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SOME LACUNARY INTERPOLATION PROBLEMS FOR THE LAGUERRE ABSCISSAS

Abstract. The problem of modified $(0, 1, \ldots, m-2, m)$ interpolation $m \geq 2$ and $(0, 1, 2, 4)$ interpolation on Laguerre abscissas are investigated.

1. Introduction

Turán and his associates [2], [10] initiated the concept of lacunary interpolation by considering the problem of existence, uniqueness, explicit representation and convergence of interpolatory polynomials of degree $\leq 2n-1$ when the values and second derivates are prescribed on the zeros of

$$\Pi_n(x) = (1-x^2)P'_{(n-1)}(x),$$

where $P_{(n-1)}(x)$ denotes the Legendre polynomial of degree $n-1$. They called this type of interpolation $(0,2)$ interpolation. Since then a lot of work has been done in this direction by several authors [4]–[9], [12]. However, it was Prasad and Saxena [6] who first investigated $(0,2)$ interpolation on the Laguerre abscissas. They showed that there exists a unique polynomial $R_n(x)$ of degree $\leq 2n-1$ satisfying

$$R_n(\xi_k) = \alpha_k, \quad R''_n(\xi_k) = \beta_k, \quad k = 0, 1, \ldots, n-1, \tag{1.1}$$

where

$$0 = \xi_0 < \xi_1 < \cdots < \xi_{n-1} < \infty \tag{1.2}$$

are the zeros of $xL'_n(x) = nL_n^{(-1)}(x)$, $L_n^{(\alpha)}(x)$ being the Laguerre polynomial of degree n, $L_n(x) \equiv L_n^{(0)}(x)$ and α_k, β_k are any preassigned numbers.

Let

$$0 < x_1 < x_2 < \cdots < x_n \tag{1.3}$$

be the zeros of $L_n^{(\alpha)}(x)$, $\alpha > -1$. In [6] it was also mentioned that $(0,2)$ interpolation polynomials on the nodes (1.3) do exist and are unique if $-1 <$

$\alpha < 1$. This result was later extended by Chak and Szabados [3]. Recently Bajpai [1] has considered the problem of $(0,1,3)$ interpolation on the nodes (1.2) and (1.3). Our aim here is to consider the problem of existence and uniqueness of the polynomials in the following cases:

(i) Find polynomials $Q_n(x)$ of degree $\leq mn-2$, $m \geq 2$, satisfying the conditions:

$$(1.4)\quad \begin{cases} Q_n^{(i)}(\xi_k) = y_k^{(i)}, & i = 0,1,2,\ldots,m-2, \quad k = 0,1,2,\ldots,n-1, \\ Q_n^{(m)}(\xi_k) = y_k^{(m)}, & k = 1,2,\ldots,n-1, \end{cases}$$

where $y_k^{(i)}$ are any preassigned real numbers and ξ_k are given by (1.2).

(ii) Find polynomials $S_n(x)$ of degree $\leq mn-1$, $m \geq 2$, satisfying the conditions:

$$(1.5)\quad S_n^{(i)}(\xi_k) = y_k^{(i)}, \quad i = 0,1,2,\ldots,m-2,m, \quad k = 0,1,2,\ldots,n-1.$$

While (ii) is the usual $(0,1,2,\ldots,m-2,m)$ interpolation, case (i) is commonly referred to as modified $(0,1,2,\ldots,m-2,m)$ interpolation on the nodes (1.2).

(iii) Find polynomials $F_n(x)$ of degree $\leq 4n-1$ satisfying the conditions:

$$(1.6)\qquad F_n^{(i)}(x_k) = y_k^{(i)}, \quad i = 0,1,2,4, \quad k = 1,2,\ldots,n,$$

where x_k are given by (1.3). This is $(0,1,2,4)$ interpolation on the nodes (1.3).

In particular, we shall prove the following theorems:

THEOREM 1.1. *There exists a unique polynomial $Q_n(x)$ of degree $\leq mn-2$, $m \geq 2$, satisfying* (1.4).

THEOREM 1.2. *There exists a unique polynomial $S_n(x)$ of degree $\leq mn-1$, $m \geq 2$, satisfying* (1.5).

