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ON A CLASS OF GENERALIZED FREDHOLM OPERATORS, VI

Abstract. Let X be a complex Banach space and T a generalized Fredholm operator on X (see [3], [4], [5], [6] and [7]). In [7] we have shown that T has a Kato decomposition (X_1, X_2). We say that a Kato decomposition (X_1, X_2) of T is *non-trivial* if $X_2 \neq \{0\}$. The main result of this paper reads as follows:

Let T be a generalized Fredholm operator with a non-trivial Kato decomposition. Then

(i) The subspace X_2 of each Kato decomposition of T is unique if and only if T has finite ascent.

(ii) The subspace X_1 of each Kato decomposition of T is unique if and only if T has finite descent.

(iii) T has a unique Kato decomposition if and only if 0 is a pole of the resolvent $(T - \lambda I)^{-1}$.

1. Introduction and notations

In this paper X always denotes a complex, infinite-dimensional Banach space. Notations and definitions not explicitly given are taken from our previous papers [3], [4], [5], [6] and [7].

DEFINITION. Let $T \in \mathcal{L}(X)$. We say that T has the *Kato decomposition* (X_1, X_2) if X_1 and X_2 are closed, T-invariant subspaces of X with

(i) $X = X_1 \oplus X_2$;
(ii) if $T_1 = T|_{X_1}$ then $T_1(X_1)$ is closed and $N(T_1) \subseteq \bigcap_{n \geq 1} T_1^n(X_1)$;
(iii) if $T_2 = T|_{X_2}$ then T_2 is nilpotent.

In [7] we have shown that each operator in $\Phi_g(X)$ has a Kato decomposition.

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PROPOSITION 1.1. *Let $T \in \Phi_g(X)$ and (X_1, X_2) a Kato decomposition of T. If $T_1 = T|_{X_1}$, $T_2 = T|_{X_2}$ and $m \geq 0$ with $T_2^m = 0$, then:*

(1) $T_1 \in \Phi(X_1)$, $j(T_1) = 0$ *and* $T_2 \in \mathcal{F}(X_2)$.
(2) $N(T^k) = N(T_1^k) \oplus X_2$ *for each* $k \geq m$.
(3) $T^n(X) = T_1^n(X)$ *for* $n \geq m$ *and* $\bigcap_{n\geq 1} T^n(X) = \bigcap_{n \geq 1} T_1^n(X_1)$.

Proof. (1) is shown in [7], Proposition 3.6, for a special Kato decomposition. The proof there works for an arbitrary Kato decomposition.

(2) $N(T^k) = N(T_1^k) \oplus N(T_2^k) = N(T_1^k) \oplus X_2$ for $k \geq m$.

(3) For $n \geq m$, $T^n(X) = T^n(X_1 \oplus X_2) = T^n(X_1) = T_1^n(X_1)$, hence $\bigcap_{n\geq 1} T_1^n(X_1) = \bigcap_{n\geq m} T_1^n(X_1) = \bigcap_{n \geq m} T^n(X) = \bigcap_{n\geq 1} T^n(X)$. ■

NOTATIONS. By X^* we denote the dual space of X and by T^* the adjoint of $T \in \mathcal{L}(X)$. If M is a subspace of X then $M^\perp$ is defined as follows:

$$M^\perp = \{\varphi \in X^* : \varphi(x) = 0 \text{ for all } x \in M\}.$$

For a subspace N of X^* we write ${}^\perp N$ for the subspace

$${}^\perp N = \{x \in X : \varphi(x) = 0 \text{ for all } \varphi \in N\}.$$

If $T \in \Phi_g(X)$, then we know from [3] and [5] that

$$T^* \in \Phi_g(X^*) \text{ and } T^n(X) \text{ is closed for each } n \geq 0.$$

Thus we also have that $(T^*)^n(X^*)$ is closed for all $n \geq 0$. We shall make frequent use of this properties (without further reference).

The proof for the following technical result is entirely elementary.

