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B. Y. CHEN INEQUALITIES FOR SLANT SUBMANIFOLDS IN COMPLEX SPACE FORMS

1. Introduction

In this article, we establish some B. Y. Chen inequalities for slant submanifolds M^n in complex space forms $\widetilde{M}^m(c)$.

In the introduction of the article "Can one live in a best world without tension?" [1], B. Y. Chen translates some problems of the world into the mathematical problems. By a "best world" we mean "a surrounding space which has the highest degree of homogeneity". Almost everyone desires to live in a "best world" without tension. The question is whether "to live in a best world without tension" is a real posibility.

The final goal of this area of research is to solve following WORLD PROBLEM: "Determine the best ways of living for all individuals who live in a best world".

In order to apply Differential Geometry effectively, we shall assume that the objects we are going to investigate are Riemannian manifolds (we need to have metric in order to distinguish the shapes of different individuals). According to work of Lie, Klein and Killing, the family of Riemannian manifolds with the highest degree of homogeneity consists of Euclidean spaces, Riemannian spheres, real projective spaces and real hyperbolic spaces. Such spaces have the highest degree of homogeneity because they have the largest groups of isometries.

Hence, a best world in terms of Differential Geometry is nothing but a Riemannian space form $\widetilde{M}^m(c)$ with constant sectional curvature, say c.

What is tension? It is a well-known fact since the time of Laplace that the tension field of a submanifold is nothing but the mean curvature vector field. Hence, the amount of tension applied to an individual at a point is simply measured by the squared mean curvature at that point.

With this specifications, the problems of the world can be translated into the following mathematical problems:

PROBLEM 1. Given a Riemannian manifold M, does there exist an isometric embedding $x : M \to \widetilde{M}^m(c)$ such that the squared mean curvature is zero everywhere?

PROBLEM 2. Given an arbitrary isometric immersion of a Riemannian manifold M into a Riemannian space form, what are the relationships between the intrinsic invariants of M and the main extrinsic invariant, namely, the squared mean curvature?

PROBLEM 3. Does there exist a sharp lower bound of the squared mean curvature for an isometric embedding of a Riemannian manifold in a Riemannian space form?

PROBLEM 4. Does every intrinsic invariant relate directly with the squared mean curvature for a submanifold in a Riemannian space form?

2. Preliminaries: Riemannian invariants

The Riemannian invariants of a Riemannian manifold are the intrinsic characteristics of the Riemanian manifold. In this section we recall a string of Riemannian invariants on a Riemannian manifold [4].

Let M be a Riemannian manifold. Denote by $k(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM, p \in M$.

For any orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the tangent space T_pM, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{i<j} k(e_i \wedge e_j). \tag{2.1}$$

We denote by

$$(\inf k)(p) = \inf\{k(\pi);\ \pi \subset T_pM, \dim \pi = 2\}, \tag{2.2}$$

and we introduce the first Chen invariant

$$\delta_M(p) = \tau(p) - (\inf k)(p). \tag{2.3}$$

Let L be a subspace of T_pM of dimension $r \geq 2$ and $\{e_1, e_2, \ldots, e_r\}$ an orthonormal basis of L. We define the scalar curvature $\tau(L)$ of the r-plane section L by

$$\tau(L) = \sum_{\alpha<\beta} k(e_\alpha \wedge e_\beta), \quad \alpha, \beta = 1, \ldots, r. \tag{2.4}$$

Given an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the tangent space T_pM, we simply denote by $\tau_{1\ldots r}$ the scalar curvature of r-plane section spanned by $e_1, \ldots, e_r$. The scalar curvature $\tau(p)$ of M at p is nothing but the scalar curvature of the tangent space of M at p. And if L is a 2-plane section, $\tau(L)$ is nothing but the sectional curvature $k(L)$ of L.

