wedge \widehat{f}(b).
\end{aligned}$$

Step 3. $\widehat{f}(a \wedge b) = \widehat{f}(a) \wedge \widehat{f}(b)$ whenever $a \in R^+$ and $b \in \mathcal{X}_{\mathcal{R}}(R, R_0)$.

Since $\mathcal{X}_{\mathcal{R}}(R, R_0)$ is generating for R^+, a is of the form $a = a_1 + \ldots + a_n$ for some $a_1, \ldots, a_n \in \mathcal{X}_{\mathcal{R}}(R, R_0)$. The proof will follow mathematical induction on n.

If $n = 1$, the statement is trivial. Suppose now that the statement holds for any $a' = a_1 + \ldots + a_i$ with $1 \le i \le n$. Put $a = a_1 + \ldots + a_n$, $u_0 = a_{n+1}$.

[3]We recall that $*_R$ is understood as that defined by (2.1).

Then there exist $v_1, \dots, v_k \in \mathcal{X}_{\mathcal{R}}(R, R_0)$ such that $b = (v_1 + \dots + v_k) + (a \wedge b)$. Since $v := v_1 + \dots + v_k \le b$, $v \in \mathcal{X}_{\mathcal{R}}(R, R_0)$. Hence $v = b - (a \wedge b)$. Since $\widehat{f}$ preserves meets in $\mathcal{X}_{\mathcal{R}}(R, R_0)$, we have $\widehat{f}(u_0 \wedge v) = \widehat{f}(u_0) \wedge \widehat{f}(v)$, so that $\widehat{f}(u_0 \wedge (b - (a \wedge b))) = \widehat{f}(u_0) \wedge \widehat{f}(b - (a \wedge b)) = \widehat{f}(u_0) \wedge (\widehat{f}(b) - (\widehat{f}(a) \wedge \widehat{f}(b)))$ when we have used induction hypothesis. By Step 1, $\widehat{f}((a + u_0) \wedge b) = \widehat{f}(a + u_0) \wedge \widehat{f}(b)$, that is, $\widehat{f}((a_1 + \dots + a_{n+1}) \wedge b) = \widehat{f}(a_1 + \dots + a_{n+1}) \wedge \widehat{f}(b)$ for any n.

Step 4. $\widehat{f}(a \wedge b) = \widehat{f}(a) \wedge \widehat{f}(b)$ whenever $a, b \in R^+$.

Let $a = a_1 + \dots + a_n$, $b = b_1 + \dots + b_k$. The proof will follow complete induction on k.

If $k = 1$, we apply Step 3. Suppose now that the assertion holds for any j with $1 \le j \le k$. Put $B = a$, $A = b_1 + \dots + b_k$, $u_0 = b_{k+1}$. By Step 3, $\widehat{f}(u_0 \wedge (B - (A \wedge B))) = \widehat{f}(u_0) \wedge \widehat{f}(B - (A \wedge B))$ and $\widehat{f}(A \wedge B) = \widehat{f}(A) \wedge \widehat{f}(B)$. Therefore the conditions of Step 1 are satisfied, so that $\widehat{f}((A + u_0) \wedge B) = \widehat{f}(A + u_0) \wedge \widehat{f}(B)$ which proves $\widehat{f}((a_1 + \dots + a_n)) \wedge (b_1 + \dots + b_{k+1})) = \widehat{f}(a_1 + \dots + a_n) \wedge \widehat{f}(b_1 + \dots + b_{k+1})$ for each n and each k.

Step 5. $\widehat{f}(a \wedge b) = \widehat{f}(a) \wedge \widehat{f}(b)$ whenever $a, b \in R$. There exist $a', a'', b', b'' \in R^+$ such that $a = a' - a''$ and $b = b' - b''$. By Step 4, $\widehat{f}((a' + b'') \wedge (b' + a'')) = \widehat{f}(a' + b') \wedge \widehat{f}(b' \wedge a'')$. Subtracting $\widehat{f}(b'')$ and $\widehat{f}(a'')$ we obtain the assertion in question.

Claim 2. $\widehat{f}$ preserves the product $\cdot$ in R.