PROPOSITION 1.2. *Let $T \in \mathcal{L}(X)$. Then*

$$N(T) \subseteq \bigcap_{n\geq 1} T^n(X) \iff \bigcup_{n\geq 1} N(T^n) \subseteq T(X)$$
$$\iff \bigcup_{n\geq 1} N(T^n) \subseteq \bigcap_{n \geq 1} T^n(X).$$

REMARK 1.3. Let $T \in \mathcal{L}(X)$. If X_1 and X_2 are T-invariant subspaces of X with $X = X_1 \oplus X_2$ and if m is an integer ≥ 0 with $(T|_{X_2})^m = 0$, then the assertions (2) and (3) of Proposition 1.1. remain valid.

2. On Kato decompositions

In this section we prove some results which are similar to the results obtained in [2] for Hilbert space operators. Throughout this section let T be an operator in $\mathcal{L}(X)$.

PROPOSITION 2.1. *Suppose that (Y_1, Y_2) and (Z_1, Z_2) are Kato decompositions of T. Then (Y_1, Z_2) and (Z_1, Y_2) are Kato decompositions of T.*

Proof. It suffices to show that (Y_1, Z_2) is a Kato decomposition of T. If $X = Y_1 + Z_2$ and $Y_1 \cap Z_2 = \{0\}$, then we are done. Put $R_i = T|_{Y_i}$ $(i = 1, 2)$ and $S_i = T|_{Z_i}$ $(i = 1, 2)$. Take integers $\mu, \nu \geq 0$ with $R_2^\mu = 0$ and $S_2^\nu = 0$. Let $n = \max\{\mu, \nu\}$.

Take $x \in Y_1 \cap Z_2$. Then $T^\nu x = S_2^\nu x = 0$, hence $S_1^\nu x = T^\nu x = 0$, thus

$$x \in N(S_1^\nu) \subseteq \bigcap_{k \geq 1} S_1^k(Z_1) \subseteq Z_1.$$

This gives $x \in Z_1 \cap Z_2 = \{0\}$. Therefore we have $Y_1 \cap Z_2 = \{0\}$.

Now we show that $X = Y_1 + Z_2$. Let $x \in X$. Then there are $y_i \in Y_i$ $(i = 1, 2)$ and $z_i \in Z_i$ $(i = 1, 2)$ such that $x = y_1 + y_2 = z_1 + z_2$. It follows that $y_1 - z_1 = z_2 - y_2$ and therefore

$$T^n(y_1 - z_1) = T^n(z_2 - y_2) = S_2^n z_2 - R_2^n y_2 = 0,$$

hence

$$y_1 - z_1 \in N(T^n) = N(S_1^n) \oplus Z_2$$

(Proposition 1.1). Thus there are $\xi_1 \in N(S_1^n)$ and $\xi_2 \in Z_2$ such that $y_1 - z_1 = \xi_1 + \xi_2$. From $y_1 - z_1 - \xi_2 = \xi_1 \in N(S_1^n)$ and Proposition 1.2 we get

$$y_1 - z_1 - \xi_2 \in \bigcap_{k \geq 1} S_1^k(Z_1) = \bigcap_{k \geq 1} T^k(X) = \bigcap_{k \geq 1} R_1^k(Y_1) \subseteq Y_1.$$

Hence $z_1 + \xi_2 \in Y_1$. This gives $z_1 \in Y_1 + Z_2$. It results that $x = z_1 + z_2 \in Y_1 + Z_2 + Z_2 = Y_1 + Z_2$. ∎

PROPOSITION 2.2. *Suppose that (Y_1, Y_2) and (Z_1, Z_2) are Kato decompositions of T. Then there is $A \in \mathcal{L}(X)^{-1}$ with*

$$TA = AT, \quad Z_1 = A(Y_1) \quad \text{and} \quad Z_2 = A(Y_2).$$

Proof. Take projections P and Q in $\mathcal{L}(X)$ with

$$P(X) = Y_1, \quad N(P) = Y_2, \quad Q(X) = Z_1, \quad N(Q) = Z_2.$$

Put $A = QP + (I - Q)(I - P)$. Since $TP = PT$ and $QT = TQ$, we get $TA = AT$.

A is injective. In fact, let $x \in N(A)$, then $QPx = -(I - Q)(I - P)x$, thus $QPx \in Q(X) \cap (I - Q)(X) = \{0\}$, hence $Px \in N(Q) \cap P(X) = Y_1 \cap Z_2 = \{0\}$ (Proposition 2.1). It follows that $0 = QPx = -(I - Q)(I - P)x = -x + Qx$, hence $x = Qx$. Since $x \in N(P)$, we get $x \in N(P) \cap Q(X) = Y_2 \cap Z_1 = \{0\}$, thus $x = 0$.