THEOREM 1.3. *Let $\alpha > -1$ and n be a positive integer. The problem of $(0,1,2,4)$ interpolation is uniquely solvable for nodes* (1.3) *if and only if*

$$(1.7)\qquad P_n^{(\alpha,((3\alpha+3)/2)-n)}\left(\frac{1}{5}\right) \neq 0,$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree n.

It may be noted that particular values $m = 2$ and $m = 3$ in Theorem 1.2 yield the cases discussed in [6] and [1], respectively.

2. Preliminaries

In this section we state a few known results which we shall use later on. From [11] we have

$$(2.1)\qquad xL_n^{(\alpha)}(x) + (\alpha + 1 - x)L_n^{(\alpha)\prime}(x) + nL_n^{(\alpha)}(x) = 0,$$

$$(2.2)\qquad L_n^{(\alpha)}(x) = \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu} \frac{(-x)^\nu}{\nu!},$$

$$(2.3)\qquad L_n^{(\alpha)}(0) = \binom{n+\alpha}{n},$$

$$(2.4)\qquad P_n^{(\alpha,\beta)}(x) = \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu}\binom{n+\beta}{\nu}\left(\frac{x-1}{2}\right)^\nu \left(\frac{x+1}{2}\right)^{n-\nu}.$$

If

$$(2.5)\qquad \omega_n(x) = xL_n'(x),$$

then

$$(2.6)\qquad \omega_n(\xi_k) = 0, k = 0, 1, \ldots, n-1,$$

$$(2.7)\qquad \omega_n'(0) = -n, \omega_n''(0) = n(n-1),$$

$$(2.8)\qquad \omega_n'(\xi_k) = \xi_k L_n''(\xi_k) = \omega_n''(\xi_k), \quad k = 1, 2, \ldots, n-1.$$

3. Some Lemmas

In order to prove our theorems we need the following lemmas.

LEMMA 3.1 (Windauer [13]). *Let $f_1, f_2, \ldots, f_K$ be functions from $\mathbb{R}$ to $\mathbb{R}$ in an interval $I \in \mathbb{R}$ which have derivatives of order $N (K, N \in \mathbb{N})$. Then, in I*

$$\left(\prod_{k=1}^{K} f_k\right)^{(N)} = \sum_{\substack{n_1, n_2, \ldots, n_K = 0 \\ n_1 + n_2 + \ldots + n_K = N}}^{N} \frac{N!}{n_1! n_2! \ldots n_k!} \prod_{k=1}^{K} f_k^{(n_k)}.$$

LEMMA 3.2. *Let $\omega_n(x)$ be as given by (2.5), then for $m \geq 2$,*

$$(3.1)\qquad [\omega_n^{m-1}(x)]^{(m-1)}|_{x=0} = (m-1)!(-n)^{m-1},$$

$$(3.2)\qquad [\omega_n^{m-1}(x)]^{(m)}|_{x=0} = \frac{(m-1)!m!}{2}(-n)^{m-2} n(n-1)$$

and for $k = 1, 2, \ldots, n-1$,

$$(3.3)\qquad [\omega_n^{m-1}(x)]^{(m-1)}|_{x=\xi_k} = (m-1)![\xi_k L_n''(\xi_k)]^{m-1},$$

$$(3.4)\qquad [\omega_n^{m-1}(x)]^{(m)}|_{x=\xi_k} = \frac{(m-1)m!}{2}[\xi_k L_n''(\xi_k)]^{m-1}.$$

Proof. The proof follows from Lemma 3.1 and (2.6)–(2.8).

LEMMA 3.3. *For* $m \geq 2$,

$$\int_{-\infty}^{0} L_n'(t) e^{(m-1)t/2} dt = 1 - \left(\frac{m+1}{m-1}\right)^n. \tag{3.5}$$

P r o o f. The proof of this lemma can be given along the same lines as in [6].

