DEMONSTRATIO MATHEMATICA
Vol. XXXII No 4 1999

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# PROJECTIVE COLLINEATION AS A PRODUCT OF SPECIAL HARMONIC HOMOLOGIES

**Abstract.** The problem of decomposition of a projective collineation into special harmonic homologies i.e. homologies with the fixed center and homologies with fundamental hyperspaces containing the fixed points, is considered. We prove that for every harmonic homology of the $n$-dimensional projective space there exists $j \leq 3$ such that the harmonic homology is a product of $j$ harmonic homologies from the given class.

## 1. Introduction

The problem of decomposition of a projective collineation into any special collineations was considered in many papers, for instance Cater [1], Ellers [2], Witczyński [4], [6], and many others.

In this paper we deal with the $n$-dimensional projective space $P^n(F)$, where $F = R$ or $F = C$, and we present the decomposition of a projective collineation $f$ into harmonic homologies satisfying some conditions presented below.

Let $s_1$ and $a_2, a_3, \ldots, a_n$ be fixed projectively independent points belonging to $P^n(F)$. Let $\Phi(n)$ be the class of harmonic homologies of the space $P^n(F)$ with the centre in the point $s_1$. Let $\Psi(n)$ be the class of harmonic homologies of the space $P^n(F)$ with fundamental hyperspaces including points $a_2, a_3, \ldots, a_n$. Let $\Theta(n) = \Phi(n) \cup \Psi(n)$.

We introduce some auxiliary notation.

Let $Z(x_1, x_2, \ldots, x_k)$ denote the subspace generated by points $x_1$, $x_2, \ldots, x_k \in P^n(F)$.

Let

$$f : (a_1, a_2, \ldots, a_k) \longmapsto (b_1, b_2, \ldots, b_k)$$

denote that $f(a_i) = b_i$, $i =, 2, \ldots, k$, where $f$ is a projective collineation of $P^n(F)$.

Let $f_{a,H}$ denote a harmonic homology with the centre $a$ and the fundamental hyperspace $H$. Matrices of collineations will be denoted by $\mathbf{G}$, $\mathbf{H}$, $\mathbf{M}$ ...

## 2. Decomposition of a harmonic homology into harmonic homologies belonging to the class $\Theta(n)$

Witczyński has proved the following theorem.

THEOREM 1. (Witczyński (1981)) *For every harmonic homology $f_{a,H}$ of $P^n(F)$ there exists $j \leq 7$, such that $f_{a,H}$ is a composition of $j$ transformations from the class $\Theta(n)$.* ■

Using the same arguments we obtain the following corollary.

COROLLARY 1. *For every harmonic homology $f_{a,H}$ of $P^n(F)$, such that $s_1 \notin H$, there exists $j \leq 3$ such that $f_{a,H}$ is a composition of $j$ transformations from the class $\Theta(n)$.* ■

In the theorems presented below we show that number seven can be diminished to three for every harmonic homology $f_{a,H}$.

THEOREM 2. *For every harmonic homology $f_{a,H}$ of $P^n(F)$, such that $a \notin Z(a_2, a_3, \dots, a_n)$ and $s_1 \in H$, there exists $j \leq 3$ such that $f_{a,H}$ is a composition of $j$ transformations from the class $\Theta(n)$.*

Proof. Assume that $f_{a,H} \notin \Theta(n)$. Let $\bar{a}$ be a point such that $\bar{a} \in Z(s_1, a)$, $\bar{a} \neq s_1$ and $\bar{a} \neq a$. Let $\bar{H}$ be a hyperplane such that $Z(a_2, a_3, \dots, a_n) \subset \bar{H}$ and $h_{\bar{a},\bar{H}}(s_1) = a$. Let $\bar{a}_{n+1} \in H$ be a point such that points $a_2, a_3, \dots, a_n$, $\bar{a}_{n+1}$ are projectively independent and $\bar{H} = Z(a_2, a_3, \dots, a_n, \bar{a}_{n+1})$. Then

$$\bar{H} \cap H = Z(\bar{a}_3, \bar{a}_4, \dots, \bar{a}_n, \bar{a}_{n+1})$$

for some points $\bar{a}_3, \bar{a}_4, \dots, \bar{a}_n$ such that $\bar{a}_3, \bar{a}_4, \dots, \bar{a}_n, \bar{a}_{n+1}$ are projectively independent. Let $f_{a,H}(\bar{a}) = \bar{a}'$. The homology $h_{\bar{a},\bar{H}}$ is defined as follows

$$h_{\bar{a},\bar{H}} : (a_2, \bar{a}_3, \dots, \bar{a}_{n+1}, s_1, a, \bar{a}') \longmapsto (a_2, \bar{a}_3, \dots, \bar{a}_{n+1}, a, s_1, \bar{a}').$$