A is surjective. In fact, let $y \in X$. Then $y = z_1 + z_2$ with $z_i \in Z_i$ $(i = 1, 2)$. From Proposition 2.1 we derive $X = Y_1 \oplus Z_2$, thus $z_1 = y_1 + w_2$ for some

$y_1 \in Y_1$ and some $w_2 \in Z_2$. We obtain

$$Py_1 = y_1, \quad Qz_1 = z_1 \quad \text{and} \quad Qw_2 = 0.$$

Thus

$$QPy_1 = Qy_1 = Q(z_1 - w_2) = z_1. \tag{2.1}$$

Since $X = Y_2 \oplus Z_1$, we get $z_2 = y_2 + w_1$, $y_2 \in Y_2$ and $w_1 \in Z_1$. Then

$$\begin{aligned}(I-Q)(I-P)y_2 &= (I-Q)y_2 = (I-Q)(z_2 - w_1) \\ &= (I-Q)z_2 - (I-Q)w_1 = z_2.\end{aligned} \tag{2.2}$$

From (2.1), (2.2), $(I-Q)(I-P)y_1 = 0$ and $QPy_2 = 0$ we obtain

$$\begin{aligned}A(y_1 + y_2) &= QPy_1 + (I-Q)(I-P)y_1 + QPy_2 + (I-Q)(I-P)y_2 \\ &= z_1 + z_2 = y.\end{aligned}$$

It remains to show that $Z_1 = A(Y_1)$ (the proof for $Z_2 = A(Y_2)$ is similar). Take $y_1 \in Y_1$. Then $Py_1 = y_1$, thus $Ay_1 = QPy_1 + (I-Q)(I-P)y_1 = QPy_1 = Qy_1 \in Q(X) = Z_1$. Hence $A(Y_1) \subseteq Z_1$.

Let $u_1 \in Z_1$. Since $X = Y_1 \oplus Z_2$, there are $v_1 \in Y_1$ and $v_2 \in Z_2$ with $u_1 = v_1 + v_2$. Then we get

$$u_1 = Qu_1 = Q(v_1 + v_2) = Qv_1 = QPv_1$$

and

$$(I-Q)(I-P)v_1 = 0.$$

Thus $u_1 = QPv_1 + (I-Q)(I-P)v_1 = Av_1 \in A(Y_1)$. It follows that $Z_1 \subseteq A(Y_1)$. ∎

PROPOSITION 2.3. *Suppose that* (Y_1, Y_2) *is a Kato decomposition of* T, $A \in \mathcal{L}(X)^{-1}$ *and* $TA = AT$. *Then* $(A(Y_1), A(Y_2))$ *is a Kato decomposition of* T.

Proof. Since $A^{-1} \in \mathcal{L}(X)$ and $TA = AT$, it is easy to see that $A(Y_1)$ and $A(Y_2)$ are closed, T-invariant subspaces of X. Furthermore, we have

$$A(Y_1) \cap A(Y_2) = \{0\} \quad \text{and} \quad X = A(Y_1) + A(Y_2).$$

Thus $X = A(Y_1) \oplus A(Y_2)$.

Put $T_i = T|_{Y_i}$ and $R_i = T|_{A(Y_i)}$ $(i = 1, 2)$. Take a convergent sequence (y_n) in $R_1(A(Y_1))$ and put $y_0 = \lim_{n\to\infty} y_n$. Then there is a sequence (x_n) in Y_1 with $y_n = R_1Ax_n = TAx_n = ATx_n$. It follows that $Tx_n = A^{-1}y_n \to A^{-1}y_0$ $(n \to \infty)$. Since $T(Y_1)$ is closed, we get $A^{-1}y_0 \in T(Y_1)$, thus $A^{-1}y_0 = Tx_0$ for some $x_0 \in Y_1$. This gives $y_0 = ATx_0 = TAx_0 = R_1Ax_0 \in R_1(A(Y_1))$. Thus $R_1(A(Y_1))$ is closed.