4. Proof of Theorem 1.1

Theorem 1.1 is proved if we could show that the polynomial $Q_n(x)$ of degree $\leq mn-2$, $m \geq 2$, which satisfies the conditions

$$Q_n^{(i)}(\xi_k) = 0, \quad i = 0,1,\ldots,m-2, \quad k = 0,1,\ldots,n-1, \tag{4.1}$$

$$Q_n^{(m)}(\xi_k) = 0, \quad k = 1,2,\ldots,n-1, \tag{4.2}$$

is identically zero. In view of (4.1), it is evident that

$$Q_n(x) = \omega_n^{(m-1)}(x) q_{n-2}(x), \tag{4.3}$$

where $q_{n-2}(x)$ is a polynomial of degree $\leq n-2$ and $\omega_n(x)$ is given by (2.5). Now, differentiating (4.3) m times and using conditions (4.2) we obtain for $k = 1,2,\ldots,n-1$

$$m[\omega_n^{m-1}(x)]^{(m-1)}|_{x=\xi_k} q_{n-2}'(\xi_k) + [\omega_n^{m-1}(x)]^{(m)}|_{x=\xi_k} q_{n-2}(\xi_k) = 0$$

which yields, on using (3.3) and (3.4),

$$m![\xi_k L_n''(\xi_k)]^{m-1} q_{n-2}'(\xi_k) + \frac{(m-1)m!}{2}[\xi_k L_n''(\xi_k)]^{m-1} q_{n-2}(\xi_k) = 0$$

or

$$q_{n-2}'(\xi_k) + \frac{(m-1)}{2} q_{n-2}(\xi_k) = 0.$$

Hence, $q_{n-2}(x) \equiv 0$. Consequently, $Q_n(x) \equiv 0$. This completes the proof of Theorem 1.1.

5. Proof of Theorem 1.2

The theorem will be proved if we show that the polynomial $S_n(x)$ of degree $mn-1$, $m \geq 2$, which satisfies the following conditions

$$S_n^{(i)}(\xi_k) = 0, \quad i = 0,1,\ldots,m-2; \quad k = 0,1,\ldots,n-1, \tag{5.1}$$

$$S_n^{(m)}(\xi_k) = 0, \quad k = 0,1,\ldots,n-1, \tag{5.2}$$

is identically zero. Since $S_n(x)$ satisfies condition (5.1), it can be written as

$$S_n(x) = [\omega_n(x)]^{m-1} g_{n-1}(x), \tag{5.3}$$

where $g_{n-1}(x)$ is a polynomial of degree $\leq n-1$ and $\omega_n(x)$ is defined by (2.5). Now, from (5.3), (5.2) and Lemma 3.2 it follows that

$$g'_{n-1}(0) = \frac{1}{2}(m-1)(n-1)g_{n-1}(0), \tag{5.4}$$

$$g'_{n-1}(\xi_k) + \frac{1}{2}(m-1)g_{n-1}(\xi_k) = 0, \quad k = 1,2,\dots,n-1. \tag{5.5}$$

From (5.5) we get

$$g_{n-1}(x) = e^{-(m-1)x/2}\Big[C\int_{-\infty}^{x} L'_n(t)e^{(m-1)t/2}dt + D\Big].$$

Since the first term of the right-hand side is a polynomial so, $D = 0$. Hence,

$$g_{n-1}(x) = Ce^{-(m-1)x/2}\int_{-\infty}^{x} L'_n(t)e^{(m-1)t/2}dt. \tag{5.6}$$

Further, from (5.6) and Lemma 3.3 it follows that

$$g_{n-1}(0) = C\left[1 - \left(\frac{m+1}{m-1}\right)^n\right], \tag{5.7}$$

$$g'_{n-1}(0) = C\frac{(m-1)}{2}\left[\left(\frac{m+1}{m-1}\right)^n - 1\right] - nC. \tag{5.8}$$

Consequently, from (5.8), (5,7) and (5.4) we obtain for $m \geq 2$

$$n\left[\frac{(m-1)}{2}\left\{\left(\frac{m+1}{m-1}\right)^n - 1\right\} - 1\right]C = 0.$$

Since

$$n\left[\frac{(m-1)}{2}\left\{\left(\frac{m+1}{m-1}\right)^n - 1\right\} - 1\right] \neq 0$$

so $C = 0$. Hence, $g_{n-1}(x) \equiv 0$. Therefore, $S_n(x) \equiv 0$.