For an integer $k \geq 0$, we denote by $S(n,k)$ the finite set which consists of k-tuples $(n_1, n_2, \ldots, n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \ldots + n_k \leq n$. Denote by S$(n)$ the set of k-tuples with $k \geq 0$ for a fixed n.

For each k-tuples $(n_1, \ldots, n_k) \in S(n)$, we introduce a Riemannian invariant defined by

$$\delta(n_1, \ldots, n_k) = \tau(p) - S(n_1, \ldots, n_k)(p), \tag{2.5}$$

where

$$S(n_1, \ldots n_k) = \inf\{\tau(L_1) + \ldots + \tau(L_k)\}. \tag{2.6}$$

$L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j, j = 1, \ldots, k$.

We define:

$$d(n_1, \ldots n_k) = \frac{n^2(n + k - 1 - \sum_{j=1}^{k} n_j)}{2(n + k - \sum_{j=1}^{k} n_j)}, \tag{2.7}$$

$$b(n_1, \ldots n_k) = \frac{1}{2}\Big[n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)\Big]. \tag{2.8}$$

We denote by H the mean curvature vector, i.e.

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{2.9}$$

where h is the second fundamental form of the submanifold M.

3. B. Y. Chen inequalities

B. Y. Chen gave the following inequality for submanifolds in real space forms.

THEOREM 3.1. *Given an m-dimensional real space form $\widetilde{M}(c)$ and an n-dimensional submanifold M, $n \geq 3$, we have*

$$\delta_M \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2 + (n+1)c\right\}. \tag{3.1}$$

The equality case of inequality (3.1) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of $T_p^{\perp}M$ such that the shape operators of M in $\widetilde{M}(c)$ at p have the following forms:

$$(3.2)\qquad A_{n+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & \mu & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix}, \quad a+b=\mu,$$

$$(3.3)\qquad A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \dots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where we denote by

$$(3.4)\qquad A_r = A_{e_r}, r = n+1, \dots, m,$$

$$(3.5)\qquad h_{ij}^r = g(h(e_i, e_j), e_r), r = n+1, \dots, m.$$

By an analogous way we prove an inequality for θ-slant submanifolds M in complex space forms $\widetilde{M}(c)$ of constant holomorphic sectional curvature c.

THEOREM 3.2. *Given an m-dimensional complex space form $\widetilde{M}(c)$ and a θ-slant submanifold M,* $\dim M = n$, $n \geq 3$, *we have*

$$(3.6)\qquad \delta_M \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2 + (n+1+3\cos^2\theta)\frac{c}{4}\right\}.$$

The equality case of the inequality (3.6) *holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \dots, e_{2m}\}$ of $T_p^{\perp}M$ such that the shape operators of M in $\widetilde{M}(c)$ at p have the forms* (3.2) *and* (3.3).

Proof. We recall the Gauss equation for the submanifold M

$$(3.7)\qquad \widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,W), h(Y,Z)) - g(h(X,Z), h(Y,W)), \quad \forall X,Y,Z,W \in \Gamma(TM),$$

where $\widetilde{R}$ denotes the curvature tensor of $\widetilde{M}(c)$ and R denotes the curvature tensor of M.

Since $\widetilde{M}(c)$ is a complex space form, then we have

$$(3.8)\qquad \widetilde{R}(X,Y,Z,W) = \frac{c}{4}\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z) + g(JX,Z)g(JY,W) - g(JX,W)g(JY,Z) + 2g(X,JY)g(Z,JW)\}, \quad \forall X,Y,Z,W \in \Gamma(TM).$$

Let $p \in M$ and $\{e_1, e_2, \ldots, e_n\}$ an orthonormal basis of T_pM and $\{e_{n+1}, \ldots, e_{2m}\}$ an orthonormal basis of $T_p^{\perp}M$. For $X = Z = e_i, Y = W = e_j$ from the equation (3.8) it follows that

$$(3.9)\quad \widetilde{R}(e_i, e_j, e_i, e_j) = \frac{c}{4}\{n^2 - n - g(Je_i, e_j)g(Je_j, e_i) + 2g(e_i, Je_j)g(e_i, Je_j)\}$$
$$= \frac{c}{4}\{n^2 - n + 3\sum_{i,j} g^2(Je_i, e_j)\}.$$