Now take $u \in N(R_1)$. We have $u = Av$ for some $v \in Y_1$. This gives $ATv = TAv = Tu = R_1u = 0$, hence $T_1v = Tv = 0$. From $N(T_1) \subseteq \bigcap_{k\geq 1} T_1^k(Y_1)$ it results that

$$u \in \bigcap_{k\geq 1} AT_1^k(Y_1) = \bigcap_{k\geq 1} T^k(A(Y_1)) = \bigcap_{k\geq 1} R_1^k(A(Y_1)).$$

It remains to show that R_2 is nilpotent. But this is straightforward, since T_2 is nilpotent. ∎

3. Kato decompositions for operators in $\Phi_g(X)$

Throughout this section we assume that T is an operator in $\Phi_g(X)$. Recall from Section 1 that T and T^* have Kato decompositions.

PROPOSITION 3.1.

(1) *If (X_1, X_2) is a Kato decomposition of T, then $(X_2^\perp, X_1^\perp)$ is a Kato decomposition of T^*.*

(2) *If (N_1, N_2) is a Kato decomposition of T^*, then $({}^\perp N_2, {}^\perp N_1)$ is a Kato decomposition of T.*

Proof. (1) It is easy to see that $X_1^\perp$ and $X_2^\perp$ are closed, T^*-invariant subspaces of X^* with $X^* = X_1^\perp \oplus X_2^\perp$. It is also easy to see that $T^*|_{X_1^\perp}$ is nilpotent, since $T|_{X_2}$ is nilpotent. Let ν be an integer ≥ 1 such that $(T^*|_{X_1^\perp})^\nu = 0$ and $(T|_{X_2})^\nu = 0$. Put $\Psi = T^*|_{X_2^\perp}$.

Since $T^* \in \Phi_g(X^*)$, it follows from [5], Proposition 1.5, that $\Psi \in \Phi_g(X_2^\perp)$, thus $\Psi(X_2^\perp)$ is closed. It remains to show that

$$N(\Psi) \subseteq \bigcap_{k\geq 1} \Psi^k(X_2^\perp).$$

By Remark 1.3 and Proposition 1.1 (3) it suffices to show that

$$N(\Psi) \subseteq \bigcap_{k\geq 1} (T^*)^k(X^*). \tag{3.1}$$

Take some integer $\mu \geq \nu$ and put $T_1 = T|_{X_1}$. From Proposition 1.2 we get

$$N(T_1^\mu) \subseteq T_1^\nu(X_1) = T^\nu(X).$$

Therefore (see Proposition 1.1 (2))

$$N(T^\mu) = N(T_1^\mu) \oplus X_2 \subseteq T^\nu(X) \oplus X_2,$$

thus (since all subspaces are closed)

$$N(\Psi^\nu) = N((T^*)^\nu) \cap X_2^\perp \subseteq (T^*)^\mu(X^*).$$

Hence $N(\Psi) \subseteq N(\Psi^\nu) \subseteq (T^*)^\mu(X^*)$ for all $\mu \geq \nu$. This shows that (3.1) holds.

(2) It is easy to see that ${}^{\perp}N_2$ and ${}^{\perp}N_1$ are closed, T-invariant subspaces of X and that $X = {}^{\perp}N_2 \oplus {}^{\perp}N_1$. Put $T_1 = T|_{{}^{\perp}N_2}$, $T_2 = T|_{{}^{\perp}N_1}$, $\Psi_1 = T^*|_{N_1}$ and $\Psi_2 = T^*|_{N_2}$.

There is some integer $\nu \geq 1$ such that $\Psi_2^{\nu} = 0$. Then it is easy to check that $T_2^{\nu} = 0$. From [5], Proposition 1.5, we get $T_1 \in \Phi_g({}^{\perp}N_2)$, hence $T_1({}^{\perp}N_2)$ is closed. It remains to show that

$$N(T_1) \subseteq \bigcap_{k \geq 1} T_1^k({}^{\perp}N_2).$$

As above it suffices to show that

$$N(T_1) \subseteq \bigcap_{k \geq 1} T^k(X). \tag{3.2}$$

Take some integer $\mu \geq \nu$. Then, by Proposition 1.2,

$$N(\Psi_1^{\mu}) \subseteq \Psi_1^{\nu}(N_1) = (T^*)^{\nu}(X^*).$$

Therefore

$$N((T^*)^{\mu}) = N(\Psi_1^{\mu}) \oplus N_2 \subseteq (T^*)^{\nu}(X^*) \oplus N_2.$$

