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ON CERTAIN RIEMANN SPACE WITH METRICAL TENSOR WITH SEPARABLE COORDINATES AND ITS APPLICATIONS

1. Introduction

The purpose of the present paper are properties of following three Riemann spaces: first endowed with metrical tensor with coordinates of functions of time t, second endowed with metrical tensor with coordinates of functions of radius r and third endowed with metrical tensor with coordinates of functions with separable variables of t and r.

We shall give an application to Einstein's theory of space-time.

2. Formulas for curvature tensor

Let V be a n-dimensional Riemann space with the metrical tensor

$$(2.1)\quad (g_{\lambda\beta}) = \operatorname{diag}(g_{11}, g_{22}, \dots, g_{nn}) \quad \text{where}$$
$$g_{11} = g_{11}(t,r), \quad g_{22} = g_{22}(t,r), \quad g_{33} = g_{33}(t,r),$$
$$g_{44} = T(\theta) g_{33}(t,r), \quad g_{55} = g_{55}(t,r), \dots, \quad g_{nn} = g_{nn}(t,r),$$
$t = x^1$, $r = x^2$, $\theta = x^3$, $(t, r, \theta, x^4, \dots, x^n) = (x^1, x^2, x^3, x^4, \dots, x^n)$ [1].

The curvature tensor is as follows:

$$(2.2)\quad \begin{cases} R^{\gamma}_{1_\gamma 1} = \dfrac{-g_{11}}{g_{\gamma\gamma}} R^{1}_{1_{\gamma\gamma}}, R^{\eta}_{2_\eta 2} = \dfrac{-g_{22}}{g_{\eta\eta}} R^{2}_{2_{\eta\eta}}, \\ R^{\eta}_{2_\eta 1} = \dfrac{-g_{11}}{g_{\eta\eta}} R^{1}_{2_{\eta\eta}} = \dfrac{-g_{22}}{g_{\eta\eta}} R^{2}_{1_{\eta\eta}} = R^{\eta}_{1_\eta 2}, \\ R^{\sigma}_{3_\sigma 3} = \dfrac{-g_{33}}{g_{\sigma\sigma}} R^{3}_{3_{\sigma\sigma}},\ R^{\nu}_{\mu\nu\mu} = \dfrac{-g_{\mu\mu}}{g_{\nu\nu}} R^{\mu}_{\mu\nu\nu} \end{cases}$$

$(\gamma = 2, 3, \dots, n;\ \eta = 3, 4, \dots, n;\ \sigma = 4, 5, \dots, n;\ \mu + 1 = \nu = 5, 6, \dots, n)$ (not sum over γ, η, σ, μ, ν), where

$$
\begin{aligned}
R^1_{122} &= \frac{1}{4g_{11}}\Big(-2\ddot{g}_{22} - 2g''_{11} + \frac{\dot{g}^2_{22}}{g_{22}} + \frac{g'^2_{11}}{g_{11}} + \frac{\dot{g}_{11}\dot{g}_{22}}{g_{11}} + \frac{g'_{11}g'_{22}}{g_{22}}\Big),\\
R^1_{1\eta\eta} &= \frac{1}{4g_{11}}\Big(-2\ddot{g}_{\eta\eta} + \frac{\dot{g}^2_{\eta\eta}}{g_{\eta\eta}} + \frac{\dot{g}_{11}\dot{g}_{\eta\eta}}{g_{11}} - \frac{g'_{11}g'_{\eta\eta}}{g_{22}}\Big),\\
R^2_{2\eta\eta} &= \frac{1}{4g_{22}}\Big(-2g''_{\eta\eta} + \frac{g'^2_{\eta\eta}}{g_{\eta\eta}} - \frac{\dot{g}_{22}\dot{g}_{\eta\eta}}{g_{11}} + \frac{g'_{22}g'_{\eta\eta}}{g_{22}}\Big),\\
R^2_{1\eta\eta} &= \frac{1}{4g_{22}}\Big(-2\dot{g}'_{\eta\eta} + \frac{\dot{g}_{22}g'_{\eta\eta}}{g_{22}} + \frac{\dot{g}_{\eta\eta}g'_{11}}{g_{11}} + \frac{\dot{g}_{\eta\eta}g'_{\eta\eta}}{g_{\eta\eta}}\Big), \qquad (2.3)\\
R^3_{344} &= \frac{1}{4g_{33}}\Big(\Big(-2\frac{d^2T}{d\theta^2} + \frac{1}{T}\Big(\frac{dT}{d\theta}\Big)^2\Big)g_{33} - \Big(\frac{\dot{g}^2_{33}}{g_{11}} + \frac{g'^2_{33}}{g_{22}}\Big)T\Big),\\
R^3_{3\nu\nu} &= \frac{1}{4g_{33}}\Big(\frac{-\dot{g}_{33}\dot{g}_{\nu\nu}}{g_{11}} - \frac{g'_{33}g'_{\nu\nu}}{g_{22}}\Big),\\
R^\mu_{\mu\nu\nu} &= \frac{1}{4g_{\mu\mu}}\Big(\frac{-\dot{g}_{\mu\mu}\dot{g}_{\nu\nu}}{g_{11}} - \frac{g'_{\mu\mu}g'_{\nu\nu}}{g_{22}}\Big),\\
&(\eta = 3,4,\dots,n;\ \mu+1 = \nu = 5,6,\dots,n)
\end{aligned}
$$

(not sum over η, μ, ν). The dot denotes differentiation with respect to t, the comma with respect to r. Remaining coordinates are null.