Let $M \subset \widetilde{M}(c)$ a θ-slant submanifold, $\dim M = n = 2k$. For $X \in \Gamma(TM)$ we have

$$JX = PX + FX, \quad PX \in \Gamma(TM), FX \in \Gamma(T^{\perp}M).$$

Let $p \in M$ and an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of T_pM with $e_2 = \dfrac{1}{\cos\theta}Pe_1, \ldots, e_{2k} = \dfrac{1}{\cos\theta}Pe_{2k-1}$.

We have $g(Je_1, e_2) = g(Pe_1, \frac{1}{\cos\theta}Pe_1) = \cos\theta$ and, in same way, $g(Je_i, e_{i+1}) = \cos\theta$ for $i = 3, 5, \ldots, 2k-1$.

The relation (3.9) implies that

$$(3.10)\qquad \widetilde{R}(e_i, e_j, e_i, e_j) = \frac{c}{4}\{n^2 - n + 3n\cos^2\theta\}.$$

Denoting by

$$(3.11)\qquad \|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)),$$

the relation (3.10) implies that

$$(3.12)\qquad \frac{c}{4}\{n^2 - n + 3n\cos^2\theta\} = 2\tau + \|h\|^2 - n^2\|H\|^2,$$

or equivalently,

$$(3.13)\qquad 2\tau = n^2\|H\|^2 - \|h\|^2 + \frac{c}{4}\{n^2 - n + 3n\cos^2\theta\}.$$

Denoting by

$$(3.14)\qquad \varepsilon = 2\tau - \frac{n^2}{n-1}(n-2)\|H\|^2 - \frac{c}{4}\{n^2 - n + 3n\cos^2\theta\},$$

we obtain

$$\varepsilon = n^2\|H\|^2(1 - \frac{n-2}{n-1}) - \|h\|^2,$$

i.e.,

$$(3.15)\qquad n^2\|H\|^2 = (n-1)(\varepsilon + \|h\|^2).$$

Let $p \in M, \pi \subset T_pM, \dim \pi = 2, \pi = sp\{e_1, e_2\}$. We define $e_{n+1} = \frac{H}{\|H\|}$ and from the relation (3.15) we obtain

$$\Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2 = (n-1)\Big(\sum_{i,j=1..n}\sum_{r=n+1..2m}(h_{ij}^r)^2 + \varepsilon\Big),$$

or equivalently,

(3.16)

$$\Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2 = (n-1)\Big\{\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{2m}(h_{ij}^r)^2 + \varepsilon\Big\}.$$

We invoke now Lemma 3.1 from [3]: "*Let $a_1, \ldots, a_n, c$ be $n+1, n \geq 3$, real numbers such that*

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 = (n-1)\Big(\sum_{i=1}^{n} a_i^2 + c\Big).$$

Then $2a_1a_2 \geq c$ with the equality holding if and only if $a_1 + a_2 = a_3 = \ldots = a_n$."

By using this Lemma we have from (3.16):

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1\ldots n}\sum_{r=n+2..2m}(h_{ij}^r)^2 + \varepsilon. \tag{3.17}$$