Since all subspaces are closed, we get

$$N(T_1^{\nu}) = N(T^{\nu}) \cap {}^{\perp}N_2 \subseteq T^{\mu}(X).$$

Hence $N(T_1) \subseteq N(T_1^{\nu}) \subseteq T^{\mu}(X)$ for each $\mu \geq 0$. This shows that (3.2) is valid. ∎

Now we come to the main results of this paper. A Kato decomposition (X_1, X_2) of T is called *non-trivial* if $X_2 \neq \{0\}$. From Proposition 2.2 we get that *if T has a non-trivial Kato decomposition, then all Kato decompositions are non-trivial.* It follows that if T has no non-trivial Kato decomposition, then T has the *unique* Kato decomposition $(X, \{0\})$.

In what follows we shall investigate the case where T has a non-trivial Kato decomposition. By m we always denote the smallest integer ≥ 0 such that

$$N(T) \cap T^m(X) = N(T) \cap T^{m+k}(X) \quad \text{for each} \quad k \geq 0$$

(see [7], Proposition 1.3). If T has a non-trivial Kato decomposition then $m > 0$. In fact, suppose to the contrary that $m = 0$. Then $N(T) = N(T) \cap T^k(X)$ for each $k \geq 0$, thus $N(T) \subseteq \bigcap_{k=1}^{\infty} T^k(X)$. Since $T(X)$ is closed, we get that $(X, \{0\})$ is a Kato decomposition. But this is a contradiction, since T has non-trivial Kato decomposition. Furthermore, we have for the special Kato decomposition (Z_1, Z_2) of T constructed in [7]:

$$(T|_{Z_2})^m = 0$$

(see [7], Corollary 2.2).

THEOREM 3.2. *Suppose that T has a non-trivial Kato decomposition. The following conditions are equivalent:*

(1) $p(T) < \infty$.
(2) *T has the SVEP in 0.*
(3) *For each Kato decomposition (X_1, X_2) of T we have that $T|_{X_1}$ is left invertible.*
(4) *For each Kato decomposition (X_1, X_2) of T we have $X_2 = N(T^m)$.*
(5) *For all Kato decompositions (X_1, X_2) and (Y_1, Y_2) of T we have $X_2 = Y_2$.*

Proof. The equivalence of (1), (2) and (3) follows from [5], Theorem 2.5, [6], Theorem 2.9, and the remark in Section 4 of [7].

Suppose that $p(T) < \infty$. Then, by [5], Proposition 1.2, $N(T) \cap T^m(X) = \{0\}$. Let (Z_1, Z_2) be the Kato decomposition of T which we have constructed in [7], Sections 2 and 3. For the subspaces Y and N_j in Section 2 of [7] we then have

$$Y = X \quad \text{and} \quad N_j = N(T^j) \quad (j \geq 0),$$

hence

$$Z_2 = N_m = N(T^m). \tag{3.3}$$

If (X_1, X_2) is an arbitrary Kato decomposition of T, take projections P and $Q \in \mathcal{L}(X)$ such that

$$P(X) = Z_1,\ N(P) = Z_2,\ Q(X) = X_1 \ \text{and} \ N(Q) = X_2.$$

Put $A = QP + (I - Q)(I - P)$. From (3.3) and Proposition 2.2 it follows that $X_2 = A(Z_2) = A(N(T^m))$. Fix some integer $\nu \geq m$ with $(T|X_2)^\nu = 0$. Since $p(T) = m$ ([5], Proposition 2.1), we get

$$X_2 \subseteq N(T^\nu) = N(T^m) = Z_2. \tag{3.4}$$

Now let $z_2 \in Z_2$. Thus $Pz_2 = 0$, $(I - Q)z_2 \in X_2 \subseteq Z_2 = (I - P)(Z_2)$, hence $(I-Q)z_2 = (I-P)(I-Q)z_2$. This equation gives $z_2 - Qz_2 = z_2 - Qz_2 - Pz_2 + PQz_2$, thus $PQz_2 = 0$. Hence $Qz_2 \in Q(X) \cap N(P) = X_1 \cap Z_2 = \{0\}$ (Proposition 2.1). It follows that $z_2 \in N(Q) = X_2$. Thus we have shown that $Z_2 \subseteq X_2$. From (3.4) and (3.3) we get

$$X_2 = Z_2 = N(T^m).$$

Thus we have shown that (1) implies (4). It is clear that (4) implies (5).

