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ON THE NONDEGENERATE HARMONIC STRUCTURE ON CERTAIN APPROXIMATIONS OF THE SIERPIŃSKI CARPET

Abstract. In this paper we prove that certain approximation of Sierpiński carpet carries exactly one nondegenerate harmonic structure. This structure, uniquely determined by the invariance property, is symmetric with respect to the isometries of the carpet. Also, the probabilistic interpretation of our results is given.

1. Introduction

In this paper we establish the existence and uniqueness of nondegenerate harmonic structure on certain approximations of the Sierpiński carpet (definitions to follow). Uniqueness of the nondegenerate harmonic structure which is invariant under natural symmetries of the fractal is true for general simple nested fractals (SNF's) (see [1], [14]). However, dropping the symmetry assumption can result in the nonuniqueness. For example, on the Vicsek set (the X-fractal) there is a whole one-parametric family of nondegenerate harmonic structures, but among these only one is symmetric (see [13]). On other fractals (Sierpiński gasket, Lindstrøm snowflake) the symmetry assumption is not necessary for uniqueness (Sabot [14] discusses this topic in detail).

In this paper we determine that on certain approximations of the Sierpiński carpet (which is not an SNF) the nondegenerate harmonic structure is unique without any *a priori* symmetry assumptions: the invariance property itself yields this uniqueness.

Nondegenerate harmonic structures on simple nested fractals give raise to regular Dirichlet forms on them. On SNF's this approach gives a construction of the Laplace operator as a limit of finite-difference operators

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(see [5], [6], [9]). Fractals like the Sierpiński carpet lack the nesting property and as a consequence such a construction would require more work. There are however reasons to believe that the 'reproduced' Dirichlet forms (corresponding to finite-difference operators) do converge, with appropriate normalizing constants, to a 'nice' symmetric form on the carpet (regularity of the limiting Dirichlet form is a problem). See the remark just after the proof of Theorem 1.

Existing construction of the Laplace operator on the carpet (or rather the Dirichlet form associated with it), due to Kusuoka and Zhou (see [11]) is related to the Brownian motion on the carpet constructed and investigated by Barlow and Bass in a series of papers (see [2] for the construction and [1] for an extensive list of references). In particular the question of uniqueness still remains open. The uniqueness of the nondegenerate harmonic structure on approximation of the carpet can be understood as a hint in this direction.

The paper is organized as follows. We devote section 1 to setting the notation and collecting the necessary preliminary facts. The existence of a nondegenerate harmonic structure is shown in section 2 (Theorem 1) and its uniqueness in section 3 (Theorem 2). We conclude the paper by explaining the probabilistic meaning of the obtained results.

2. Preliminaries

As we want this paper to be self-contained we start with setting the notation and some preliminary results. The facts collected here can be found in [1], [4], [8].

Suppose that $\Psi = \{\psi_1, \dots, \psi_8\}$ is the hyperbolic iterated functions system which defines the Sierpiński carpet (see figure 1):

$$\psi_1(x) = \tfrac{1}{3}\left(x - \left(\tfrac{1}{2}, 0\right)\right) + \left(\tfrac{1}{2}, 0\right);$$

$$\psi_2(x) = \tfrac{1}{3}\left(x - (1, 0)\right) + (1, 0);$$

$$\psi_3(x) = \tfrac{1}{3}\left(x - \left(1, \tfrac{1}{2}\right)\right) + \left(1, \tfrac{1}{2}\right);$$

$$\psi_4(x) = \tfrac{1}{3}\left(x - (1, 1)\right) + (1, 1);$$

$$\psi_5(x) = \tfrac{1}{3}\left(x - \left(\tfrac{1}{2}, 1\right)\right) + \left(\tfrac{1}{2}, 1\right);$$

$$\psi_6(x) = \tfrac{1}{3}\left(x - (0, 1)\right) + (0, 1);$$

$$\psi_7(x) = \tfrac{1}{3}\left(x - \left(0, \tfrac{1}{2}\right)\right) + \left(0, \tfrac{1}{2}\right);$$

$$\psi_8(x) = \tfrac{1}{3}x.$$

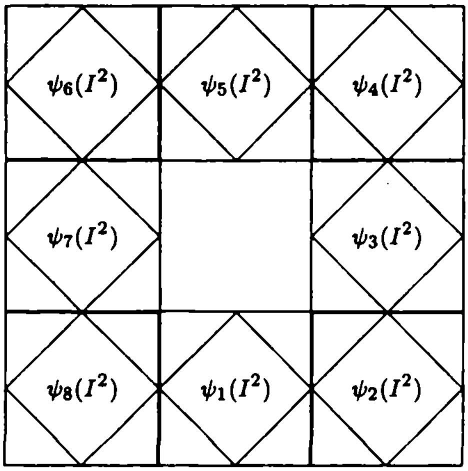

Figure 1: Transformations that define the Sierpiński carpet

These mappings are compositions of scaling with factor $\frac{1}{3}$ and translation. W is the Hutchinson operator associated with these contractions, i.e. the mapping acting on subsets of the plane as follows

$$\mathbf{R}^2 \supset \mathcal{A} \mapsto W(\mathcal{A}) \stackrel{def}{=} \bigcup_{i=1}^{8} \psi_i(\mathcal{A}).$$

By Hutchinson's results we know that if we restrict our attention to compact subsets of the plane, $\mathcal{H}(\mathbf{R}^2)$ and endow this space with the Hausdorff metric ρ_H, then

$$W : (\mathcal{H}(\mathbf{R}^2), \rho_H) \to (\mathcal{H}(\mathbf{R}^2), \rho_H)$$

is a contraction with contraction factor $\frac{1}{3}$. Consequently W has a unique fixed point $\mathcal{C}$ and this fixed point is asymptotically stable — for any compact nonempty set $\mathcal{A} \subset \mathbf{R}^2$ we have

$$W^n(\mathcal{A}) \stackrel{n\to\infty}{\longrightarrow} \mathcal{C} \quad \text{in the Hausdorff metric,}$$

$\mathcal{C}$ is called he attractor of the system $\{\psi_1, \ldots, \psi_8\}$.