Let us define a harmonic homology $g_{s_1,H_1}$ as follows

$$g_{s_1,H_1} : (s_1, \bar{a}_3, \dots, \bar{a}_{n+1}, \bar{a}, \bar{a}') \longmapsto (s_1, \bar{a}_3, \dots, \bar{a}_{n+1}, \bar{a}', \bar{a}).$$

Then

$$f_{a,H} = h_{\bar{a},\bar{H}} g_{s_1,H_1} h_{\bar{a},\bar{H}}.$$

■

THEOREM 3. *For every harmonic homology $f_{a,H}$ of $P^n(F)$, such that $a \in Z(a_2, a_3, \dots, a_n)$ and $s_1 \in H$, there exists $j \leq 2$ such that $f_{a,H}$ is a composition of $j$ transformations from the class $\Theta(n)$.*

Proof. There is no loss of generality in assuming that $a = a_2$. Let $\bar{a}_{n+1} \in H$ be a point such that the points $a_2, a_3, \dots, a_n, \bar{a}_{n+1}$ are projectively independent. Let $\bar{H} = Z(a_2, a_3, \dots, a_n, s_1)$ and $H_1 = Z(a_2, a_3, \dots, a_n, \bar{a}_{n+1})$. Then

$$H_1 \cap H = Z(\bar{a}_3, \bar{a}_4, \dots, \bar{a}_n, \bar{a}_{n+1})$$

and

$$\bar{H} \cap H = Z(\bar{a}_3, \bar{a}_4, \ldots, \bar{a}_n, s_1)$$

or some points $\bar{a}_3, \bar{a}_4, \ldots, \bar{a}_n$ such that $\bar{a}_3, \bar{a}_4, \ldots, \bar{a}_n, \bar{a}_{n+1}$ are projectively ndependent.

Let $y \in Z(s_1, a_2)$ be a point such that $y \neq s_1$ and $y \neq a_2$. Let $y' = f_{a,H}(y)$; note that $y'$ belongs to the line $Z(s_1, a_2)$. Let us define homologies $h_{\bar{a}_{n+1},\bar{H}}$ and $g_{s_1,H_1}$ as follows:

$$g_{s_1,H_1} : (s_1, \bar{a}_3, \ldots, \bar{a}_{n+1}, a_2, y, y') \longmapsto (s_1, \bar{a}_3, \ldots, \bar{a}_{n+1}, a_2, y', y)$$

and

$$h_{\bar{a}_{n+1},\bar{H}} : (s_1, \bar{a}_3, \ldots, \bar{a}_{n+1}, a_2, x, x') \longmapsto (s_1, \bar{a}_3, \ldots, \bar{a}_{n+1}, a_2, x', x),$$

where $x$ is a choosen point such that $x \in Z(s_1, \bar{a}_{n+1})$ and $x \neq s_1$ and $x \neq \bar{a}_{n+1}$ and $g_{s_1,H_1}(x) = x'$. We have

$$f_{a_2,H} : (\bar{a}_3, \ldots, \bar{a}_{n+1}, y, y', x) \longmapsto (\bar{a}_3, \ldots, \bar{a}_{n+1}, y', y, x)$$

thus

$$f_{a,H} = h_{\bar{a}_{n+1},\bar{H}}\, g_{s_1,H_1} = g_{s_1,H_1}\, h_{\bar{a}_{n+1},\bar{H}}.$$

∎

Theorems 2, 3 and Corollary 1 show that for every harmonic homology of the space $P^n(F)$ there exists $j \leq 3$ such that the harmonic homology is a composition of $j$ homologies belonging to the class $\Theta(n)$. The theorem below shows that the number three cannot be diminished in the case of the space $P^2(F)$. First we will show two lemmas about matrices of harmonic homologies with a fixed point.

