DEMONSTRATIO MATHEMATICA
Vol. XXXII No 4 1999

Richard F. Patterson

A CHARACTERIZATION FOR THE LIMIT POINTS OF DOUBLE SEQUENCES

Abstract. In 1944 R.P Agnew characterized limit points of single dimensional sequences by proving the following: Let A be regular and let x_n be a bounded complex sequence, then there exists a subsequence y_n of x_n such that the set L_y of limit points of the transform Y_n of y_n includes the set L_x of limit points of the sequence x_n. In this paper we shall use the definition of Pringsheim limit points in [6] to present a multidimensional analogues of Agnew result in [1].

1. Introduction

In [2]–[5], and [8] the four dimensional matrix transformation $(Ax)_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}$ was studied extensively by Robison and Hamilton. In their work and throughout this paper, the class of four dimensional matrices and double sequences are of real valued entry unless specified otherwise. Here we consider the behavior of four dimensional matrix transformations on the spaces of subsequences of a double sequence. Such a four dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence (defined below) into a P-convergent sequence with the same P-limit. In this papers we shall present an extension of Agnew's theorem in [1] concerning limit points and summability of subsequences. We give a double sequence analogue of Agnew's theorem as follows: Let A be an RH-regular matrix and $x_{k,l}$ be a bounded double complex sequence; then there exists a subsequence $y_{k,l}$ of $x_{k,l}$ such that the set P_y of Pringsheim limit points of the transform $Y_{k,l}$ of $y_{k,l}$ includes the set P_x of Pringsheim limit points of x.

2. Definitions, notations and preliminary results

DEFINITION 2.1 (Pringsheim, [7]). A double sequence $x = [x_{k,l}]$ has *Pringsheim limit* L (denoted by P-lim $x = L$) provided that given $\epsilon > 0$ there

1991 *Mathematics Subject Classification*: Primary 40B05; Secondary 40C05
Key words and phrases: divergent double sequences, subsequences of a double sequences, Pringsheim limit point, P-convergent, P-divergent, RH-regular.

exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. We shall describe such an x more briefly as *P-convergent*.

DEFINITION 2.2 (Pringsheim, [7]). A double sequence x is called *definite divergent*, if for every (arbitrarily large) $G > 0$ there exist two natural numbers n_1 and n_2 such that $|x_{n,k}| > G$ for $n \geq n_1, k \geq n_2$.

DEFINITION 2.3. The double sequence $[y]$ is a double *subsequence* of the sequence $[x]$ provided that there exist two increasing double index sequences $\{n_j\}$ and $\{k_j\}$ such that if $z_j = x_{n_j,k_j}$, then y is formed by

$$\begin{array}{cccc} z_1 & z_2 & z_5 & z_{10} \\ z_4 & z_3 & z_6 & - \\ z_9 & z_8 & z_7 & - \\ - & - & - & - \end{array}$$

A two dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The Silverman-Toeplitz theorem [9], [10] characterizes the regularity of two dimensional matrix transformations. In 1926 Robison presented a four dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded. The definition of the regularity for four dimensional matrices will be stated next, followed by the Robison-Hamilton characterization of the regularity of four dimensional matrices.

DEFINITION 2.4 ([6]). The four dimensional matrix A is said to be *RH-regular* if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

THEOREM 2.1 (Hamilton [2], Robison [8]). *The four dimensional matrix A is RH-regular if and only if*

RH_1 : $P\text{-}\lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*

RH_2 : $P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$;

RH_3 : $P\text{-}\lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each l;*

RH_4 : $P\text{-}\lim_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each k;*

RH_5 : $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ *is P-convergent; and*

RH_6 : *there exist finite positive integers A and B such that* $\sum_{k,l>B} |a_{m,n,k,l}| < A$.

DEFINITION 2.5 ([6]). A number β is called a *Pringsheim limit point* of the double sequence $x = [x_{n,k}]$ provided that there exists a subsequence $= [y_{n,k}]$ of $[x_{n,k}]$ that has Pringsheim limit β: P-$\lim y_{n,k} = \beta$.

REMARK 2.1. The definition of a Pringsheim limit point can also be stated as follows: β is a Pringsheim limit point of x provided that there exist two increasing index sequences $\{n_i\}$ and $\{k_i\}$ such that $\lim_i x_{n_i,k_i} = \beta$.

DEFINITION 2.6 ([6]). A double sequence x is *divergent in the Pringsheim sense* (P-divergent) provided that x does not *converge in the Pringsheim sense* (P-convergent).