(4) implies (1): Let (Z_1, Z_2) be the Kato decomposition as above. Corollary 2.2 (4) in [7] gives $\{0\} = T^m(X) \cap Z_2$, hence

$$T^m(X) \cap N(T) \subseteq T^m(X) \cap N(T^m) = T^m(X) \cap Z_2 = \{0\}.$$

Now use [6], Proposition 2.1, to obtain $p(T) < \infty$.

(5) implies (1): Again let (Z_1, Z_2) be the special Kato decomposition from [7]. Our hypothesis says that

$$Y_2 = Z_2 \text{ for each Kato decompositon } (Y_1, Y_2) \text{ of } T. \tag{3.5}$$

Assume to the contrary that $p(T) = \infty$. Then

$$N(T) \cap \bigcap_{k \geq 1} T^k(X) \neq \{0\}$$

([5], Proposition 2.5). Put $X_0 = \bigcap_{k \geq 1} T^k(X)$. By Theorem 4.8 in [3] we have $T(X_0) = X_0$. Take $x_0 \in N(T) \cap X_0$ with $x_0 \neq 0$. Then we get $x_1, \ldots, x_{m-1} \in X_0$ such that

$$Tx_j = x_{j-1} \quad (j = 1, \ldots, m-1) \quad \text{and} \quad Tx_0 = 0. \tag{3.6}$$

Furthermore, we have from Proposition 1.1

$$x_0, \ldots, x_{m-1} \in Z_1, \tag{3.7}$$

since $Z_2 \neq \{0\}$, $Z_1^{\perp} \neq \{0\}$. Hence there is a functional $\varphi \in Z_1^{\perp}$ with

$$\varphi \neq 0. \tag{3.8}$$

Take a projection $P \in \mathcal{L}(X)$ with $P(X) = Z_1$ and $(I-P)(X) = N(P) = Z_2$. Define the operator $R \in \mathcal{L}(X)$ by

$$Rx = (I-P)x + \sum_{n=0}^{m-1} \varphi(T^n(I-P)x)x_n \quad (x \in X).$$

Since R is the sum of an operator with closed range and a finite-dimensional operator, R has closed range. Put $Y_2 = R(X)$.

Since $TP = PT$, we obtain from (3.7) that for $x \in X$

$$\begin{aligned} TRx &= (I-P)Tx + \sum_{n=0}^{m-1} \varphi(T^n(I-P)x)Tx_n \\ &= (I-P)Tx + \sum_{n=1}^{m-1} \varphi(T^{n-1}(I-P)Tx)x_{n-1} \\ &= (I-P)Tx + \sum_{n=0}^{m-2} \varphi(T^n(I-P)Tx)x_n. \end{aligned} \tag{3.9}$$

Because of $Z_2 = (I-P)(X)$ and $(T|_{Z_2})^m = 0$, it follows that

$$\varphi(T^{m-1}(I-P)Tx)x_{m-1} = \varphi(T^m(I-P)x)x_{m-1} = 0,$$

thus, by (3.9), $TRx = RTx$. This shows that $T(Y_2) \subseteq Y_2$. From $(T|_{(I-P)(X)})^m = 0$ and $T^m x_j = 0$ $(j = 0, \ldots, m-1)$ we derive $T^m(Rx) = 0$ for all $x \in X$. Hence $T|_{Y_2}$ is nilpotent.

Next we show that $Z_1 \cap Y_2 = \{0\}$. Let $x \in Z_1 \cap Y_2$, then $x = Rz$ for some $\in X$, thus

$$x = (I - P)z + \sum_{n=0}^{m-1} \varphi(T^n(I - P)z)x_n,$$

hence

$$x - \sum_{n=0}^{m-1} \varphi(T^n(I - P)z) = (I - P)z.$$

This and (3.7) show that $(I - P)z \in Z_1 \cap Z_2 = \{0\}$, therefore $(I - P)z = 0$, thus $x = 0$.