From the Gauss equation for $X = Z = e_1, Y = W = e_2$, we obtain

$$\begin{aligned}
k(\pi) &= \frac{c}{4}(1+3\cos^2\theta) + \sum_{r=n+1}^{2m}[h_{11}^r h_{22}^r - (h_{12}^r)^2] \\
&\geq \frac{c}{4}(1+3\cos^2\theta) + \frac{1}{2}\Big[\sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1..n}\sum_{r=n+2..2m}(h_{ij}^r)^2 + \varepsilon\Big] \\
&\quad + \sum_{r=n+2}^{2m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m}(h_{12}^r)^2 \\
&= \frac{c}{4}(1+3\cos^2\theta) + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{i,j=1..n}\sum_{r=n+2..2m}(h_{ij}^r)^2 + \frac{1}{2}\varepsilon \\
&\quad + \sum_{r=n+2}^{2m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m}(h_{12}^r)^2 \\
&= \frac{c}{4}(1+3\cos^2\theta) + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m}\sum_{i,j>2}(h_{ij}^r)^2
\end{aligned}$$

$$+\frac{1}{2}\sum_{r=n+2}^{2m}(h_{11}^r+h_{22}^r)^2+\sum_{j>2}[(h_{1j}^{n+1})^2+(h_{2j}^{n+1})^2]+\frac{1}{2}\varepsilon$$
$$\geq \frac{c}{4}(1+3\cos^2\theta)+\frac{\varepsilon}{2},$$

or equivalently,

$$k(\pi)\geq \frac{c}{4}(1+3\cos^2\theta)+\frac{\varepsilon}{2}. \tag{3.18}$$

From the relation (3.13), it follows that

$$\inf k \geq \frac{c}{4}(1+3\cos^2\theta)+\tau-\frac{c}{8}\{n^2-n+3n\cos^2\theta\}-\frac{n^2(n-2)}{2(n-1)}\|H\|^2. \tag{3.19}$$

The relation (3.19) implies that

$$\delta_M \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2+(n+1+3\cos^2\theta)\frac{c}{4}\right\}, \tag{3.20}$$

where δ_M is defined by the formula (2.3).

This relation represents the inequality to prove.

The case of equality at a point $p\in M$ holds if and only if it achieves the equality in the previous inequality and we have the equality in the Lemma:

$$\begin{cases} h_{ij}^{n+1}=0,\ \forall i\neq j, i,j>2,\\ h_{ij}^r=0,\ \forall i\neq j, i,j>2, r=n+1,\dots,2m,\\ h_{11}^r+h_{22}^r=0,\ \forall r=n+2,\dots,2m,\\ h_{1j}^{n+1}=h_{2j}^{n+1}=0,\ \forall j>2,\\ h_{11}^{n+1}+h_{22}^{n+1}=h_{33}^{n+1}=\dots=h_{nn}^{n+1}.\end{cases}$$

We may choose $\{e_1,e_2\}$ such that $h_{12}^{n+1}=0$ and we denote by $a=h_{11}^r, b=h_{22}^r, \mu=h_{33}^{n+1}=\dots=h_{nn}^{n+1}$.

It follows that the shape operators take the desired forms.

Applying Theorem 3.2, we may obtain B. Y. Chen inequality for totally real submanifolds in complex space forms.

REMARK. For totally real submanifolds we have $\theta=\frac{\pi}{2}$.

COROLLARY. *Given an m-dimensional complex space form $\widetilde{M}(c)$ and a totally real submanifold M, $\dim M=n, n\geq 3$, we have*

$$\delta_M \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2+(n+1)\frac{c}{4}\right\}. \tag{3.21}$$

The equality case of inequality (3.21) *is identical with the equality case of inequality* (3.6) *from Theorem* 3.2.

Another B. Y. Chen inequality for submanifolds in real space forms is given by the following [4]

THEOREM 3.3. *Given an m-dimensional space form $\widetilde{M}(c)$ and an n-dimensional submanifold M,$n \geq 3$, we have*

$$\delta(n_1, \ldots, n_k) \leq d(n_1, \ldots, n_k) \|H\|^2 + b(n_1, \ldots, n_k)c. \tag{3.22}$$

Next we prove a generalization of Theorem 3.2 in terms of Chen invariants.