Let $a, b \in R^+$. There exist $a_1, \dots, a_n, b_1, \dots, b_m \in \mathcal{X}_{\mathcal{R}}(R, R_0)$ such that $a = a_1 + \dots + a_n$ and $b = b_1 + \dots + b_m$. Then $a \cdot b = \sum_{i=1}^n \sum_{j=1}^m a_i \cdot b_j$ and $a_i \cdot b_j \in \mathcal{X}_{\mathcal{R}}(R, R_0)$. Calculate $\widehat{f}(a \cdot b) = \sum_{i=1}^n \sum_{j=1}^m \widehat{f}(a_i \cdot b_j) = \sum_{i=1}^n \sum_{j=1}^m \widehat{f}(a_i) \cdot \widehat{f}(b_j) = (\sum_{i=1}^n \widehat{f}(a_i)) \cdot (\sum_{j=1}^m \widehat{f}(b_j)) = \widehat{f}(a) \cdot \widehat{f}(b)$.

If now $a, b \in R$, then $a = a_1 - a_2$ and $b = b_1 - b_2$, where $a_1, a_2, b_1, b_2 \in R^+$. Then $\widehat{f}(a \cdot b) = \widehat{f}(a_1 \cdot b_1 + a_1 \cdot b_2 - a_2 \cdot b_1 + a_2 \cdot b_2) = \widehat{f}(a_1 \cdot b_1) - \widehat{f}(a_1 \cdot b_2) - \widehat{f}(a_2 \cdot b_1) + \widehat{f}(a_2 \cdot b_2) = \widehat{f}(a_1) \cdot \widehat{f}(b_1) - \widehat{f}(a_1) \cdot \widehat{f}(b_2) - \widehat{f}(a_2) \cdot \widehat{f}(b_1) + \widehat{f}(a_2) \cdot \widehat{f}(b_2) = \widehat{f}(a) \cdot \widehat{f}(b)$.

Consequently, we have proved that $\widehat{f}$ is a morphism from $(R; +, \cdot, 0, \le)$ into $(R'; +, \cdot, 0, \le)$ such that $\mathcal{X}_{\mathcal{R}}(\widehat{f}) = f$. ∎

Theorem 4.2. *The functor $\mathcal{X}_{\mathcal{R}}$ defines a categorical equivalence of the category $\mathcal{PR}$ of ℓ-rings and the category $\mathcal{PBCK}$ of product BCK-algebras.*

Proof. According to [MaL, Thm IV.4.1], to prove that $\mathcal{X}_{\mathcal{R}}$ is an equivalence of the categories in question, it is necessary and sufficient to show that $\mathcal{X}_{\mathcal{R}}$ is faithful and full, and each object X from $\mathcal{PBCK}$ is isomorphic to $\mathcal{X}_{\mathcal{R}}(R, R_0')$ for some object (R, R_0) in $\mathcal{PR}$.

Due to Theorem 4.1, $\mathcal{X}_{\mathcal{R}}$ is faithful and full, and by Theorem 3.2, there exists an object (R, R_0) in $\mathcal{PR}$ such that $\mathcal{X}_{\mathcal{R}}(R, R_0)$ is isomorphic with X which proves Theorem. ■

Remark 4.3. Theorem 4.2 holds also for the category of product BCK-algebras which are also associative. In this case $\mathcal{APR}$ has to be the category of associative elements from $\mathcal{PR}$.

Finally we compare product BCK-algebras and MV*-algebras introduced in [BDG]. We recall that Cornish [Cor] introduced so-called conical BCK-algebras, which by [Dvu 1, Rem 3.4] can be defined by the equivalent way: A commutative BCK-algebra $(X; *, 0)$ is said to be *conical* iff it is with the relative cancellation property and if $x + y$ is defined in X for all $x, y \in X$. It is possible to show that $\mathcal{X}(G, G_0)$ is conical iff $G_0 = G^+$, and due to Cornish [Cor] the category of conical BCK-algebras is categorical equivalent to the category of all Abelian ℓ-groups.