Now we show that $X = Z_1 + Y_2$. Let $x \in X$, then $x = z_1 + z_2$ with $z_1 = Px$ and $z_2 = (I - P)x$. It follows that

$$x = \underbrace{\Big(z_1 - \sum_{n=0}^{m-1} \varphi(T^n z_2)x_n\Big)}_{\in Z_1} + \underbrace{\Big(z_2 + \sum_{n=0}^{m-1} \varphi(T^n z_2)x_n\Big)}_{\in Y_2},$$

hence $x \in Z_1 + Y_2$.

We summarize: Y_2 is a closed, T-invariant subspace, $T|_{Y_2}$ is nilpotent, and $X = Z_1 \oplus Y_2$. Therefore, (Z_1, Y_2) is a Kato decomposition of T. From (3.5) we obtain $Z_2 = Y_2$. The definition of R and (3.7) show then that

$$\sum_{n=0}^{m-1} \varphi(T^n(I - P)x)x_n \in Z_1 \cap Z_2 = \{0\} \text{ for all } x \in X.$$

Hence $\sum_{n=0}^{m-1} \varphi(T^n v)x_n = 0$ for all $v \in Z_2$. Since $T^{m-1}x_j = 0$ for $j = 0, \dots, m-2$ and $T^{m-1}x_{m-1} = x_0$, we get

$$0 = T^{m-1}\Big(\sum_{n=0}^{m-1} \varphi(T^n v)x_n\Big) = \varphi(T^{m-1}v)x_0 \text{ for all } v \in Z_2.$$

Thus $\sum_{n=0}^{m-2} \varphi(T^n v)x_n = 0$ for each $v \in Z_2$. Applying T^{m-2} to this equation, we get $\varphi(T^{m-2}v)x_0 = 0$. Similar arguments give

$$\varphi(T^j v)x_0 = 0 \text{ for } j = m-3, \dots, 1, 0.$$

Hence $\varphi(v)x_0 = 0$. Since $x_0 \neq 0$, we get

$$\varphi(v) = 0 \text{ for all } v \in Z_2,$$

thus $\varphi \in Z_2^{\perp}$. Since $\varphi \in Z_1^{\perp}$, we obtain $\varphi = 0$. But this contradicts (3.8). ∎

The next theorem follows from Proposition 3.1 and Theorem 3.2 by duality. Recall that

$$q(T) = p(T^*), \quad p(T) = q(T^*) \quad ([5], \text{Proposition 2.2}),$$

$$T^* \text{ has the SVEP in } 0 \iff q(T) < \infty \quad ([5], \text{Proposition 2.5})$$

and

$$q(T) < \infty \iff T|_{X_1} \text{ is right invertible}$$
$$\text{for each Kato decomposition } (X_1, X_2) \text{ of } T$$

([6], Theorem 2.9, and [7], Remark in Section 4).

THEOREM 3.3. *Suppose that T has a non-trivial Kato decomposition. The following assertions are equivalent:*

(1) $q(T) < \infty$.
(2) T^* *has the SVEP in* 0.
(3) *For each Kato decomposition* (X_1, X_2) *of* T *we have* $X_1 = T^m(X)$.
(4) *For each Kato decomposition* (X_1, X_2) *of* T *we have that* $T|_{X_1}$ *is right invertible.*
(5) *For all Kato decompositions* (X_1, X_2) *and* (Y_1, Y_2) *of* T *we have* $X_1 = Y_1$.

The following theorem characterizes operators in $\Phi_g(X)$ which have a non-trivial *unique* Kato decompostion.

THEOREM 3.4. *If T has a non-trivial Kato decomposition, then following conditions are equivalent:*

(1) $0 < p(T) = q(T) < \infty$.
(2) $(T - \lambda I)^{-1}$ *has a pole at* $\lambda = 0$.
(3) 0 *is an isolated point of* $\sigma(T)$.
(4) 0 *is a boundary point of* $\sigma(T)$.
(5) T *has a unique Kato decomposition.*
(6) T *and* T^* *have the SVEP in* 0.
(7) *For each Kato decomposition* (X_1, X_2) *of* T *we have* $T|_{X_1} \in \mathcal{L}(X_1)^{-1}$.
(8) $X = T^m(X) \oplus N(T^m)$.
(9) *For each Kato decomposition* (X_1, X_2) *of* T *we have* $X_1 = T^m(X)$ *and* $X_2 = N(T^m)$.