LEMMA 1. *Let $s_1 = (0,0,1)$. Then a harmonic homology $h$ of $P^2(F)$ such that $h(s_1) = s_1$, has in some coordinate system one of the matrices*

$$\mathbf{G}_1 = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \mp 1 \end{pmatrix} \quad \mathbf{G}'_1 = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \mp 1 & 0 \\ 0 & 0 & \mp 1 \end{pmatrix} \quad \mathbf{G}''_1 = \begin{pmatrix} \mp 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \mp 1 \end{pmatrix}$$

$$\mathbf{G}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & -1 \end{pmatrix} \qquad \mathbf{G}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & y & -1 \end{pmatrix}$$

$$\mathbf{G}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & -1 \end{pmatrix} \qquad \mathbf{G}_5 = \begin{pmatrix} 1 & 0 & 0 \\ x & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{G}_6 = \begin{pmatrix} 1 & 0 & 0 \\ x & -1 & 0 \\ y & 0 & -1 \end{pmatrix} \qquad \mathbf{G}_7 = \begin{pmatrix} 1 & x & 0 \\ 0 & -1 & 0 \\ 0 & y & 1 \end{pmatrix}$$

$$\mathbf{G}_8 = \begin{pmatrix} 1 & x & 0 \\ 0 & -1 & 0 \\ y & \frac{xy}{2} & -1 \end{pmatrix} \qquad \mathbf{G}_9 = \begin{pmatrix} z & \frac{1-z^2}{x} & 0 \\ x & -z & 0 \\ \frac{-xy}{z+t} & y & t \end{pmatrix},$$

*where* $t = \pm 1$, $x, y, z \in F$ *and for the matrix* $\mathbf{G}_9$, $z \neq -t$ *and* $x \neq 0$.

Proof. Matrix $\mathbf{G}$ of a harmonic homology $h$ such that $h(s_1) = s_1$, satisfies the conditions:

$$\mathbf{G}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{GG} = d\mathbf{I}$$

for some constants $c, d$, with $cd \neq 0$, where $\mathbf{I}$ denotes the identity matrix. Solving above equations we obtain the assertion. ■

LEMMA 2. *Let* $s_1 = (0,0,1)$. *Then the harmonic homology of* $P^2(F)$ *with the centre* $s_1$ *has one of the matrices*

$$\mathbf{G}_1 = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \mp 1 \end{pmatrix} \qquad \mathbf{G}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & -1 \end{pmatrix}$$

$$\mathbf{G}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & y & -1 \end{pmatrix} \qquad \mathbf{G}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & -1 \end{pmatrix},$$

*where* $x, y \in F$.

Proof. The matrix $\mathbf{M}$ of a harmonic homology with the centre $s_1$ satisfies the conditions mentioned in the proof of Lemma 1. Moreover

$$\mathbf{M}s_1 = cs_1$$

and for every point $a$, $a \neq ds_1$, where $d \in F - \{0\}$

$$\mathbf{M}a \neq ca,$$

where $c \neq 0$. An easy computation shows that only matrices $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{G}_3$, $\mathbf{G}_4$ from those mentioned in Lemma 1 have these properties. ■

THEOREM 4. *There exsists a harmonic homology* $f_{a,H} \notin \Theta(2)$ *of the space* $P^2(F)$, *which is not a composition of two harmonic homologies belonging to the class* $\Theta(2)$.

Proof. Consider three cases.

**I.** Consider two transformations $g_{s_1,H_1}$ and $g_{s_1,H_2}$ belonging to the class $\Phi(2)$. Let $f_{a,H}$ be a harmonic homology such that $a \neq s_1$ and $s_1 \notin H$. Then $f_{a,H} \neq g_{s_1,H_1} \cdot g_{s_1,H_2}$ independently on a choice of fundamental lines $H_1, H_2 \subset P^2(F)$.

**II.** Consider two transformations $h_{c_1,\bar{H}_1}$ and $h_{c_2,\bar{H}_2}$ belonging to the class $\Psi(2)$, thus $a_2 \in \bar{H}_2$ i $a_2 \in \bar{H}_1$. Let $f_{a,H}$ be a harmonic homology such that $a \neq a_2$ and $a_2 \notin H$. Then $f_{a,H} \neq h_{c_1,\bar{H}_1}h_{c_2,\bar{H}_2}$ independently on a choice of points $c_1$, $c_2$ and fundamental straight lines $H_1$ and $H_2$ such that $a_2 \in H_1 \cap H_2$.