Denote by $\mathcal{PCBCK}$ the category of all product conical BCK-algebras, and by $\mathcal{PCBCK}_u$ the category of all product conical BCK-algebras whose objects are pairs (X, u), where u is a fixed quasi strong unit such that $u \cdot u \leq u$, and morphisms is any BCK-homomorpshim preserving the product and fixed quasi strong units.

An MV-algebra $(M; \oplus, \odot, ^*, 0, 1)$ is said to be *perfect* if, for each element $x \in M$, either $x \in \mathrm{Rad}(M)$ or $x^* \in \mathrm{Rad}(M)$.

According to [BeDi], we say that a perfect MV-algebra M is with *principal radical* if there is an element $a \in \mathrm{Rad}(M)$ such that the ideal of M generated by a coincides with $\mathrm{Rad}(M)$. The category MV_{pr} of perfect MV-algebras with principal radical is a category whose objects are pairs (M, a), where M is a perfect MV-algebra with a fixed element $a \in \mathrm{Rad}(M)$ generating $\mathrm{Rad}(M)$, as an ideal, and morphisms are MV-homomorphisms preserving fixed elements a in radicals.[4] Due to [BeDi, Prop 27], this category is equivalent with the category MV of all MV-algebras, or equivalently with the category of all unital Abelian ℓ-groups (Mundici's representation).

We recall that in [BDG] an MV*-algebra M has been introduced, which can be defined equivalently as follows: M is a perfect MV-algebra with a binary operation $* : \mathrm{Rad}(M) \times \mathrm{Rad}(M) \to \mathrm{Rad}(M)$ such that (i) it is associative, (ii) $a * (b + c) = (a * b) + (a * c)$, $(b + c) * a = (b * a) + (c * a)$, $a, b, c \in \mathrm{Rad}(M)$.

Now let MV_p^* be the category of MV*-algebras whose morphisms are MV-homomorphisms preserving a binary operation $*$ on radicals. According to [DiLe, Thm 3.5], the category MV_p of perfect MV-algebras is categorically equivalent with the category of all ℓ-groups (not necessarily unital),

[4] We recall that any MV-homomorphism maps radicals into radicals, use (2.2).

and MV^*_p is categorically equivalent to the category of all associative ℓ-rings [BDG, Thm 2.6] (not necessarily unital). Let MV^*_{pr} denote the category having as objects pairs (A, g), where A is a perfect MV*-algebra with principal radical with a distinguished generator g such that $g * g \le g$, and morphisms are MV^*_{pr}-homomorphisms preserving the distinguished generator of the radical.

Finally, let $\mathcal{R}$ denote the category of associative ℓ-rings, where the objects are associative ℓ-rings, and let $\mathcal{R}_u$ be the category of associative ℓ-rings with a fixed strong unit u such that $u \cdot u \le u$, i.e., objects are pairs (R, u), where u is a fixed strong unit in R, and morphisms are ℓ-rings morphisms preserving fixed strong units.

THEOREM 4.4. (1) *The categories* MV^*_p *of perfect* MV*-*algebras,* $\mathcal{PCBCK}$ *of product conical BCK-algebras, and* $\mathcal{R}$ *of associative* ℓ*-rings are categorically equivalent.*

(2) *The categories* MV^*_{pr}, $\mathcal{PCBCK}_u$, *and* $\mathcal{R}_u$ *are categorical equivalent.*

Proof. It follows ideas developed in above and in [BDG], and from the observation that an element u of X is a quasi strong unit iff the ideal of X generated by u is equal to X. ■

5. Ideals in product BCK-algebras and BCKf-algebras

Let X be a product BCK-algebra. A non-empty subset I of X is said to be a $\cdot$-*ideal* of X if

(i) $0 \in I$;
(ii) $a * b \in I$, $a \in X$, and $b \in I$, entail $a \in I$;
(iii) $a \in I$ and $b \in X$ entail $a \cdot b \in I$ and $b \cdot a \in I$.

We denote by $\mathcal{I}_p(X)$ the set of all $\cdot$-ideals of X.