In particular we may take $\mathcal{A} = V^{(0)} = \{x_1, x_2, x_3, x_4\}$, where $x_1 = (\frac{1}{2}, 0)$, $x_2 = (1, \frac{1}{2})$, $x_3 = (\frac{1}{2}, 1)$, $x_4 = (0, \frac{1}{2})$ (midpoints of the edges of the unit square). Then

$$V^{(n)} \stackrel{def}{=} W^n(V^{(0)}) \stackrel{n\to\infty}{\longrightarrow} \mathcal{C}$$

in the Hausdorff metric.

Observe that points x_1, x_2, x_3, x_4 are fixed points of mappings $\psi_1, \psi_3, \psi_5, \psi_7$ respectively and are the essential fixed points of the system Ψ in the sense of Lindstrøm ([12]).

A *conductivity matrix* on $V^{(0)}$ is, by definition (see [1], [3]), a square matrix $A = (a_{ij})_{i,j=1}^{4}$, whose entries (called 'conductances between vertices') satisfy the following conditions:

1. $\forall i \neq j \;\; a_{ij} \geq 0$;
2. $\forall i \neq j \;\; a_{ij} = a_{ji}$;
3. $\forall i = 1,2,3,4 \;\; \sum_{j=1}^{4} a_{ij} = 0$.

We equip $V^{(0)}$ with natural graph structure associated with A :

$$E_A^{(0)} = \{\langle x_i, x_j \rangle : a_{ij} > 0,\; i,j = 1, \ldots, 4,\; i \neq j\}.$$

The matrix A is called *irreducible* if $(V^{(0)}, E_A^{(0)})$ is connected as a graph.

The conductivity matrix A gives raise to its Dirichlet form $\mathcal{E}_A^{(0)}$:

$$\text{for } f : V^{(0)} \to \mathbf{R} \qquad \mathcal{E}_A^{(0)}(f,f) \stackrel{def}{=} \frac{1}{2} \sum_{i,j=1}^{4} a_{ij} \cdot (f(x_i) - f(x_j))^2$$

and to its Dirichlet operator $\Delta_A^{(0)}$:

$$\Delta_A^{(0)} f(x_i) \stackrel{def}{=} \frac{1}{2} \sum_{j=1}^{4} a_{ij} \cdot (f(x_j) - f(x_i)),\; i = 1,2,3,4.$$

They are related through:

$$\text{(1)} \qquad \forall f,g : V^{(0)} \to \mathbf{R} \qquad \mathcal{E}_A^{(0)}(f,g) = -\langle \Delta_A^{(0)} f, g \rangle_{L^2(V^{(0)})}.$$

With the iterated functions system Ψ we now 'reproduce' the form $\mathcal{E}_A^{(0)}$ to the form $\mathcal{E}_A^{(1)}$, acting on functions $g : V^{(1)} \to \mathbf{R}$. This form is defined as follows:

$$\text{(2)} \qquad \mathcal{E}_A^{(1)}(g,g) \stackrel{def}{=} \sum_{i=1}^{8} \mathcal{E}_A^{(0)}(g \circ \psi_i, g \circ \psi_i)$$

(the Dirichlet form associated with the 'reproduced' conductivity matrix $A^{(1)}$ on $V^{(1)}$). The Dirichlet operator associated with this form will be denoted by $\Delta_A^{(1)}$.

By $\mathcal{R}$ we denote the correspondence $\mathcal{E}_A^{(0)} \mapsto \mathcal{E}_A^{(1)}$. It is called the *reproduction map*.

Next, if a Dirichlet form $\tilde{\mathcal{E}}$ (or a conductivity matrix $\tilde{A}$) on $V^{(1)}$ is given, it can be mapped to a Dirichlet form $\tilde{\mathcal{E}}^{(0)}$ on $V^{(0)}$ in a following way

$$\text{for } \; f : V^{(0)} \to \mathbf{R},$$

$$\text{(3)} \qquad \tilde{\mathcal{E}}^{(0)}(f,f) \stackrel{def}{=} \inf\{\tilde{\mathcal{E}}(g,g) : g : V^{(1)} \to \mathbf{R}, g|_{V^{(0)}} = f\}.$$

This infimum is attained on a unique function g, called the harmonic extension of f with respect to the conductivity matrix $\tilde{A}$. This harmonic extension satisfies

$$g|_{V^{(0)}} = f;$$
$$\text{for } x \in V^{(1)} \setminus V^{(0)}, \quad \Delta_A^{(1)} g(x) = 0. \tag{4}$$

The operation $\widetilde{\mathcal{E}} \mapsto \widetilde{\mathcal{E}}^{(0)}$ is called the decimation map and will be denoted by $\mathcal{D}$.

Consider now the composition $\mathcal{D} \circ \mathcal{R}$. It maps Dirichlet forms on $C(V^{(0)})$ $(= \{f : V^{(0)} \to \mathbf{R}\})$ into Dirichlet forms on $C(V^{(0)})$.

DEFINITION 1. A conductivity matrix A is called a harmonic structure on $V^{(0)}$, if it is an 'eigenform' of the mapping $\mathcal{D} \circ \mathcal{R}$, i.e. if there exists a real number $\lambda > 0$ such that

$$(\mathcal{D} \circ \mathcal{R})(\mathcal{E}_A^{(0)}) = \lambda \cdot \mathcal{E}_A^{(0)}.$$

If this conductivity matrix is nondegenerate, then the harmonic structure is called nondegenerate, too.