THEOREM 3.4. *Given an m-dimensional complex space form $\widetilde{M}(c)$ and an n-dimensional θ-slant submanifold M,$n \geq 3$, we have*

$$\delta(n_1, \ldots, n_k) \leq d(n_1, \ldots, n_k) \|H\|^2 + b(n_1, \ldots, n_k)\frac{c}{4} + 3n\cos^2\theta\frac{c}{8} - 6\sum_{j=1}^{k} m_j \cos^2\theta\frac{c}{8}, \tag{3.23}$$

where $n_j = 2m_j + \varphi_j, \varphi_j \in \{0, 1\}, \forall j = 1, \ldots, k$.

Let $\widetilde{M}(c)$ be a complex space form, $\dim_{\mathbb{C}} \widetilde{M}(c) = m$ and $M \subset \widetilde{M}(c)$ an n-dimensional submanifold.

For any $p \in M$ and for any $X \in T_pM$, we have $JX = PX + FX, PX \in T_pM, FX \in T_p^{\perp}M$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM. We put

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j). \tag{3.24}$$

Let $L \subset T_pM$ be a subspace of T_pM,$\dim L = r$. We put

$$\Psi(L) = \sum_{1 \leq i < j \leq r} g^2(Pu_i, u_j), \tag{3.25}$$

where $\{u_1, \ldots, u_r\}$ is an orthonormal basis of L.

In order to prove Theorem 3.4, we will use the following Lemma [4].

LEMMA. *Let $\widetilde{M}(c)$ be a complex space form, $\dim_{\mathbb{C}} \widetilde{M}(c) = m$ and $M \subset \widetilde{M}(c)$ an n-dimensional submanifold. Let $n_1, \ldots, n_k$ be integers ≥ 2 satisfying $n_1 < n, n_1 + \ldots + n_k \leq n$. For $p \in M$, let $L_j \subset T_pM$ a subspace of T_pM, $\dim L_j = n_j, \forall j = 1, \ldots, k$. Then we have*

$$\tau(p) - \sum_{j=1}^{k} \tau(L_j) \leq d(n_1, \ldots, n_k) \|H\|^2 + \frac{1}{8}\Big[n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1) + 3\|P\|^2 - 6\sum_{j=1}^{k} \Psi(L_j)\Big]c. \tag{3.26}$$

Proof of Lemma. From the Gauss equation for the submanifold M, denoting by $\widetilde{R}$ the curvature tensor of $\widetilde{M}(c)$ and denoting by R the curvature tensor of M, we obtain the relation

$$\widetilde{R}(X,Y,Z,W) = \frac{c}{4}\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)+ \tag{3.8}$$
$$+ g(JX,Z)g(JY,W) - g(JX,W)g(JY,Z) + 2g(X,JY)g(Z,JW)\},$$
$$\forall X,Y,Z,W \in \Gamma(TM).$$

Let $\widetilde{M}(c)$ be an m-dimensional complex space form with $\dim_{\mathbb{C}} \widetilde{M}(c) = m$ and M an n-dimensional submanifold with $n \geq 3$. Let $p \in M$ and $\{e_1,\dots,e_n\}$ an orthonormal basis of T_pM; from the previous relation for $X = Z = e_i, Y = W = e_j$, we have

$$2\tau = n^2 \|H\|^2 - \|h\|^2 + \{n(n-1) + 3\|P\|^2\}\frac{c}{4}. \tag{3.27}$$

Denoting by

$$\eta = 2\tau - 2d(n_1,\dots,n_k)\|H\|^2 - \{n(n-1) + 3\|P\|^2\}\frac{c}{4}, \tag{3.28}$$

we obtain

$$n^2\|H\|^2 = (\eta + \|h\|^2)\gamma, \tag{3.29}$$

where we denote

$$\gamma = n + k - \sum_{j=1}^{k} n_j. \tag{3.30}$$

From the Gauss equation we obtain

$$\tau(L_j) = \{n_j(n_j-1)+6\Psi(L_j)\}\frac{c}{8}+ \sum_{r=n+1}^{2m}\sum_{\alpha_j<\beta_j}[h^r_{\alpha_j\alpha_j}h^r_{\beta_j\beta_j}-(h^r_{\alpha_j\beta_j})^2]. \tag{3.31}$$

