DEMONSTRATIO MATHEMATICA
Vol. XXXII No 4 1999

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ON DUAL BADE THEOREM IN LOCALLY CONVEX $C(K)$-MODULES

Introduction

It is the purpose of this paper to generalize Dual Bade Theorem indicated in [1, Theorem 9. 12] to barrelled locally convex Hausdorff spaces. In [1], Y. Abramovich, E. L. Arenson and A. K. Kitover asked the following question. Let X be a Banach $C(K)$-module or a Banach lattice. Is it true that $Z(X') = W(X')$? They gave a positive solution to this problem in Theorem 9. 12 called Dual Bade Theorem in [1].

Throughout, X denotes a barrelled locally convex Hausdorff space, $L(X)$ denotes the set of all continuous linear operators from X into X, and X' denotes the topological dual of X. We denote by I the identity operator. Let $C(K)$ be the set of all continuous real or complex valued functions defined on a compact Hausdorff space K with the supremum norm. We say that X is a barrelled locally convex $C(K)$-module if the bilinear mapping $(p) : C(K) \times X \to X, (a,x) \to a.x$ satisfies the following conditions:

(1) $1.x = x$ for all $x \in X, 1 \in C(K)$,
(2) $(a.b).x = a.(b.x)$, for each $a, b \in C(K), x \in X$,
(3) the bilinear mapping (p) is separately continuous.

Then, this is accomplished by the following bilinear mappings in two steps:

$$(A)\ X \times X' \to C(K)', (x, x') \to \mu_{x,x'}(a) = x'(a.x), a \in C(K),$$
$$(B)\ C(K)'' \times X' \to X', (a, x') \to (a.x')(x) = a(\mu_{x,x'}), x \in X.$$

The bilinear mapping $(p) : C(K) \times X \to X$ defines a mapping $m : C(K) \to L(X), m(a)x = a.x$, which is norm to strong operator topology continuous unital ($m(1) = I$) algebra homomorphism. Since $C(K)$ is an AM-space with unit, $C(K)''$ is isomorphic to some $C(S)$ with S a hyperstonian, [2, Theorem 15. 7], [4]. It is well-known that X' is a locally convex $C(S)$-module. From

[4], it is known that the linear mapping $m^* : C(K)'' \to L(X')$ defined by $m^*(a)x' = a.x'$, for each $a \in C(K)'', x' \in X'$ satisfies the following properties:

(i) For each $a \in C(K)'', m^*(a)$ is continuous from $X'[\sigma(X',X)]$ into $X'[\sigma(X',X)]$.
(ii) m^* is a $w^* - w^*$ operator topology continuous unital algebra homomorphism.
(iii) For each $a \in C(K)$, $m^*(a) = (m(a))^*$, where $(m(a))^*$ is the adjoint of $m(a)$.

For unexplained notion and terminology we refer to [1], [2], [5]. We introduce the following definitions which can be found in [1] for a Banach space case.

DEFINITION. Let X be a locally convex $C(K)$-module, $x \in X$. Then

$$\Delta(x) = Cl_X\{a.x : a \in C(K), \|a\| \leq 1\}$$

where $'Cl'_X$ denotes the closure in X. Let Y be a linear subspace of X. Then Y is called an ideal if $\Delta(x) \in Y$ for each $x \in Y$. Let Y be an ideal in X. Then the set

$$Z(Y) = \{T : Y \to Y | (\exists \lambda \geq 0)(\forall y \in Y)(Ty \in \lambda \Delta(y))\}$$

is called the center of Y. Fix $x \in X$. We define the set $X(x)$ by

$$X(x) = \overline{span\{a.x : a \in C(K)\}},$$

where the closure is taken with respect to the topology in X. Similar definitions can be done for locally convex $C(S)$-module X'.