Proof. The equivalence of (1) and (2) follows from [1], Satz 101.2. From [3], Theorem 4.8 (f), we get the equivalence of (2), (3), and (4). The equivalence of (1) and (5) follows from Theorem 3.2, Theorem 3.3 and [1], Satz 72.3. Use [1], Satz 72.3, and [5], Theorem 2.5, to obtain the equivalence of (6) and (1). From Theorems 3.2 and 3.3 we get that (6) and (7) are equivalent. Satz 72.4 in [1] gives the equivalence of (8) and (1). (5) and (9) are equivalent by Theorem 3.2 and Theorem 3.3. ∎

In [3], Theorem 4.4, we have shown that if $A \in \mathcal{L}(X)$ and $0 < p(A) = q(A) < \infty$, then $A^p \in \Phi_g(X)$, where $p = p(A)$. But, in general, it does *not* follow that $A \in \Phi_g(X)$. By [3], Example 1.7 (b), there is an operator

$A \in \mathcal{L}(X)$ with $A^2 = 0$, but $A \notin \Phi_g(X)$ (observe that $p(A) = q(A) = 2$). This example also shows that a nilpotent operator need not belong to $\Phi_g(X)$.

Now we are in a position to characterize chain-finite operators which belong to $\Phi_g(X)$.

COROLLARY 3.5. *Let $A \in \mathcal{L}(X)$ and $0 < p(A) = q(A) < \infty$. Put $p = p(A)$. Then the following conditions are equivalent:*

(1) $A \in \Phi_g(X)$.
(2) $\dim(A(X) \cap N(A^{p-1})) < \infty$.

Proof. Put $X_1 = A^p(X)$, $X_2 = N(A^p)$. Then, by [1], Satz 101.2, $X = X_1 \oplus X_2$. Furthermore, we have

$$A(X_1) = A^{p+1}(X) = A^p(X) = X_1 \text{ is closed,} \tag{3.10}$$

$$A(X_2) = A(X) \cap N(A^{p-1}) \subseteq X_2 \tag{3.11}$$

and

$$A|_{X_1} \subseteq \mathcal{L}(X_1)^{-1}. \tag{3.12}$$

Put $A_1 = A|_{X_1}$ and $A_2 = A|_{X_2}$.

(1) $\Longrightarrow$ (2): Since $p > 0$ and $X = X_1 \oplus X_2$, we get from (3.10) and (3.11) that (X_1, X_2) is the (unique) non-trivial Kato decomposition of A. Proposition 1.1 shows that $A_2 \in \mathcal{F}(X_2)$, thus (2) follows from (3.11).

(2) $\Longrightarrow$ (1): From (3.10) and (3.11) we see that X_1 and X_2 are closed, A-invariant subspaces of X. (3.12) gives $A_1 \in \Phi_g(X_1)$. Since (2) holds, we obtain from (3.11) that $A_2 \in \mathcal{F}(X_2) \subseteq \Phi_g(X_2)$. Now use Proposition 1.5 in [5] to obtain $A \in \Phi_g(X)$. ∎

COROLLARY 3.6. *Suppose that $A \in \mathcal{L}(X)$ is nilpotent, $A \neq 0$, $n \in \mathbb{N}$, $A^n = 0$ and $A^{n-1} \neq 0$. Then the following conditions are equivalent:*

(1) $A \in \Phi_g(X)$.
(2) $\dim(A(X) \cap N(A^{n-1})) < \infty$.
(3) $A \in \mathcal{F}(X)$.

Proof. Since $X = A^n(X) \oplus N(A^n)$ and $A^{n-1} \neq 0$, we have $p(A) = q(A) = n > 0$. Corollary 3.5 shows that (1) and (2) are equivalent. Because of $\mathcal{F}(X) \subseteq \Phi_g(X)$, (3) implies (1). It remains to show that (1) implies (3). Since A is nilpotent, $\sigma_\Phi(A) = \sigma(A) = \{0\}$, hence A is a Riesz operator. If (1) holds, then it follows from [4], Theorem 3.7, that $A \in \mathcal{F}(X)$. ∎

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