We shall prove that

$$\sum_{j=1}^{k}\sum_{r=n+1}^{2m}\sum_{\alpha_j<\beta_j}[h^r_{\alpha_j\alpha_j}h^r_{\beta_j\beta_j} - (h^r_{\alpha_j\beta_j})^2] \geq \frac{\eta}{2}, \tag{3.32}$$

where η is defined by the relation (3.28).

Let $p \in M$ and $\{e_1,\dots,e_n\}$ an orthonormal basis of T_pM; let $L_1,\dots,L_k$ be k mutually orthogonal subspaces of T_pM, $\dim L_j = n_j$, defined by:

$L_1 = sp\{e_1,\dots,e_{n_1}\}$,
$L_2 = sp\{e_{n_1+1},\dots,e_{n_1+n_2}\}$,
....
$L_k = sp\{e_{n_1+\dots+n_{k-1}+1},\dots,e_{n_1+\dots+n_{k-1}+n_k}\}$.

Let $e_{n+1}=\frac{H}{\|H\|}, e_{n+1}\in T_p^{\perp}M$. We denote by $a_i=h_{ii}^{n+1}=g(h(e_i,e_i),e_{n+1})$ and from the relation (3.29) we obtain

$$(3.33)\quad \Big(\sum_{i=1}^{n}a_i\Big)^2=\gamma[\eta+\sum_{i\neq j}(h_{ij}^{n+1})^2+\sum_{i=1}^{n}(h_{ii}^{n+1})^2+\sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2].$$

We denote by $D_j, j=1,\dots,k$ the sets:
$D_1=\{1,\dots,n_1\}$,
$D_2=\{n_1+1,\dots,n_1+n_2\}$,
....
$D_k=\{n_1+\dots+n_{k-1}+1,\dots,n_1+\dots+n_{k-1}+n_k\}$.
Also, we denote by
$b_1=a_1$,
$b_2=a_2+\dots+a_{n_1}$,
$b_3=a_{n_1+1}+\dots+a_{n_1+n_2}$,
....
$b_{k+1}=a_{n_1+\dots+n_{k-1}+1}+\dots+a_{n_1+n_2+\dots+n_{k-1}+n_k}$,
$b_{k+2}=a_{n_1+\dots+n_k+1}$,
....
$b_{\gamma+1}=a_n$.
Then the relation (3.33) implies that

$$\Big(\sum_{i=1}^{\gamma+1}b_i\Big)^2=\gamma[\eta+\sum_{i=1}^{\gamma+1}{b_i}^2+\sum_{i\neq j}(h_{ij}^{n+1})^2+\sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2$$
$$-2\sum_{\alpha_1<\beta_1}a_{\alpha_1}a_{\beta_1}-\dots-2\sum_{\alpha_k<\beta_k}a_{\alpha_k}a_{\beta_k}],$$

with $\alpha_j,\beta_j\in D_j, \forall j=1,\dots,k$.

Applying Lemma 3.1. from [3] we have

$$\sum_{\alpha_1<\beta_1}a_{\alpha_1}a_{\beta_1}+\dots+\sum_{\alpha_k<\beta_k}a_{\alpha_k}a_{\beta_k}\geq\frac{1}{2}[\eta+\sum_{i\neq j}(h_{ij}^{n+1})^2+\sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2].$$

The previous relation implies that

$$\sum_{j=1}^{k}\sum_{r=n+1}^{2m}\sum_{\alpha_j<\beta_j}[h_{\alpha_j\alpha_j}^{r}h_{\beta_j\beta_j}^{r}-(h_{\alpha_j\beta_j}^{r})^2]$$
$$\geq\frac{\eta}{2}+\frac{1}{2}\sum_{r=n+1}^{2m}\sum_{(\alpha,\beta)\notin D^2}(h_{\alpha\beta}^{r})^2+\sum_{r=n+2}^{2m}\sum_{\alpha_j\in D_j}(h_{\alpha_j\alpha_j}^{r})^2\geq\frac{\eta}{2},$$

where we denote by $D^2=(D_1\times D_1)\cup\dots\cup(D_k\times D_k)$.