3. The case of separable variables

Let us take Riemann spaces $\widetilde{V}$, $\widetilde{\widetilde{V}}$ with metrical tensors

$$
\begin{aligned}
&\mathrm{diag}(A_1, A_2, A_3, A_3, A_5, \dots, A_n), \qquad (3.1)\\
&A_\alpha = A_\alpha(t),\ (\alpha = 1,2,3,5,\dots,n),\\
&A_3 = A_2;
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{diag}(B_1, B_2, B_3, TB_3, B_5, \dots, B_n), \qquad (3.2)\\
&B_\alpha = B_\alpha(r),\ (\alpha = 1,2,3,5,\dots,n),\\
&T = T(\theta).
\end{aligned}
$$

The curvature tensors $K^\varphi_{\lambda\beta\varrho}$, $L^\varphi_{\lambda\beta\varrho}$ $(\lambda,\beta,\varrho,\varphi = 1,2,\dots,n)$, Riccitensors $K_{\lambda\lambda}$, $L_{\lambda\lambda}$ $(\lambda = 1,2,\dots,n)$, scalar curvatures K, L of spaces $\widetilde{V}$, $\widetilde{\widetilde{V}}$ can be calculated from (2.3), (2.2).

Let we take a Riemann space V with the metrical tensor

$$
\begin{aligned}
&\mathrm{diag}(A_1B_1, A_2B_2, A_3B_3, TA_3B_3, A_5B_5, \dots, A_nB_n),\ A_3 = A_2, \qquad (3.3)\\
&A_\alpha = A_\alpha(t), B_\alpha = B_\alpha(r)(\alpha = 1,2,3,5,\dots,n),\ T = T(\theta).
\end{aligned}
$$

The formulas of the curvature tensor $R^{\varphi}_{\lambda\beta\varrho}$, of Ricci tensor $R_{\lambda\beta}$ and of scalar curvature R of V are as follows:

$$(3.4)\quad R^{\varphi}_{\lambda\beta\varrho} = a^{\varphi}_{\lambda\beta\varrho}K^{\varphi}_{\lambda\beta\varrho} + b^{\varphi}_{\lambda\beta\varrho}L^{\varphi}_{\lambda\beta\varrho}\ (\lambda,\beta,\varrho,\varphi = 1,2,\dots,n),$$
$$R_{\lambda\lambda} = a_{\lambda\lambda}K_{\lambda\lambda} + b_{\lambda\lambda}L_{\lambda\lambda},$$

$$(3.5)\quad R_{12} = \frac{2B_3'}{B_3}K^2_{133} + \frac{A_3B_\nu'}{A_\nu B_\nu}K^2_{1\nu\nu} + \frac{2\dot{A}_3B_2}{A_3B_3}L^2_{133} + \frac{2\dot{A}_\nu B_2}{A_\nu B_\nu}L^2_{1\nu\nu},$$
$$(\lambda = 1,2,\dots,n;\nu = 5,6,\dots,n)$$

$$(3.6)\quad R = \frac{K}{B_1} + \frac{L}{A_3},$$

where $a^{\varphi}_{\lambda\beta\varrho}$, $a_{\lambda\lambda}$ are functions of the variables r; $b^{\varphi}_{\lambda\beta\varrho}$, $b_{\lambda\lambda}$ are functions of the variable t; $K^{\varphi}_{\lambda\beta\varrho}$, $L^{\varphi}_{\lambda\beta\varrho}$, $K_{\lambda\lambda}$, $L_{\lambda\lambda}$, K, L are the coordinates of curvature tensors, Ricci tensors and scalars curvatures of $\tilde{V}$, $\tilde{\tilde{V}}$ respectively; all are calculated from (2.2), (2.3).

THEOREM. *When the Riemann spaces* $\tilde{V}$, $\tilde{\tilde{V}}$ *are local-Euclidean, or Ricci flat (for* $R_{12} = 0$ *in* (3.5)), *or with null scalar curvature, then the Riemann space* V *is local-Euclidean, or Ricci flat, or with null scalar curvature respectively.*

Proof. The theorem follows immediately from formulas (3.4), (3.5), (3.6).