We first prove that on $V^{(0)}$ such a structure exists (theorem 1). This theorem does not come as a surprise, as existence of nondegenerate harmonic structures was established on several graphs (finitely ramified fractals for example). Another question is if this structure is unique. The general answer is no, see the example of Vicsek set in [13], worked out by Metz. On the Vicsek set there exists a whole one-parametric family of nondegenerate harmonic structures. However, among them only one is symmetric. On the other hand, some other fractals (Sierpiński gasket, Lindstrøm snowflake) admit only one nondegenerate harmonic structure altogether. It turns out that the Sierpiński carpet falls into the latter class — there is only one nondegenerate harmonic structure on it (up to a multiplicative constant of course).

3. Existence of a nondegenerate harmonic structure

Harmonic structures related to symmetric random walks on symmetric state-spaces must share their symmetry properties. In this section we show that a nondegenerate symmetric harmonic structure on $V^{(0)}$ exists. From the proof it will follow that it is unique up to a multiplicative constant. Moreover we get the precise value of the eigenvalue λ. We have the following:

THEOREM 1. *There exists a unique, up to a multiplicative constant, symmetric nondegenerate harmonic structure on* $V^{(0)}$, *with eigenvalue*

$$\lambda_0 = \frac{11 - \sqrt{73}}{6} \approx 0.4093\ldots$$

Proof. By 'symmetric' structures we mean those that remain invariant under the symmetries of the set $V^{(0)}$. Therefore entries of a symmetric (with respect to the symmetries of V_0) conductivity matrix satisfy:

$$a_{13} = a_{24} = \alpha \geq 0$$

and

$$a_{12} = a_{23} = a_{34} = a_{14} = \beta \geq 0.$$

If $\beta = 0$ then this structure is degenerate: the graph $(V^{(0)}, E_A^{(0)})$ is not connected.

Suppose then $\beta > 0$; without loss of generality we can assume $\beta = 1$. Denote the resulting matrix by A_α and its Dirichlet form by $\mathcal{E}^{\alpha,(0)}$.

For an arbitrary $f : V^{(0)} \to \mathbf{R}$, setting $B_i = f(x_i)$, $i = 1, 2, 3, 4$, we have

$$\mathcal{E}^{\alpha,(0)}(f, f) = (B_2 - B_1)^2 + (B_3 - B_2)^2 + (B_4 - B_3)^2 + (B_1 - B_4)^2$$
$$\alpha(B_3 - B_1)^2 + \alpha(B_4 - B_2)^2.$$

Consider now the reproduced form $\mathcal{E}^{\alpha,(1)}$, defined for $f : V^{(1)} \to \mathbf{R}$ according to (2) by

$$\mathcal{E}^{\alpha,(1)}(f, f) = \sum_{i=1}^{8} \mathcal{E}^{\alpha,(0)}(f \circ \psi_i, f \circ \psi_i).$$

We investigate its decimation, $\mathcal{D}\mathcal{E}^{\alpha,(1)}$.

Suppose $f : V^{(1)} \to \mathbf{R}$ is given. We will find

$$\mathcal{D}\mathcal{E}^{\alpha,(1)}(f, f) = \inf\{\mathcal{E}^{\alpha,(1)}(g, g) : g : V^{(1)} \to \mathbf{R},$$
$$g(x_i) = f(x_i) = B_i, i = 1, 2, 3, 4\}.$$

Let g be the harmonic extension of f (the unique function realizing this infimum).

According to notations from the figure 2, denoting $a_i = g(p_i), b_i = g(q_i), c_i = g(r_i), d_i = g(s_i), e_i = g(t_i)$ we arrive at the following system of equations that the values of g must satisfy:

$$(a_1 - B_1) + (a_1 - c_1) + (a_1 - d_1) + (a_1 - b_4) + \alpha(a_1 - b_1) + \alpha(a_1 - e_4) = 0,$$
$$(b_1 - B_1) + (b_1 - c_1) + (b_1 - e_1) + (b_1 - a_2) + \alpha(b_1 - a_1) + \alpha(b_1 - d_2) = 0,$$
$$(c_1 - a_1) + (c_1 - b_1) + \alpha(c_1 - B_1) = 0,$$
$$(d_1 - e_4) + (d_1 - a_1) + \alpha(d_1 - b_4) = 0,$$
$$(e_1 - b_1) + (e_1 - d_2) + \alpha(e_1 - a_2) = 0,$$

plus the next 15 equations obtained by cyclic substitution ($1 \leftarrow 2$, $2 \leftarrow 3$, $3 \leftarrow 4$, $4 \leftarrow 1$). Eliminating e_i's and d_i's and using symmetry one reaches the

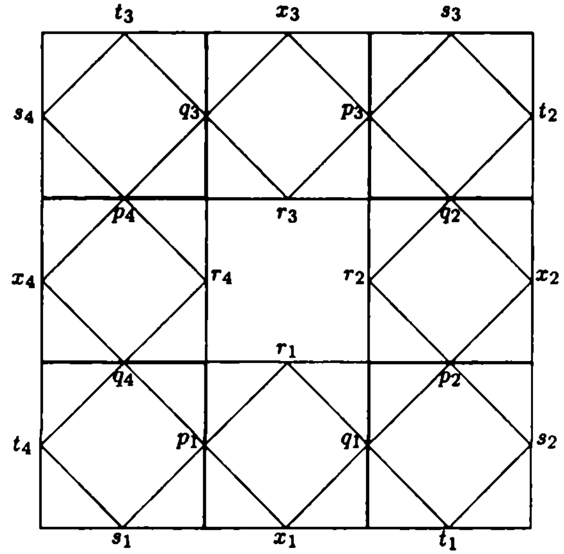

Figure 2: Notation for the proof of Theorem 1

solution:

$$\begin{aligned}
a_1 &= \tfrac{7\alpha^2+42\alpha+59}{3(\alpha+3)(5\alpha+11)}B_1 + \tfrac{(3\alpha+7)(\alpha+1)}{3(\alpha+3)(5\alpha+11)}B_2 + \tfrac{2(\alpha+1)(\alpha+2)}{3(\alpha+3)(5\alpha+11)}B_3 + \tfrac{3\alpha^2+20\alpha+29}{3(\alpha+3)(5\alpha+11)}B_4,\\
b_1 &= \tfrac{7\alpha^2+42\alpha+59}{3(\alpha+3)(5\alpha+11)}B_1 + \tfrac{3\alpha^2+20\alpha+29}{3(\alpha+3)(5\alpha+11)}B_2 + \tfrac{2(\alpha+1)(\alpha+2)}{3(\alpha+3)(5\alpha+11)}B_3 + \tfrac{(3\alpha+7)(\alpha+1)}{3(\alpha+3)(5\alpha+11)}B_4,\\
c_1 &= \tfrac{15\alpha^2+62\alpha+59}{3(\alpha+3)(5\alpha+11)}B_1 + \tfrac{2}{5\alpha+11}B_2 + \tfrac{4(\alpha+1)}{3(\alpha+3)(5\alpha+11)}B_3 + \tfrac{2}{5\alpha+11}B_4,\\
d_1 &= \tfrac{(\alpha+3)(3\alpha+7)}{3(\alpha+3)(5\alpha+11)}B_1 + \tfrac{2(\alpha+1)}{3(5\alpha+11)}B_2 + \tfrac{(\alpha+1)(3\alpha+5)}{3(\alpha+3)(5\alpha+11)}B_3 + \tfrac{7\alpha+13}{3(5\alpha+11)}B_4,\\
e_1 &= \tfrac{(\alpha+3)(3\alpha+7)}{3(\alpha+3)(5\alpha+11)}B_1 + \tfrac{7\alpha+13}{3(5\alpha+11)}B_2 + \tfrac{(\alpha+1)(3\alpha+5)}{3(\alpha+3)(5\alpha+11)}B_3 + \tfrac{2(\alpha+1)}{3(5\alpha+11)}B_4,
\end{aligned}$$

and so on, cyclically.

Careful computation gives:

$$\begin{aligned}
&\left(\mathcal{D}\mathcal{E}^{\alpha,(1)}\right)(f,f) =\\
&= \tfrac{4(\alpha+1)(\alpha+3)(23\alpha+53)}{[3(\alpha+3)]^2(5\alpha+11)}\left\{(B_2-B_1)^2+(B_3-B_2)^2+(B_4-B_3)^2+(B_1-B_4)^2\right\}\\
&\quad + \tfrac{32(\alpha+1)(\alpha+3)(2\alpha+5)}{[3(\alpha+3)]^2(5\alpha+11)}\left\{(B_2-B_1)(B_4-B_3)+(B_3-B_2)(B_1-B_4)\right\}\\
&\quad + \tfrac{8(\alpha+1)(\alpha+3)(11\alpha+23)}{[(3(\alpha+3)]^2(5\alpha+11)}\left\{(B_2-B_1)(B_3-B_2)+(B_3-B_2)(B_4-B_3)\right.\\
&\qquad \left.+(B_4-B_3)(B_1-B_4)+(B_1-B_4)(B_2-B_1)\right\}.
\end{aligned}$$

In view of the identity

$$(B_2 - B_1)(B_3 - B_2) + (B_3 - B_2)(B_4 - B_3) +$$
$$(B_4 - B_3)(B_1 - B_4) + (B_1 - B_4)(B_2 - B_1) =$$
$$-\tfrac{1}{2}\left\{(B_2 - B_1)^2 + (B_3 - B_2)^2 + (B_4 - B_3)^2 + (B_1 - B_4)^2\right\}$$
$$-(B_2 - B_1)(B_4 - B_3) - (B_1 - B_4)(B_3 - B_2)$$

we have

$$\left(\mathcal{D}\mathcal{E}^{\alpha,(1)}\right)(f,f) =$$
$$= \tfrac{4\cdot 6\cdot(\alpha+1)(2\alpha+5)}{9(\alpha+3)(5\alpha+11)}\left\{(B_2 - B_1)^2 + (B_3 - B_2)^2 + (B_4 - B_3)^2 + (B_1 - B_4)^2\right\}$$
$$-\tfrac{3\cdot 8(\alpha+1)^2}{9(\alpha+3)(5\alpha+11)}\left\{(B_2 - B_1)(B_4 - B_3) + (B_3 - B_2)(B_1 - B_4)\right\}.$$

But

$$(B_2 - B_1)^2 + (B_3 - B_2)^2 + (B_4 - B_3)^2 + (B_1 - B_4)^2 =$$
$$\tfrac{2}{2+\alpha}\mathcal{E}^{\alpha,(0)}(f,f) + \tfrac{2\alpha}{2+\alpha}\{(B_2 - B_1)(B_4 - B_3) + (B_3 - B_2)(B_1 - B_4)\},$$

so

$$\left(\mathcal{D}\mathcal{E}^{\alpha,(1)}\right)(f,f) =$$
$$= a_1\mathcal{E}^{\alpha,(0)}(f,f) + a_2\{(B_2 - B_1)(B_4 - B_3) + (B_3 - B_2)(B_1 - B_4)\},$$

where

$$a_1 = \frac{4\cdot 6(\alpha+1)(2\alpha+5)}{9(\alpha+3)(5\alpha+11)}\cdot\frac{2}{2+\alpha}$$

and

$$a_2 = \left[\frac{2\alpha}{2+\alpha}\cdot\frac{4\cdot 6(\alpha+1)(2\alpha+5)}{9(\alpha+3)(5\alpha+11)} - \frac{3\cdot 8(\alpha+1)^2}{9(\alpha+3)(5\alpha+11)}\right].$$