Thus we have proved the relation (3.32). From the relation (3.31), we obtain

$$\sum_{j=1}^{k}\tau(L_j) \geq \frac{\eta}{2} + \sum_{j=1}^{k}\{n_j(n_j-1)+6\Psi(L_j)\}\frac{c}{8} = \tau - d(n_1,\dots,n_k)\|H\|^2$$
$$-\{n(n-1)+3\|P\|^2\}\frac{c}{8} + \sum_{j=1}^{k}\{n_j(n_j-1)+6\Psi(L_j)\}\frac{c}{8},$$

or equivalently,

$$\tau - \sum_{j=1}^{k}\tau(L_j) \leq d(n_1,\dots,n_k)\|H\|^2 + \frac{c}{8}\{n(n-1)+3\|P\|^2$$
$$-\sum_{j=1}^{k}\{n_j(n_j-1)+6\Psi(L_j)\}\}.$$

This relation represents the inequality to prove, or equivalently,

$$\delta(n_1,\dots,n_k) \leq d(n_1,\dots,n_k)\|H\|^2 + b(n_1,\dots,n_k)\frac{c}{4} + \frac{c}{8}\{3\|P\|^2 - 6\sum_{j=1}^{k}\Psi(L_j)\}. \tag{3.34}$$

Proof of Theorem 3.4. Let $\widetilde{M}(c)$ be a complex space form, $\dim_{\mathbb{C}}\widetilde{M}(c)=m$ and $M\subset\widetilde{M}(c)$ a θ-slant n-dimensional submanifold, $n\geq 3, n=2k$. Let $p\in M$ and $\{e_1,\dots,e_n\}$ an orthonormal basis of T_pM.

For any $X\in T_pM$ we have $JX=PX+FX, PX\in T_pM, FX\in T_p^{\perp}M$.

We choose $e_2=\frac{1}{\cos\theta}Pe_1,\dots,e_{2k}=\frac{1}{\cos\theta}Pe_{2k-1}$.

We have $g(Pe_1,e_2)=g(Pe_1,\frac{1}{\cos\theta}Pe_1)=\cos\theta$ and, in same way, $g(Je_i,e_{i+1})=\cos\theta$ for $i=3,5,\dots,2k-1$.

It follows that $\|P\|^2=n\cos^2\theta$.

Let $p\in M$ and $\{e_1,\dots,e_n\}$ an orthonormal basis of T_pM; let $L_1,\dots,L_k$ be k mutually orthogonal subspaces of T_pM, $\dim L_j=n_j$, defined by:

$L_1=sp\{e_1,\dots,e_{n_1}\}$,

$L_2=sp\{e_{n_1+1},\dots,e_{n_1+n_2}\}$,

....

$L_k=sp\{e_{n_1+\dots+n_{k-1}+1},\dots,e_{n_1+\dots+n_{k-1}+n_k}\}$.

In same way, it follows that $\Psi(L_j)=m_j\cos^2\theta$, where $n_j=2m_j+\varphi_j, \varphi_j\in\{0,1\}, \forall j=1,\dots,k$.

From (3.26) we obtain inequality (3.23).

COROLLARY. *Given an m-dimensional complex space form $\widetilde{M}(c)$ and a totally real submanifold $M, \dim M = n, n \geq 3$ we have*

$$\delta(n_1, \ldots, n_k) \leq d(n_1, \ldots, n_k) \|H\|^2 + b(n_1, \ldots, n_k)\frac{c}{4}. \tag{3.35}$$

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