Let B be a Boolean algebra in $C(K)''$. Then $m^*(B) = B'$ is an equicontinuous Boolean algebra of projections in $L(X')$, [4]. We denote by $\langle B' \rangle$ the linear span of B' in $L(X')$. With respect to induced w^*-operator topology $\langle B' \rangle$ is a locally solid convex Riesz space, [3], [4]. Fix $x' \in X'$. By $\langle B' \rangle(x')$ we denote the linear subspace of X' generated by $\{Ex' : E \in B'\}$. Define $\phi_{x'} : \langle B' \rangle \to \langle B' \rangle(x')$ by $\phi_{x'}(T) = Tx'$. Note that $\phi_{x'}$ is a Riesz homomorphism and $\langle B' \rangle(x')$ is a locally convex solid Riesz space, [3], [4]. An equicontinuous Boolean algebra M of projections in the locally convex space X is strongly equicontinuous if $\{E_n\}$ converges to zero in $L(X)$ whenever $\{E_n\} \subseteq M$ is a disjoint sequence. Note that $m^*(B) = B'$ is strongly equicontinuous in $L(X')$, [4]. Let A be a weakly closed algebra of continuous linear operators on X. By $LatA$ we denote the set of all closed subspaces of X that are left invariant by every element of A. The set of operators that leave invariant every space in $LatA$ is denoted by $AlgLatA$. The algebra A is called reflexive if $A = AlgLatA$. Since X is a barrelled space, X' is $\sigma(X',X)$-quasicomplete [5, page 148]. Therefore, applying the Corollary 5. 6 in [3], we obtain the following result.

COROLLARY 1 [4, Corollary II. 14]. *Let X be a barelled locally convex $C(K)$-module. Then the following are equivalent:*

(i) $T \in AlgLatm^*(B)$ *and* $T \in L(X')$,
(ii) $T \in \overline{\langle B' \rangle} = \overline{m^*(C(K))}$,

where closure is taken with respect to w^* operator topology.

We introduce the following notations for later references.

$W(X') = \{T \in L(X') : TY \subseteq Y$ for each w^*-closed ideal Y of $X'\}$ and $Z(X')$ is the center of the dual space X'.

THEOREM 2. *Let X be a barrelled locally convex $C(K)$-module and let $T \in L(X')$. Then the following statements are equivalent:*

(i) $T \in W(X') = AlgLatm^*(C(K))$,
(ii) $\mu_{x,x'} = 0 \Rightarrow \mu_{x,Tx'} = 0$, *where* $x \in X, x' \in X'$.

Proof. (i) $\Rightarrow$ (ii). Let $T \in W(X')$ and $x.x' = 0$. For each $a \in C(K)$, $x'(ax) = 0$, i.e., $x'(y) = 0\ \forall y \in X(x)$. Hence $x' \in X(x)^\circ$ (the polar of $X(x)$). Therefore, $TX(x)^\circ \subseteq X(x)^\circ$, i.e., $Tx'(y) = 0\ \forall y \in X(x)$. Thus, $(x.Tx')(a) = 0\ \forall a \in C(K)$ and $x.Tx' = 0$.

(ii) $\Rightarrow$ (i). Let Y be a $\sigma(X',X)$-closed ideal in X' and denote by Y° its polar in X. Clearly, Y° is a closed ideal in X. If $x.x' = 0$, $x \in Y^\circ$, then by (ii) $x.Tx' = 0$, i.e., $(x.Tx')(a) = 0\ \forall a \in C(K)$. This means that $T'Y^\circ \subseteq Y^\circ$, ($T'$ is the adjoint of T). Taking polar both sides we have $TY \subseteq Y$, i.e., $T \in W(X')$.

Now we can state and prove the main theorem that generalizes Theorem 9. 12 in [1].

THEOREM 3. *Let X be a barrelled locally convex $C(K)$-module and let $T \in L(X')$. Then the following are equivalent:*

(i) $T \in Z(X')$,
(ii) $T \in W(X')$.

Proof. (i) $\Rightarrow$ (ii). Let $T \in Z(X')$ and let Y be a $\sigma(X',X)$-closed ideal in X'. For each $y \in Y, \Delta(y) \in Y$ and there exists $0 \leq \lambda$ such that for all $y \in Y, Ty \in \lambda\Delta(y) \subseteq Y$. This means that $TY \subseteq Y$. Therefore, $T \in W(X')$.

(ii) $\Rightarrow$ (i).For this purpose, let $T \in W(X')$ and by the above corollary 1 there exists a net $\{a_\alpha\}$ in $C(K)$, $\|a_\alpha\| \leq 1$ such that $m^*(a_\alpha)x' \to Tx'$ $\forall x' \in X'$ in the $\sigma(X',X)$. For each $x' \in X'$ $m^*(a_\alpha)x' \in \Delta(x')$. Since $\Delta(x')$ is $\sigma(X',X)$ closed, it follows that $T \in Z(X')$.

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Received June 23, 1998.