We are looking for α such that the coefficient a_2 in the expression above vanishes. Therefore α must solve the equation

$$\frac{2\alpha}{(\alpha+2)}(2\alpha+5) = (\alpha+1).$$

This equation has only one positive solution,

$$\alpha_0 = \frac{\sqrt{73}-7}{6} \approx 0.2573\ldots$$

For this choice of $\alpha = \alpha_0$

$$(\mathcal{D}\circ\mathcal{R})\mathcal{E}^{\alpha_0,(0)} = \lambda_0\cdot\mathcal{E}^{\alpha_0,(0)}$$

with $\lambda_0 = \frac{11-\sqrt{73}}{6} \approx 0.4093\ldots$ A_{α_0} is the unique (up to a multiplicative constant) symmetric nondegenerate harmonic structure which we were looking for. □

REMARKS. In the case of simple nested fractals existence of a nondegenerate harmonic structure on $V^{(0)}$ is sufficient for constructing a regular Dirichlet form on the complete fractal — as the decimation invariance is honestly inherited in subsequent steps. See [1], [5], [6], [9], [10]. It is clear that for infinitely ramified fractals like the Sierpiński carpet there is no obvious reason why the just established one-step decimation invariance should carry over to next steps. In order to pass to the limit with n one should find proper constants to normalize with and this is still an open question (for SNF's the n-th constant would be just λ_0^n).

4. Uniqueness of the nondegenerate harmonic structure

Now we prove that the harmonic structure from the previous section is unique not only among the symmetric ones, but altogether unique.

THEOREM 2. *Suppose that A is a nondegenerate conductivity matrix on $V^{(0)}$ such that for some $\lambda > 0$*

$$(\mathcal{D} \circ \mathcal{R})\mathcal{E}_A^{(0)} = \lambda \cdot \mathcal{E}_A^{(0)}. \tag{5}$$

Then $\lambda = \lambda_0$ and $A = A_{\alpha_0}$.

Proof. Suppose that $A = (a_{ij})_{i,j=1}^4$ is a conductivity matrix satisfying (5).

First, by [7] we know that the eigenvalue in (5) is uniquely determined by geometric properties of $V^{(0)}$, so $\lambda = \lambda_0 = \frac{11-\sqrt{73}}{6}$ (in fact all we will need for uniqueness is that $\lambda_0 \neq 1$ and $\lambda_0 \neq \frac{2}{3}$).

Irreducibility of A is equivalent to (see [13]):

$$\Big(W \subset V^{(0)},\ \mathcal{E}_A^{(0)}(\mathbf{1}_W, \mathbf{1}_W) = 0\Big) \Rightarrow W = V^{(0)}.$$

We have also (see [13]):

1. $\ker \Delta_A^{(0)}$ consists of constant functions only,
2. If $\beta_1, \beta_2, \beta_3, \beta_4$ are real numbers such that $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 0$, then there exists a unique function g_β satisfying

$$g(x_1) = 0,\ \ (\Delta_A^{(0)} g)(x_i) = \beta_i,\ i = 1, 2, 3, 4. \tag{6}$$

For brevity we write $\Delta_A^{(0)} g = (\beta_1, \beta_2, \beta_3, \beta_4)$.

Let f, g, h be the unique functions on $V^{(0)}$ such that

$$f(x_1) = 0,\quad \Delta_A^{(0)} f = (1,\ -1,\ 0,\ 0), \tag{7}$$

$$g(x_1) = 0,\quad \Delta_A^{(0)} g = (1,\ 0,\ -1,\ 0), \tag{8}$$

(9) $$h(x_1) = 0, \quad \Delta_A^{(0)} h = (1,\, 0,\, 0,\, -1).$$

Suppose $g_1 : V^{(1)} \to \mathbf{R}$ is the harmonic extension of g with respect to the conductivity matrix A, i.e. the unique function on $V^{(1)}$ such that

(10) $$\begin{gathered} g_1|_{V^{(0)}} = g, \\ \forall x \in V^{(1)} \setminus V^{(0)} \;\; \Delta_A^{(1)} g(x) = 0. \end{gathered}$$

The function g_1 is realizing the infimum in the variational problem (3), so

$$\mathcal{E}_A^{(1)}(g_1, g_1) = (\mathcal{D} \circ \mathcal{R}) \mathcal{E}_A^{(0)}(g, g).$$

As A and $A^{(1)}$ are both irreducible, we have (lemma 2.5 of [13]) that

$$\forall \phi : V^{(0)} \to \mathbf{R} \qquad \sum_{x \in V^{(0)}} \Delta_A^{(0)} \phi(x) = 0$$

and

$$\forall \phi : V^{(1)} \to \mathbf{R} \qquad \sum_{x \in V^{(1)}} \Delta_A^{(1)} \phi(x) = 0$$

(indeed:

$$\begin{aligned} \left| \textstyle\sum_{x \in V^{(0)}} \Delta_A^{(0)} \phi(x) \right| &= \left| \langle \Delta_A^{(0)} \phi, \mathbf{1}_{V^{(0)}} \rangle_{L^2(V^{(0)})} \right| = \left| \mathcal{E}_A^{(0)}(\phi, \mathbf{1}_{V^{(0)}}) \right| \\ &\leq \left(\left| \mathcal{E}_A^{(0)}(\phi, \phi) \cdot \mathcal{E}_A^{(0)}(\mathbf{1}_{V^{(0)}}, \mathbf{1}_{V^{(0)}}) \right| \right)^{\frac{1}{2}} = 0 \end{aligned}$$

and similarly for the other identity).