4. An application to Einstein's theory of space-time

Let be a space-time with the following metrical tensor

$$(4.1)\qquad \operatorname{diag}(A_1B_1, -A_2B_2, -A_3B_3, -TA_3B_3).$$

We shall investigate a particular case of (4.1), namely for $A_1 = 1$, $B_1 = 1$, $A_2 = A_3 = S^2(t)$, $B_2 = B_2(r)$, $B_3 = B_3(r)$, $T = \sin^2\theta$. Then the space-time has the following metrical tensor:

$$(4.2)\qquad \operatorname{diag}(1, -S^2B_2, -S^2B_3, -S^2B_3\sin^2\theta).$$

We give a solution of Einstein's equation on the space-time with metrical tensor (4.2).

We shall use a method analogous to Friedman's solution in [1], [2].

On the base of the formulas in (2.3) for $n = 4$ we obtain the following four independence coordinates of the curvature tensor for (4.2)

$$(4.3)\qquad \begin{cases} R^1_{122} = S\ddot{S}B_2,\ R^1_{133} = S\ddot{S}B_3, \\ R^2_{233} = \dfrac{1}{4B_2}\left(4\dot{S}^2B_2B_3 - 2B_3'' + \dfrac{B_2'B_3'}{B_2} + \dfrac{B_3'^2}{B_3}\right), \\ R^3_{344} = \left(1 + \dot{A}^2B_3 - \dfrac{B_3'^2}{4B_2B_3}\right)\sin^2\theta. \end{cases}$$

By the method in [1], [2] we obtain the for space-time (4.2) and the curvature tensor (4.3) the following system of two equations of Einstein's equation

$$(4.4)\quad \begin{aligned} &\frac{3\ddot{S}}{S} = \frac{1}{2}\varkappa(\varrho + 3p), \\ &S\ddot{S} + 2\dot{S}^2 - \frac{3B_3''}{4B_2B_3} + \frac{3B_2'B_3'}{8B_2^2B_3} + \frac{B_3'^2}{4B_2B_3^2} + \frac{1}{2B_3} = -\frac{1}{2}\varkappa(\varrho - p)S^2. \end{aligned}$$

Next after the elimination S we obtain the following equation

$$(4.5)\qquad \dot{S}^2 + k(r) = \frac{8\pi G}{3}\varrho S^2,$$

$\varkappa = -8\pi G$, G-gravitationconstant, where

$$(4.6)\qquad k(r) = \frac{-3B_3''}{8B_2B_3} + \frac{3B_2'B_3'}{16B_2^2B_3} + \frac{B_3'^2}{8B_2B_3^2} + \frac{1}{4B_3}.$$

The differential equation (4.5) assumes the following form:

$$(4.7)\qquad \dot{S}^2 + k(r) = \frac{A^2}{S},$$

where $k(r)$ is defined in (4.6), $A^2 = \frac{8\pi G\varrho_0 S_0^2}{3}$, $A > 0$.

Equation (4.7) will be called the generalized equation of the Friedman's equation. To solve equation (4.7), we take the following parametrical solution (4.8)

$$t = A^2\left(\frac{-\sin\frac{\psi}{2}}{k(r)}\sqrt{1 - k(r)\sin^2\frac{\psi}{2}} + \frac{1}{k(r)\sqrt{k(r)}}\arcsin\left(\sqrt{k(r)}\sin\frac{\psi}{2}\right)\right),$$

$$S = A^2\sin^2\frac{\psi}{2}$$

for $0 < k(r) \le \frac{1}{\sin^2\frac{\psi}{2}}$, where ψ is parameter.

EXAMPLE 1. For the generalized Reissner–Nordstrom space-time [2]

$$(4.9)\quad \begin{aligned} &\operatorname{diag}\left(1, \frac{-S^2}{E^2}, -S^2r^2, -S^2r^2\sin^2\theta\right), \\ &E = 1 - \frac{r_0}{r} + \frac{Kr_0^2}{r^2};\ r_0, K = \text{constans},\ r_0 > 0 \end{aligned}$$

we obtain

$$(4.10)\qquad k(r) = \frac{5K^2r_0^4}{4r^6} - \frac{7Kr_0^3}{4r^5} + (2K+1)\frac{r_0^2}{2r^4} - \frac{r_0}{4r^3}\ [2],$$

therefore the solution (4.8) for r_0, K = constans surface $(t, S) \to r = m(t, S)$ over the plane $\{(t, S) : t > 0, S > 0\}$. Since $k(r) > 0$, we call the model (4.8) with relatives (4.9), (4.10) the generalized closed Friedman's model [2].

EXAMPLE 2. For the space-time of Robertson–Walker [1]

$$\operatorname{diag}\left(1, \frac{-R^2}{1-kr^2}, -R^2r^2, -R^2r^2\sin^2\theta\right), \ R = R(t), \ k = \text{constans} \tag{4.11}$$

on the base of (4.6) we obtain $k(r) = k =$ constans. Since $k > 0$, so $k = 1$ and we obtain on the plane $\{(t, R) : t > 0, R > 0\}$ a curve, which is cycloid, so it represents a closed model of Friedman.

COROLLARY. *The solution* (4.8) *with* (4.6) *represents a generalized closed Friedman model.*

References

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