REMARK 2.2. Definition 2.6 can also be stated as follows: A double sequence x is P-divergent provided that either x contains at least two subsequences with distinct finite limit points or x contains an unbounded subsequence. Also note that, if x contains an unbounded subsequence then x also contains a definite divergent subsequence. In addition, note that if the double sequence x contains at most a finite number of unbounded rows andór columns then every subsequence of x is bounded. Also the finite number of unbounded rows andór columns does not affect the P-convergence or P-divergence of x and its subsequences.

3. Main Results

The following is a double sequence analogue of the following theorem of Agnew [1]: Let A be regular and let x_n be a bounded complex sequence. Then there exists a subsequence y_n of x_n such that the set L_y of limit points of the transform Y_n of y_n includes the set L_x of limit points of the sequence x_n.

THEOREM 3.1. *Let A be an RH-regular matrix and $\{x_{k,l}\}$ be a bounded double complex sequence. Then there exists a subsequence $\{y_{k,l}\}$ of $\{x_{k,l}\}$ such that the set P_y of Pringsheim limit points of the transform $\{Y_{k,l}\}$ of $\{y_{k,l}\}$ includes the set P_x of Pringsheim limit points of x.*

Proof. Since P_x is a closed set and the set of complex numbers is separable, there exists a countably infinite or finite subset E of P_x such that $\bar{E} = P_x$. Let $\{u_{i,j}\}$ be a double sequence in E that contains all elements of E with $\{u_{i,j}\}$ defined so that each P-limit point of x is a P-limit point of $\{u_{i,j}\}$. Thus for $i, j = 1, 2, \ldots$ let $x^{r,s}_{k_i,l_j}$ $r, s = 1, 2, \ldots$ be a subsequence of x with P-limit points $\bar{u}_{i,j}$, where $\bar{u}_{i,j}$ is a P-limit point of x. Let $\epsilon_{i,j}$ be a bounded double sequence such that P-$\lim_{i,j} \epsilon_{i,j} = 0$. Therefore there exist four increasing index sequences,

$$m_1 < m_2 < m_3 < \cdots, \quad k_1 < k_2 < k_3 < \cdots,$$

$$n_1 < n_2 < n_3 < \cdots, \quad l_1 < l_2 < l_3 < \cdots,$$

which we use to define

$$\sum_1 := \sum_{k,l=1}^{k_i,l_j} a_{m_i,n_j,k,l},$$

$$\sum_2 := \sum_{k\le k_i; l_j<l<l_{j+1}} a_{m_i,n_j,k,l},$$

$$\sum_3 := \sum_{k\le k_i; l\ge l_{j+1}} a_{m_i,n_j,k,l},$$

$$\sum_4 := \sum_{k_i<k<k_{i+1}; l\le l_j} a_{m_i,n_j,k,l},$$

$$\sum_5 := \sum_{k_i<k<k_{i+1}; l\ge l_j} a_{m_i,n_j,k,l},$$

$$\sum_6 := \sum_{k\ge k_{i+1}; l\le l_j} a_{m_i,n_j,k,l},$$

$$\sum_7 := \sum_{k\ge k_{i+1}; l_j<l<l_{j+1}} a_{m_i,n_j,k,l},$$

$$\sum_8 := \sum_{k\ge k_{i+1}; l\ge l_{j+1}} a_{m_i,n_j,k,l},$$

$$\sum_9 := \sum_{k,l\in b_{i,j}} a_{m_i,n_j,k,l}$$

such that for $i, j = 1, 2, \cdots$ and for m_i and n_j sufficiently large we have

$$\Big|\sum_1\Big| = \Big|\sum_{k,l=1}^{k_i,l_j} a_{m_i,n_j,k,l}\Big| \le \sum_{k,l=1}^{k_i,l_j} |a_{m_i,n_j,k,l}| < \epsilon_{i,j}$$

by RH_1,

$$\Big|\sum_2 + \sum_3\Big| = \Big|\sum_{k\le k_i; l>l_j} a_{m_i,n_j,k,l}\Big| \le \sum_{k\le k_i; l>l_j} \Big|a_{m_i,n_j,k,l}\Big| < \epsilon_{i,j}$$

by RH_4 and the fact that k is finite,

$$\Big|\sum_5\Big| = \Big|\sum_{k_i<k<k_{i+1}; l\ge l_j} a_{m_i,n_j,k,l}\Big| \le \sum_{k_i<k<k_{i+1}; l\ge l_j} \Big|a_{m_i,n_j,k,l}\Big| < \epsilon_{i,j}$$

by RH_4 and the fact that k is finite,

$$\Big|\sum_4 + \sum_6\Big| = \Big|\sum_{k>k_i; l\le l_j} a_{m_i,n_j,k,l}\Big| \le \sum_{k>k_i; l\le l_j} \Big|a_{m_i,n_j,k,l}\Big| < \epsilon_{i,j}$$