In particular, applying these properties to $g_1 \circ \psi_i$ ($i = 1, \ldots, 8$) on $V^{(0)}$ and to g_1 on $V^{(1)}$, we have

(11) $$\sum_{x \in V^{(0)}} \Delta_A^{(0)} (g_1 \circ \psi_i)(x) = 0, \quad i = 1, \ldots, 8,$$

(12) $$\sum_{x \in V^{(1)}} \Delta_A^{(1)} g_1(x) = 0.$$

The invariance property (5) written in terms of the Dirichlet operators can be read as

(13) $$\lambda_0 \Delta_A^{(0)} g(x_i) = \Delta_A^{(1)} g_1(x_i) = \Delta_A^{(0)} (g_1 \circ \psi_{2i-1})(x_i), \; i = 1, 2, 3, 4.$$

Using (10), (11), (12) and (13) we arrive at the following:

$$\tag{14} \begin{aligned} \Delta_A^{(0)}(g_1 \circ \psi_1) &= (\lambda_0,\, c_g,\, 0,\, -c_g - \lambda_0), \\ \Delta_A^{(0)}(g_1 \circ \psi_2) &= (0,\, 0,\, c_g,\, -c_g), \\ \Delta_A^{(0)}(g_1 \circ \psi_3) &= (-c_g,\, 0,\, c_g,\, 0), \\ \Delta_A^{(0)}(g_1 \circ \psi_4) &= (-c_g,\, 0,\, 0,\, c_g), \\ \Delta_A^{(0)}(g_1 \circ \psi_5) &= (0,\, -c_g,\, -\lambda_0, c_g + \lambda_0) \\ \Delta_A^{(0)}(g_1 \circ \psi_6) &= (c_g + \lambda_0,\, -c_g - \lambda_0,\, 0,\, 0), \\ \Delta_A^{(0)}(g_1 \circ \psi_7) &= (c_g + \lambda_0,\, 0,\, -c_g - \lambda_0,\, 0), \\ \Delta_A^{(0)}(g_1 \circ \psi_8) &= (0,\, c_g + \lambda_0,\, -c_g - \lambda_0, 0), \end{aligned}$$

where c_g is some constant (we will later see that $c_g = -1/2\lambda_0$).

Property (6) applied to these equations gives:

$$\tag{15} \begin{aligned} g_1 \circ \psi_1 &= (\lambda_0 + c_g) \cdot h - c_g \cdot f + s_1, \\ g_1 \circ \psi_2 &= c_g \cdot (h - g) + s_2, \\ g_1 \circ \psi_3 &= -c_g \cdot g + s_3, \\ g_1 \circ \psi_4 &= -c_g \cdot h + s_4, \\ g_1 \circ \psi_5 &= c_g \cdot f + \lambda_0 \cdot g - (c_g + \lambda_0) \cdot h + s_5, \\ g_1 \circ \psi_6 &= (c_g + \lambda_0) \cdot f + s_6, \\ g_1 \circ \psi_7 &= (c_g + \lambda_0) \cdot g + s_7, \\ g_1 \circ \psi_8 &= (c_g + \lambda_0) \cdot (g - f) + s_8. \end{aligned}$$

Constants $s_1, \ldots, s_8$ can be determined from compatibility conditions (such as for example $g_1 \circ \psi_1(x_2) = g_1 \circ \psi_2(x_4)$). However, since their value will be irrelevant for our purposes we skip this computation.

Next, we have

LEMMA 1. *The following relations*

$$\tag{16} f(x_3) = g(x_2),\ f(x_4) = h(x_2),\ h(x_3) = g(x_4)$$

hold true.

Proof. Consider the Dirichlet form $\mathcal{E}_A^{(0)}$ and recall that f, g, h satisfy (7), (8) and (9). Therefore

$$\mathcal{E}_A^{(0)}(f, g) = \langle -\Delta_A^{(0)} f, g \rangle = g(x_2) = \langle -\Delta_A^{(0)} f, g \rangle = f(x_3),$$

and similarly for the other equalities. □

For later use we note

$$\mathcal{E}_A^{(0)}(f,f) = f(x_2),\ \mathcal{E}_A^{(0)}(f,g) = g(x_2),$$

$$\mathcal{E}_A^{(0)}(g,g) = g(x_3),\ \mathcal{E}_A^{(0)}(f,h) = h(x_2), \tag{17}$$

$$\mathcal{E}_A^{(0)}(h,h) = h(x_4),\ \mathcal{E}_A^{(0)}(h,g) = g(x_4).$$

As g_1 agrees with g on $V^{(0)}$ and $g(x_1) = 0$, we have the following compatibility conditions:

$$\begin{aligned} g(x_2) = {} & (g_1 \circ \psi_1(x_2) - g_1 \circ \psi_1(x_1)) + (g_1 \circ \psi_2(x_3) - g_1 \circ \psi_2(x_4)) \\ & + (g_1 \psi_3(x_2) - g_1 \circ \psi_3(x_1)), \\ & \text{(along the edge } \langle x_1, x_2 \rangle), \end{aligned}$$

$$\begin{aligned} g(x_3) - g(x_2) = {} & (g_1 \circ \psi_3(x_3) - g_1 \circ \psi_3(x_2)) + (g_1 \circ \psi_4(x_4) - g_1 \circ \psi_4(x_1)) \\ & + (g_1 \circ \psi_5(x_3) - g_1 \circ \psi_5(x_2)), \\ & \text{(along the edge } \langle x_2, x_3 \rangle), \end{aligned}$$

$$\begin{aligned} g(x_4) = {} & (g_1 \circ \psi_1(x_4) - g_1 \circ \psi_1(x_1)) + (g_1 \circ \psi_8(x_3) - g_1 \circ \psi_8(x_2)) \\ & + (g_1 \circ \psi_7(x_4) - g_1 \circ \psi_7(x_1)), \\ & \text{(along the edge } \langle x_1, x_4 \rangle), \end{aligned}$$

$$\begin{aligned} g(x_3) - g(x_4) = {} & (g_1 \circ \psi_7(x_3) - g_1 \circ \psi_7(x_4)) + (g_1 \circ \psi_6(x_2) - g_1 \circ \psi_6(x_1)) \\ & + (g_1 \circ \psi_5(x_3) - g_1 \circ \psi_5(x_4)) \\ & \text{(along the edge } \langle x_4, x_3 \rangle). \end{aligned}$$