6. Proof of Theorem 1.3

The claim that the problem of $(0,1,2,4)$ interpolation is uniquely solvable for the nodes (1.3) is equivalent to the statement that if for a polynomial $F_n(x)$ of degree $\leq 4n-1$ we have

$$F_n^{(i)}(x_k) = 0, i = 0,1,2, k = 1,2,\dots,n, \tag{6.1}$$

$$F_n^{(4)}(x_k) = 0, k = 1,2,\dots,n, \tag{6.2}$$

then $F_n(x)$ is identically zero. Due to (6.1), one can see that

$$F_n(x) = [L_n^{(\alpha)}(x)]^3 f_{n-1}(x), \tag{6.3}$$

where $f_{n-1}(x)$ is a polynomial of degree $\leq n-1$. Now, from (6.3), (6.2), and (2.1) we obtain for $k = 1, 2, \ldots, n$

$$2x_k f'_{n-1}(x_k) - 3(\alpha + 1 - x_k) f_{n-1}(x_k) = 0. \tag{6.4}$$

Thus, from (6.4) it follows that

$$2x f'_{n-1}(x) - 3(\alpha + 1 - x) f_{n-1}(x) = a_n L_n^{(\alpha)}(x), \tag{6.5}$$

where a_n is a constant. Let

$$f_{n-1}(x) = \sum_{k=0}^{n-1} a_k x^k, \tag{6.6}$$

then (6.5), (6.6) and (2.2) yield

$$\sum_{k=1}^{n-1} \left[2k a_k - 3(\alpha+1) a_k + 3a_{k-1} - a_n \binom{n+\alpha}{n-k} \frac{(-1)^k}{k!} \right] x^k \\ -3(\alpha+1) a_0 + 3a_{n-1} x^n = a_n \binom{n+\alpha}{n} + a_n \frac{(-x)^n}{n!}. \tag{6.7}$$

Consequently, from (6.7) we obtain the following system of linear equations in $a_0, a_1, \ldots, a_n$

$$\begin{cases} -3(\alpha+1) a_0 - \binom{n+\alpha}{n} a_n = 0, & \\ 3a_{k-1} + [2k - 3(\alpha+1)] a_k - \dfrac{(-1)^k}{k!} \binom{n+\alpha}{n-k} = 0, & k = 1, 2, \ldots, n-1, \\ 3a_{n-1} - \dfrac{(-1)^n}{n!} a_n = 0. & \end{cases} \tag{6.8}$$

The determinant of (6.8) is given by

$$D_n(\alpha) = \begin{vmatrix} -3(\alpha+1) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\binom{n+\alpha}{n} \\ 3 & -1-3\alpha & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{(-1)^2}{1!}\binom{n+\alpha}{n-1} \\ 0 & 3 & 1-3\alpha & 0 & \cdots & 0 & 0 & 0 & \frac{(-1)^3}{2!}\binom{n+\alpha}{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 3 & 2n-5-3\alpha & \frac{(-1)^n}{(n-1)!}\binom{n+\alpha}{1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 3 & \frac{(-1)^{n+1}}{(n)!}\binom{n+\alpha}{0} \end{vmatrix}. \tag{6.9}$$

On expanding (6.9) with respect to the last column we obtain

$$(6.10)\quad D_n(\alpha) = (-1)^{n+1}\binom{n+\alpha}{n}3^n + \frac{(-1)^{n+1}}{1!}\binom{n+\alpha}{n-1}(-3\alpha-3)3^{n-1} + \frac{(-1)^{n+1}}{2!}\binom{n+\alpha}{n-2}(-3\alpha-3)(-3\alpha-1)3^{n-2}+\dots + \frac{(-1)^{n+1}}{n!}\binom{n+\alpha}{0}(-3\alpha-3)(-3\alpha-1)(-3\alpha+1)\dots(-3\alpha+2n-5)$$

$$= (-1)^{n+1}\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{3^{n-k}}{k!}(-2)^k\Big(\frac{-3\alpha+3}{2}\Big)\Big(\frac{-3\alpha+1}{2}\Big)\dots\Big(\frac{3\alpha+5}{2}-k\Big)$$

$$= (-1)^{n+1}\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{(3\alpha+3)/2}{k}(-2)^k 3^{n-1}.$$

Now, from (6.10) and (2.4) we conclude that

$$D_n(\alpha) = (-1)^{n+1}5^n P_n^{(\alpha,(3\alpha+3)/2-n)}\Big(\frac{1}{5}\Big).$$

Consequently, the condition (1.7) is equivalent to that of the unique solvability of (6.8). This completes the proof of the theorem.

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