by RH_3 and the finiteness of l, and finally

$$\left|\sum_7\right| = \left|\sum_{l_j<l<l_{j+1};k\geq k_{i+1}} a_{m_i,n_j,k,l}\right| \leq \sum_{l_j<l<l_{j+1};k\geq k_{i+1}} \left|a_{m_i,n_j,k,l}\right| < \epsilon_{i,j}$$

by RH_3 and the finiteness of l,

$$\left|\sum_8\right| = \left|\sum_{k\geq k_{i+1};l\geq l_{j+1}} a_{m_i,n_j,k,l}\right| < \epsilon_{i,j}$$

by RH_6. Thus by the above equality we have

$$\sum_{k,l=1}^{\infty,\infty} a_{m_i,n_j,k,l} = \sum_1 + \cdots + \sum_8 + \sum_9,$$

and

$$\sum_9 = \sum_{k,l=1}^{\infty,\infty} a_{m_i,n_j,k,l} - (\sum_1 + \cdots + \sum_8).$$

Therefore by RH_2

$$\sum_9 = 1 + \epsilon_{i,j}.$$

Having selected the subsequence $y_{k,l}$ from x for fixed i and j with $k \leq k_i$ and $l \leq l_j$, we now choose $y_{k,l}$ with $k_i < k < k_{i+1}$ and $l_j < l < l_{j+1}$ from $x^{r,s}_{k_i,l_j}$ such that $y_{r,s}$ is the predecessor of $y_{k,l}$ in x whenever $r < k$ or $s < l$ and

$$\left|y_{k,l} - \bar{u}_{i,j}\right| < \epsilon_{i,j}.$$

Note that the double sequence $y_{k,l}$ is bounded because $x_{k,l}$ is bounded. Therefore by the inequalities above we obtain

$$\begin{aligned}(Ay)_{m_i,n_j} &= \sum_{k,l=1}^{\infty,\infty} a_{m_i,n_j,k,l}y_{k,l} = \sum_1 + \sum_2 + \cdots + \sum_9 \\ &= \epsilon_{i,j} + \sum_9.\end{aligned}$$

Also, we obtain

$$\begin{aligned}\sum_9 &= \sum_{k,l\in b_{i,j}} a_{m_i,n_j,k,l}(y_{k,l} - \bar{u}_{i,j} + \bar{u}_{i,j}) \\ &= \bar{u}_{i,j} \sum_{k,l\in b_{i,j}} a_{m_i,n_j,k,l} + \sum_{k,l\in b_{i,j}} a_{m_i,n_j,k,l}(y_{k,l} - \bar{u}_{i,j}) \\ &= \bar{u}_{i,j}(1 + \epsilon_{i,j}) + (1 + \epsilon_{i,j})\epsilon_{i,j} \\ &= \bar{u}_{i,j} + \epsilon_{i,j}.\end{aligned}$$

Hence,

$$(Ay)_{m_i,n_j} = \bar{u}_{i,j} + \epsilon_{i,j},$$

which implies that P_y includes the set P_x. ∎

Acknowledgements. This paper is based on the author's doctoral dissertation, written under the supervision of Prof. J. A. Fridy at Kent State University. I am extremely grateful to him for his encouragement and advice.

References

[1] R. P. Agnew, *Summability of subsequences*, Bull. Amer. Math. Soc. 50 (1944), 596–598.
[2] H. J. Hamilton, *Transformations of multiple sequences*, Duke Math. J., 2 (1936), 29–60.
[3] H. J. Hamilton, *A generalization of multiple sequences transformation*, Duke Math. J. 4 (1938), 343–358.
[4] H. J. Hamilton, *Change of dimension in sequence transformation*, Duke Math. J. 4 (1938), 341–342.
[5] H. J. Hamilton, *Preservation of partial limits in multiple sequence transformations*, Duke Math. J. 5 (1939), 293–297.
[6] R. F. Patterson, *Analogues of some fundamental theorems of summability theory*, to be published in the international Journal of Mathematics and Mathematical Sciences.
[7] A. Pringsheim, *Zur theorie der zweifach unendlichen Zahlenfolgen*, Math. Ann. 53 (1900), 289–321.
[8] G. M. Robison, *Divergent double sequences and series*, Amer. Math. Soc. Trans. 28 (1926), 50–73.
[9] L. L. Silverman, *On the definition of the sum of a divergent series*, unpublished thesis, University of Missouri Studies, Mathematics Series.
[10] O. Toeplitz, *Über Allgenmeine Linear Mittelbrildungen*, Prace Matemalyczno Fizyczne (Warsaw), 22 (1911).

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NORTH FLORIDA
4565 St. John's Bluff Rd. S.
JACKSONVILLE, FL 32224 USA
E-mail: pattersr@mathcs.duq.edu

Received November 18, 1998.