Substituting (15) for the values of $g_1 \circ \psi_i$, using (16) and rearranging we obtain:

$$\begin{aligned} & (1 + c_g)g(x_2) + c_g g(x_3) - 2c_g g(x_4) = \\ & \qquad = -c_g f(x_2) + (\lambda_0 + c_g)h(x_2) - c_g h(x_4), \\ & (-1 + \lambda_0 - 2c_g)g(x_2) + (1 + c_g - \lambda_0)g(x_3) + (\lambda_0 + c_g)g(x_4) = \\ & \qquad = -c_g f(x_2) + (\lambda_0 + c_g)h(x_2) - c_g h(x_4), \\ & (2c_g + 2\lambda_0)g(x_2) - (c_g + \lambda_0)g(x_3) + (1 - c_g - \lambda_0)g(x_4) = \\ & \qquad (c_g + \lambda_0)f(x_2) - c_g f(x_4) + (\lambda_0 + c_g)h(x_4), \\ & -c_g g(x_2) + (1 - c_g - 2\lambda_0)g(x_3) + (2c_g + 3\lambda_0 - 1)g(x_4) = \\ & \qquad (c_g + \lambda_0)f(x_2) - c_g f(x_4) + (\lambda_0 + c_g)h(x_4). \end{aligned}$$

Comparing the first equation with the second and the third equation with the fourth we arrive at:

$$(2 + 3c_g - \lambda_0)g(x_2) - (1 - \lambda_0)g(x_3) - (3c_g + \lambda_0)g(x_4) = 0,$$

$$(3c_g + 2\lambda_0)g(x_2) - (1 - \lambda_0)g(x_3) + (2 - 3c_g - 4\lambda_0)g(x_4) = 0.$$

Since $2 - 3\lambda_0 \neq 0$ and $\lambda_0 \neq 1$, these equations yield

(18) $$g(x_2) = g(x_4) \quad \text{and} \quad g(x_3) = 2g(x_2).$$

Recall now that $\Delta_A^{(0)} g = (1, 0, -1, 0)$, so

$$a_{12}\, g(x_2) + a_{13}\, g(x_3) + a_{14}\, g(x_4) = 1,$$

$$-(a_{12} + a_{23} + a_{24})g(x_2) + a_{23}\, g(x_3) + a_{24}\, g(x_4) = 0,$$

$$a_{24}\, g(x_2) + a_{34}\, g(x_3) - (a_{14} + a_{24} + a_{34})g(x_4) = 0.$$

Substitute (18) in these equations, obtaining

(19) $$a_{12} = a_{23} \quad \text{and} \quad a_{14} = a_{34}.$$

Applying this reasoning to functions $\bar{f}$, $\bar{g}$, $\bar{h}$ defined by

(20) $$\bar{f}(x_2) = 0, \;\; \Delta_A^{(0)} \bar{f} = (0,\, 1,\, -1,\, 0),$$

(21) $$\bar{g}(x_2) = 0, \;\; \Delta_A^{(0)} \bar{g} = (0,\, 1,\, 0,\, -1),$$

(22) $$\bar{h}(x_2) = 0, \;\; \Delta_A^{(0)} \bar{h} = (-1,\, 1,\, 0,\, 0)$$

we conclude that

$$a_{23} = a_{34} \quad \text{and} \quad a_{12} = a_{14}.$$

The last relations together with (19) give

(23) $$a_{12} = a_{23} = a_{34} = a_{14}.$$

As a corollary we obtain that (we know that the picture is symmetric)

(24) $$c_g = -\frac{1}{2}\lambda_0.$$

To complete the proof it remains to be shown that

$$a_{13} = a_{24}.$$

Without loss of generality we may assume that

(25) $$a_{12} = a_{23} = a_{34} = a_{14} = 1, \;\; a_{13} = \gamma \geq 0, \;\; a_{24} = \delta \geq 0.$$

Consider $\mathcal{E}_A^{(1)}(g_1, g_1)$. Since the structure is assumed to be invariant, one has

(26) $$\mathcal{E}_A^{(1)}(g_1, g_1) = \lambda_0 \cdot \mathcal{E}_A^{(0)}(g, g).$$

Explicit formulae for g_1 are known (equations (15)), so in view of (24) we obtain that

$$\mathcal{E}_A^{(1)}(g_1,g_1) = \sum_{i=1}^{8} \mathcal{E}_A^{(0)}(g_1\circ\psi_i, g_1\circ\psi_i) \tag{27}$$
$$= \frac{\lambda_0^2}{2}\left[\, 2\mathcal{E}_A^{(0)}(f,f) + 4\mathcal{E}_A^{(0)}(g,g) + 2\mathcal{E}_A^{(0)}(h,h) \right.$$
$$\left. -3\mathcal{E}_A^{(0)}(f,g) + 2\mathcal{E}_A^{(0)}(f,h) - 3\mathcal{E}_A^{(0)}(h,g)\right].$$