Let $(R; +, \cdot, 0, \le)$ be an ℓ-ring. An L-*ideal* of R is a non-void subset J of R such that

(i) $x, y \in J$ entails $x \pm y \in J$;
(ii) $x \in R$, $y \in J$, $|x| \le |y|$ entail $x \in J$;
(iii) $x \in J$ and $y \in R$ entail $x \cdot y \in J$ and $y \cdot x \in J$.

We denote by $\mathcal{I}_L(R)$ the set of all L-ideals of R.

Let $(X; *, 0)$ be a commutative BCK-algebra with the relative cancellation property and let $(G(X), h)$ be its universal group. Given a subset I of X, let $h_0(I)$ be the ℓ-ideal of $G(X)$ generated by the image $h(I)$ of I in $G(X)$.

THEOREM 5.1. *Let* $(R; +, \cdot, 0, \le)$ *be an* ℓ*-ring with* $\emptyset \ne R_0 \subseteq R^+$ *such that* $a, b \in R_0$ *entails* $a *_R b \in R_0$ *and* $a \cdot b \in R_0$, *and let* $X = \mathcal{X}_\mathcal{R}(R, R_0)$. *Then* *the correspondence Φ defined by*

$$\Phi(I) := h_0(I),\ I \in \mathcal{I}_p(X), \tag{5.1}$$

where h is the embedding of X into R, is an order-isomorphism from the set $\mathcal{I}_p(X)$ of $\cdot$-ideals of the product BCK-algebra X, ordered by inclusion, onto $\mathcal{I}_L(R)$. The inverse isomorphism Ψ is given by

$$\Psi(H) := X \cap H,\ H \in \mathcal{I}_L(R). \tag{5.2}$$

Proof. In [Dvu 2, Thm 2.5] we have proved that the mappings (5.1) and (5.2) define inclusion isomorphisms between the set of ideals of X and ℓ-ideals of R. We show that they preserve also "$\cdot$-ideal property".

We assert that $x \in \Phi(I)$ iff $x = x_1 + \ldots + x_j - y_1 - \ldots - y_k$, where $x_i, y_s \in I$. It is clear that if x has a given form, then $x \in \Phi(I)$. Conversely, let $x \in \Phi(I)$. Assume that $x \geq 0$. Then there exist $x_1, \ldots, x_n \in X$ such that $x = x_1 + \ldots + x_n$. We show that $x_1, \ldots, x_n \in I$. The proof will follow mathematical induction on n. If $n = 1$, the statement is evident. Assume thus that whenever $x \in \Phi(I)$ and $x = x_1 + \ldots + x_i$ for $i \leq n$, then $x_1, \ldots, x_i \in I$. Now let $x = x_1 + \ldots + x_n + x_{n+1}$. Then $0 \leq x_{n+1} \leq x$ so that $x_{n+1} \in \Phi(I)$ and $x - x_{n+1} \in \Phi(I)$ and by induction $x_1, \ldots, x_n \in I$. Hence $x - x_1 = x_2 + \ldots + x_{n+1}$ which proves also $x_{n+1} \in I$.

If x is an arbitrary element of $\Phi(I)$, then $x = x^+ - x^-$ and $x^+, x^- \in \Phi(I)$, so that $x = x_1 + \ldots + x_j - y_1 - \ldots - y_k$.

Now let $x \in \Phi(I)$ and $y \in R$, then $x = x_1 + \ldots + x_j - y_1 - \ldots - y_k$ and $y = u_1 + \ldots + u_s - v_1 - \ldots - v_k$, where $x_1, \ldots, x_j, y_1, \ldots, y_k \in I$ and $u_1, \ldots, u_s, v_1, \ldots, v_k \in X$. Then evidently $x \cdot y,\ y \cdot x \in \Phi(I)$, which proves that Φ preserves $\cdot$-ideals. ∎

A product BCK-algebra X is said to be a BCKf-*algebra* if, for all $a, b, c \in X$,

$$a \wedge b = 0 \text{ implies } (a \cdot c) \wedge b = 0 = (c \cdot a) \wedge b. \tag{5.3}$$

For example, any linearly ordered product BCK-algebra is a BCKf-algebra.