**III.** The only case remaining concerns the compositions of the form $g_{s_1,H_1}h_{c_1,\bar{H}_1}$ and $h_{c_1,\bar{H}_1}g_{s_1,H_1}$, where $g_{s_1,H_1} \in \Phi(2)$ and $h_{c_1,\bar{H}_1} \in \Psi(2)$.

Take the coordinate system in $F^3$ such that $s_1 = (0,0,1)$ and $a_2 = (1,0,0)$ and $\bar{H}_1 = Z((1,0,0),(0,1,0))$. Then the matrix of the homology $h_{c_1,\bar{H}_1}$ has the form

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & v \\ 0 & 1 & w \\ 0 & 0 & -1 \end{pmatrix}$$

for some $v, w \in F$. In this case we do not consider the homologies with fundamental line $Z((1,0,0),(0,0,1))$ because if $\bar{H}_1 = Z((1,0,0),(0,0,1))$, then $g_{s_1,H_1}h_{c_1,\bar{H}_1}(s_1) = h_{c_1,\bar{H}_1}g_{s_1,H_1}(s_1) = s_1$. Hence we cannot obtain a homology $f_{a,H}$ such that $s_1 \neq a$ and $s_1 \notin H$.

Consider the products $\mathbf{HG}_i$ and $\mathbf{G}_i\mathbf{H}$, $i = 2,3,4$ (since the homology $g$ with the matrix $\mathbf{G}_1$ belongs to $\Phi(2) \cap \Psi(2)$, and the compositions $hg$ and $gh$, where $h \in \Psi(2)$, were considered in the case II, we can ommit the matrix $\mathbf{G}_1$). Let $\mathbf{M} \in \{\mathbf{HG}_i | i = 2,3,4\} \cup \{\mathbf{G}_i\mathbf{H} | i = 2,3,4\}$; then $\mathbf{M}$ is a matrix of a harmonic homology if and only if $\mathbf{MM} = c\mathbf{I}$ for some $c \in F$ and $c \neq 0$. Calculations show that then

$$\mathbf{HG}_i = \mathbf{G}_i\mathbf{H} = \mathbf{I}, \quad i = 2,3,4.$$

By the cases I, II and III, we infer that if the harmonic homology $f_{a,H}$ is such that $a \neq s_1$ and $a \neq a_2$ and $s_1 \notin H$ and $a_2 \notin H$, then there do not exist $h_1, h_2 \in \Theta(2)$ such that $f_{a,H} = h_1h_2$. ∎

## 3. Decomposition of a projective collineations in $P^n(F)$ into harmonic homologies belonging to the class $\Theta(n)$

Let us recall some theorems on generating of the group of projective collineations by harmonic homologies.

THEOREM 5 (Witczynski [4]). *For every projective collineation in $P^2(F)$ there exists $j \leq 3$ such that this projective collineation is a composition of $j$ harmonic homologies.* ∎

THEOREM 6 (Witczynski [5]). *For every projective collineation in $P^3(C)$ there exists $j \leq 4$ such that this projective collineation is a composition of $j$ harmonic homologies.* ∎

THEOREM 7 (Witczynski [3]). *For every projective collineation in* $P^n(F)$ *there exists* $j \leq 6E(\frac{n+1}{2}) + 1$, *where* $E(a)$ *denotes the integer part of* $a$, *such that this projective collineation is a composition of* $j$ *harmonic homologies.* ∎

By Theorems 2, 3, 5, 6 and 7 and Corollary 1 we obtain the corollaries.

COROLLARY 2. *The group of projective collineations of* $P^2(F)$ *is generated by the class* $\Theta(2)$. *The rank of generation is not greater than 9.* ∎

COROLLARY 3. *The group of projective collineations of* $P^3(C)$ *is generated by the class* $\Theta(3)$. *The rank of generation is not greater than 12.* ∎

COROLLARY 4. *The group of projective collineations of* $P^n(F)$, $n > 2$, *is generated by the class* $\Theta(n)$. *The rank of generation is not greater than* $18E(\frac{n+1}{2}) + 3$. ∎

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*Received January 15, 1999.*