Solving the problems (7), (8), (9) with conductances given by (25) we get:

$$\begin{aligned} f(x_2) = h(x_4) &= \frac{(2+\delta)(2+\gamma)-1}{4(1+\delta)(1+\gamma)}, \\ f(x_3) = h(x_3) &= \frac{1}{2(1+\gamma)}, \\ f(x_4) = h(x_2) &= \frac{1+\delta(2+\gamma)}{4(1+\delta)(1+\gamma)}, \\ g(x_2) = g(x_4) &= \frac{1}{2(1+\gamma)}, \\ g(x_3) &= \frac{1}{1+\gamma}. \end{aligned} \tag{28}$$

Comparing (26) and (27), in view of (17) we obtain that

$$\lambda_0^2(2f(x_2) + 2g(x_3) + f(x_4) - 3g(x_4)] = \lambda_0 g(x_3).$$

Using (28) and rearranging we obtain that

$$\lambda_0 \cdot \left[\frac{1}{2} + \frac{2(2+\delta)(2+\gamma) - 1 + \delta(2+\gamma)}{4(1+\delta)}\right] = 1. \tag{29}$$

Similar reasoning performed for $\bar{f}, \bar{g}, \bar{h}$ (defined by (20), (21), (22)) leads to identical equation, but with δ and γ interchanged:

$$\lambda_0 \cdot \left[\frac{1}{2} + \frac{2(2+\delta)(2+\gamma) - 1 + \gamma(2+\delta)}{4(1+\gamma)}\right] = 1. \tag{30}$$

Therefore

$$\frac{2(2+\delta)(2+\gamma) - 1 + \gamma(2+\delta)}{(1+\gamma)} = \frac{2(2+\delta)(2+\gamma) - 1 + \delta(2+\gamma)}{(1+\delta)},$$

and further

$$(\delta-\gamma)[5 + 4(\gamma+\delta) + 3\gamma\delta] = 0.$$

Since γ and δ are not allowed to be negative, we end up with

$$\gamma = \delta.$$

and the proof of Theorem 2 is completed. □

5. The probabilistic interpretation

Let $(\Omega, \mathcal{M}, \mathbf{P})$ be a given probability space. Suppose that $(X_n^{\alpha_0,(0)})_{n\geq 0}$ is the random walk on $V^{(0)}$ associated with the irreducible conductivity matrix A_{α_0}, i.e. the discrete time Markov chain on $V^{(0)}$ with transition probabilities:

$$p_{i,j} = \frac{a_{ij}}{\sum_{k\neq i} a_{ik}}, \quad i,j = 1,2,3,4, \ i \neq j,$$

so

$$p_{1,2} = p_{1,4} = p_{2,1} = p_{2,3} = p_{3,2} = p_{3,4} = p_{4,3} = p_{4,1} \stackrel{def}{=} p_2 = \tfrac{1}{2+\alpha_0},$$
$$p_{1,3} = p_{2,4} = p_{3,1} = p_{4,2} \stackrel{def}{=} p_3 = \tfrac{\alpha_0}{2+\alpha_0}.$$

This Markov chain is then 'reproduced' to a Markov chain $(X_n^{\alpha_0,(1)})_{n\geq 0}$ on $V^{(1)}$. Its transition probabilities basically coincide with those on the 0-level, with 'choice of small square we are about to enter' added. Rigorously speaking, its transition probabilities are as follows. Let $x \in V^{(1)}$. Then $x = \psi_i(x_k)$ for some $i \in \{1, \ldots 8\}$ and $x_k \in V^{(0)}$. If there is only one such i, then for $y \in V^{(1)}$ we have

$$p_{x,y}^{(1)} = \begin{cases} p_{k,l} & \text{for } y = \psi_i(x_k), \\ 0 & \text{else.} \end{cases}$$

If there are two distinct indices i_1, i_2 and two different x_{k_1}, x_{k_2} such that $x = \psi_{i_1}(x_{k_1}) = \psi_{i_2}(x_{k_2})$, then

$$p_{x,y}^{(1)} = \begin{cases} \frac{1}{2}p_{k_1,l} & \text{for } y = \psi_{i_1}(x_l), \\ \frac{1}{2}p_{k_2,l} & \text{for } y = \psi_{i_2}(x_l), \\ 0 & \text{else.} \end{cases}$$

REMARK. The construction above can be carried out for an arbitrary irreducible conductivity matrix A.

We will now be keeping track of successive hits of $V^{(0)}$ by the random walk $(X_n^{\alpha_0,(1)})_{n\geq 0}$. Let $\sigma = \inf\{n \geq 0 : X_n^{\alpha_0,(1)} \in V^{(0)} \setminus \{X_0^{\alpha_0,(1)}\}\}$. Then $\{T_n\}$ given by

$$T_0 = 0$$
$$T_{n+1} = T_n + \theta_{T_n} \circ \sigma, \quad n = 0,1,2,\ldots$$

is the sequence of successive hits o f $V^{(0)}$ by the random walk $(X_n^{\alpha_0,(1)})_{n\geq 0}$. Set

$$\tilde{p}_{i,j} = \mathbf{P}_{x_i}\left[X_\sigma^{\alpha_0,(1)} = x_j\right], \quad i,j = 1,2,3,4.$$

The invariance of the Dirichlet form associated with A_{α_0} translated in terms of the transition probabilities means that $\tilde{p}_{i,j} = p_{i,j}$, so that the distributions of the Markov chains $(X_n^{\alpha_0,(0)},)_{n\geq 0}$ and $(X_{T_n}^{\alpha_0,(1)})_{n\geq 0}$ are identical. It means that the random walk $(X_n^{\alpha_0,(0)})_{n\geq 0}$ is *decimation invariant*. Our result (Theorem 2) says that it is the unique decimation invariant random walk on $V^{(0)}$.

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