THEOREM 5.2. *Let $(R; +, \cdot, 0, \leq)$ be an ℓ-ring with $\emptyset \neq R_0 \subseteq R^+$ such that $a, b \in R_0$ entails $a *_R b \in R_0$ and $a \cdot b \in R_0$. Let $X = \mathcal{X}_{\mathcal{R}}(R, R_0)$. Then X is a BCKf-algebra if and only if R is an f-ring.*

Proof. One direction is trivial. Suppose now that X is a BCKf-algebra, and let for $a, b, c \in R$ we have $a \wedge b = 0$ and $c \geq 0$. Let $x \leq a \cdot c$ and $x \leq b$ and express $a = \sum_i a_i$, $b = \sum_j b_j$, and $c = \sum_k c_k$, where $a_i, b_j, c_k \in X$.

Since $x \leq \sum_{i,k} a_i \cdot c_k$, the Riesz decomposition property, [Goo, Prop. 2.2] entails $x = \sum_{i,k} c_{ik}$, where $c_{ik} \leq a_i \cdot c_k$. Then $c_{ik} \leq \sum_j b_j$ and applying again the Riesz decomposition property, we have $c_{ik} = \sum_j c_{ijk}$ where $c_{ijk} \leq b_j$

for any i and any k. Since $a_i \wedge b_j = 0$, we conclude that $c_{ijk} \leq a_i \cdot c_k$ and $c_{ijk} \leq b_j$, so that $c_{ijk} = 0$ for all i, j, k. Hence, $c_{ik} = 0$ for all i, k which yields $x = 0$.

In a similar way we can prove that $(c \cdot a) \wedge b = 0$. ∎

COROLLARY 5.3. *In any BCKf-algebra X we have for all $a, b, c \in X$*

$$a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c)$$
$$(b \vee c) \cdot a = (b \cdot a) \vee (c \cdot a)$$
$$a \cdot (b \wedge c) = (a \cdot b) \wedge (a \cdot c)$$
$$(b \wedge c) \cdot a = (b \cdot a) \wedge (c \cdot a)$$
$$a \wedge b = 0 \Rightarrow a \cdot b = 0.$$

Proof. Use Theorem 5.2 and [Bir, Cor XVII.5.1, Lem XVII.5.1]. ∎

Given a non-void subset A of X, we define

$$A^{\perp} := \{a \in X : a \wedge x = 0 \text{ for any } x \in A\}.$$

THEOREM 5.4. *Let $(R; +, \cdot, 0, \leq)$ be an ℓ-ring with $\emptyset \neq R_0 \subseteq R^+$ such that $a, b \in R_0$ entails $a *_R b \in R_0$ and $a \cdot b \in R_0$. Let $X = \mathcal{X}_{\mathcal{R}}(R, R_0)$. Then X is a BCKf-algebra if and only if, for every subset A of X, the set $A^{\perp}$ is a $\cdot$-ideal of X.*

Proof. Let X be a BCKf-algebra. Let $a * b \in A^{\perp}$, $b \in A^{\perp}$ and $x \in A$. Then $a \wedge b \in A^{\perp}$. We show that $a \in A^{\perp}$. Let $z \leq a$ and $z \leq x$. Then $z \leq a = a \wedge b + a * b$. The Riesz decomposition property entails that $z = z_1 + z_2$, where $z_1 \leq a \wedge b$ and $z_2 \leq a * b$. Since we also have $z_2 \leq x$, we conclude $z_2 = 0$, and $z_1 \leq a \wedge b \in A^{\perp}$, consequently, $z_1 = z_2 = 0 = z$ which yields $a \in A^{\perp}$.

Take now $a \in A^{\perp}$, $b \in X$, and $x \in A$. Then $a \wedge x = 0$, which gives $(a \cdot b) \wedge x = (b \cdot a) \wedge x = 0$.

Conversely, assume that for any non-void $A \subseteq X$, $A^{\perp}$ is a $\cdot$-ideal of X. Let $a \wedge b = 0$, $c \in X$. Then $a \in \{b\}^{\perp}$. Since $\{b\}^{\perp}$ is a $\cdot$-ideal of X, $a \cdot a$, $c \cdot a \in \{b\}^{\perp}$, i.e., $(a \cdot c) \wedge b = (c \cdot a) \wedge b = 0$. ∎

THEOREM 5.5. *The class of BCKf-algebras is equationally definable.*

Proof. Any product BCK-algebra is equationally definable, Theorem 3.1. The implication (5.1) is equivalent to the identities

$$(c \cdot (x * (x \wedge y))) \wedge (y * (x \wedge y)) = 0,$$
$$((x * (x \wedge y)) \cdot c) \wedge (y * (x \wedge y)) = 0.$$

Hence the class of BCKf-algebras is equationally definable. ∎

COROLLARY 5.6. *Any BCKf-algebra is a subdirect product of subdirectly irreducible BCKf-algebras.*

Proof. It follows from Theorem 5.5 and [Bir, Thm VIII.15]. ∎

PROPOSITION 5.7. *Any subdirectly irreducible BCKf-algebra is linearly ordered.*

Proof. Let X be an irreducible BCKf-algebra and suppose that X is not linearly ordered. Then there exist two non-zero elements $a, b \in X$ such that $a \wedge b = 0$. Put $I = \{b\}^{\perp}$ and $J = I^{\perp}$. Then $I \cap J = \{0\}$, and $a \in I$ and $b \in J$. By Theorem 5.4, I and J are both $\cdot$-ideals of X. The irreducibility of X entails that $I = \{0\}$ or $J = \{0\}$ which is a contradiction. Therefore, X is linearly ordered. ∎

THEOREM 5.8. *A product BCK-algebra is a BCKf-ring if and only if it is a subdirect product of linearly ordered product BCK-algebras.*

Proof. By Corollary 5.6 and Proposition 5.7, any BCKf-algebra is a subdirect product of linearly ordered product BCK-algebras.

Conversely, any linearly ordered product BCK-algebra is an BCKf-algebra. By Theorem 5.5, (5.1) holds in every subdirect product of linearly ordered BCKf-algebras, i.e., X is a BCKf-algebra whenever X is a subdirect product of linearly ordered product BCK-algebras. ∎

A weaker form of BCKf-algebras are *almost* BCKf-*algebras*, i.e., such product BCK-algebras X that $a \wedge b = 0$, $a, b \in X$, imply $a \cdot b = 0$. Similarly, an associative ℓ-ring R is said to be an *almost* f-*ring* if $x \wedge y = 0$ implies $x \cdot y = 0$. We recall that not every almost f-ring has to be an f-ring [Bir, § XVII.6].

A non-zero element u of a BCK-algebra X is a *weak unit* if $u \wedge a \neq 0$. Similarly a non-zero positive element u of an ℓ-algebra R is a weak unit if $x \wedge u \neq 0$ for any positive element $x \in R$.

THEOREM 5.9. *Let $(R; +, \cdot, 0, \leq)$ be an associative ℓ-ring with $\emptyset \neq R_0 \subseteq R^+$ such that $a, b \in R_0$ entails $a *_R b \in R_0$ and $a \cdot b \in R_0$. Let $X = \mathcal{X}_{\mathcal{R}}(R, R_0)$. Then X is an associative almost BCKf-algebra if and only if R is an almost f-ring. If X has a unity e, then X is an almost BCKf-algebra if and only if e is a weak unit for R.*

Proof. One direction is evident. Suppose now that X is an almost BCKf-algebra, and let $a \wedge b = 0$ for $a, b \in R$. Then $a = \sum_i a_i$ and $b = \sum_j b_j$, where $a_i, b_j \in X$. Since $a_i \wedge b_j = 0$ for all i, j, we have $a \cdot b = \sum_{i,j} a_i \cdot b_j = 0$.

Now let e be a unity for X. Then e is a unity for R, and e is a weak unit for X iff e is a weak unit for R. Applying [Bir, Thm XVII.12], R is an almost f-ring iff e is a weak unit for R. ∎

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