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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF DIFFERENCE EQUATIONS OF SECOND ORDER

Abstract. Asymptotic properties of solutions of difference equation of the form

$$\Delta^2 x_n = a_n \varphi(x_{n+k})\psi(x_{n+p}) + b_n$$

are studied.

Let N denote the set of positive integers, let Z be the set of integers and let R be the set of real numbers. In this paper we consider the difference equation of the form

$$(E) \qquad \Delta^2 x_n = a_n \varphi(x_{n+k})\psi(x_{n+p}) + b_n, \quad n \in N$$

where $k, p \in Z$, $a_n, b_n \in R$, $\varphi, \psi : R \to R$ are given. The asymptotic behavior of solutions will be investigated. The results obtained here generalize some results of A. Drozdowicz, J. Popenda [2], [3] and J. Migda, M. Migda [6].

By a solution of equation (E) we mean a sequence $\{x_n\}$ if there exists $q \in N$ such that the equation (E) is satisfied for all $n \geq q$.

We start our investigations with a useful lemma, given here without proof, which is elementary.

LEMMA 1. *Assume the series $\sum_{n=1}^{\infty} n|a_n|$ is convergent and let $r_n = \sum_{j=n}^{\infty} a_j$. Then the series $\sum_{n=1}^{\infty} r_n$ is absolutely convergent and $\sum_{n=1}^{\infty} r_n = \sum_{n=1}^{\infty} na_n$.*

THEOREM 1. *Assume that the functions φ, ψ are continuous and the series $\sum_{n=1}^{\infty} na_n$, $\sum_{n=1}^{\infty} nb_n$ are absolutely convergent. Then for every $c \in R$ there exists a solution of (E) convergent to c.*

Proof. Fix $c \in R$. Choose a number $a > 0$. Let

$$X = [c-a, c+a] \times [c-a, c+a].$$

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Then the formula

$$d((t,s),(g,h)) = \max\{|t-g|,|s-h|\}$$

defines a metric on X which is equivalent to the standard euclidean metric. Hence (X,d) is a compact metric space and the function $\mu : X \to R$ defined by

$$\mu(t,s) = \varphi(t)\ \psi(s)$$

is bounded and uniformly continuous on (X,d). Hence there is a constant $M > 1$ such that $|\varphi(t)\psi(s)| < M$ for every $t,s \in [c-a,c+a]$.

For $n \in N$ let

$$\alpha_n = |a_n| + |b_n|, \quad r_n = \sum_{j=n}^{\infty} \alpha_j, \quad \varrho_n = \sum_{j=n}^{\infty} r_j.$$

By Lemma 1 the series $\sum_{n=1}^{\infty} r_n$ is convergent. Hence $\varrho_n < \infty$ for any $n \in N$, moreover $\lim \varrho_n = 0$. There exists an index $q \in N$ such that $M\varrho_n < a$ for all $n \geq q$. Let BS denote the Banach space of all bounded sequence $x : N \to R$ equiped with the "sup" norm. Let

$$T = \{x \in \text{BS} \ : \ x_1 = \ldots = x_q = c \quad \text{and} \quad |x_n - c| \geq M\varrho_n \quad \text{for} \quad n \leq q\}.$$

Obviously, T is a convex and closed subset of BS. Choose an $\varepsilon > 0$. Then there exists $m \in N$ such that $M\varrho_n < \varepsilon$ for any $n \geq m$. For $n = 1,\ldots,m$ let G_n denote a finite ε-net for the interval $[c - M\varrho_n, c + M\varrho_n]$ and let

$$G = \{x \in T : x_n = c \quad \text{for} \quad n \geq m \quad \text{and} \quad x_n \in G_n \quad \text{for} \quad q < n < m\}.$$

Then G is a finite ε-net for T. Hence T is a complete and totally bounded metric space and so T is compact.

If $x \in T$ then $x_n \in [c-a,c+a]$ for any $n \in N$. Hence $|\varphi(x_n)\psi(x_k)| < M$ for all $x \in T$, $n,k \in N$.

Assume that $x \in T$. For $n \in N$ let

$$\beta_n = a_n\varphi(x_{n+k})\psi(_{n+p}) + b_n, \qquad u_n = \sum_{j=n}^{\infty} \beta_j.$$

Then $|\beta_n| \leq M\alpha_n$. Thus

$$|u_n| \leq \sum_{j=n}^{\infty} |\beta_j| \leq \sum_{j=n}^{\infty} M\alpha_j = Mr_n.$$

Since the series $\sum_{j=1}^{\infty} r_j$ is absolutely convergent, the series $\sum_{j=1}^{\infty} u_j$ is absolutely convergent, too. Now we define the sequence $A(x)$ by

$$A(x)(n) = \begin{cases} c & \text{for } n < q \\ c + \sum_{j=n}^{\infty} u_j & \text{for } n \geq q. \end{cases}$$

If $n \geq q$ then

$$|A(x)(n) - c| = \Big|\sum_{j=n}^{\infty} u_j\Big| \leq \sum_{j=n}^{\infty} |u_j| \leq \sum_{j=n}^{\infty} M r_j = M \varrho_n.$$

Hence $A(x) \in T$ and therefore $A(T) \subseteq T$.

Fix $\varepsilon > 0$. Since the function μ is uniformly continuous on X there exists > 0 such that for $(t, s), (g, h) \in X$, $|t - q| < \delta$, $|s - h| < \delta$ we have

$$|\varphi(t)\psi(s) - \varphi(g)\psi(h)| = |\mu(t, s) - \mu(g, h)| < \varepsilon.$$

Now, let $z \in T$ be such that $\|x - z\| < \delta$. Then $|x_n - z_n| < \delta$ for any $n \in N$. Denote by $\gamma_n = a_n \varphi(z_{n+k})\psi(z_{n+p}) + b_n$, and $v_n = \sum_{j=n}^{\infty} \gamma_j$. Then

$$\|A(x) - A(z)\| = \sup_{n \geq q} \Big|\sum_{j=n}^{\infty} u_j - \sum_{j=n}^{\infty} v_j\Big| \leq \sum_{j=q}^{\infty} |u_j - v_j|.$$

Since

$$|u_j - v_j| = \Big|\sum_{j=n}^{\infty} \beta_s - \sum_{s=j}^{\infty} \gamma_s\Big| \leq \sum_{s=j}^{\infty} |\beta_s - \gamma_s|$$

and

$$|\beta_s - \gamma_s| = |a_s \varphi(x_{s+k})\psi(x_{s+p}) - a_s \varphi(z_{s+k})\psi(z_{s+p})| \leq \varepsilon \alpha_s,$$

then $|u_j - v_j| \leq \varepsilon r_j$. Hence

$$\|A(x) - A(z)\| \leq \sum_{j=q}^{\infty} \varepsilon \varrho_q.$$

The latter means that the map $A : T \to T$ is continuous. By Schauder's fixed point theorem ([4] Theorem 3.6.1), there exists $x \in T$ such that $A(x) = x$. So we have

$$x_n = c + \sum_{j=n}^{\infty} u_j$$

for any $n \geq q$. Hence

$$\Delta x_n = c + \sum_{j=n+1}^{\infty} u_j - c - \sum_{j=n}^{\infty} u_j = -u_n.$$

for $n \geq q$. Therefore, for every $n \geq q$ we obtain

$$\Delta^2 x_n = -u_{n+1} + u_n = -\sum_{j=n+1}^{\infty} \beta_j + \sum_{j=n}^{\infty} \beta_j = \beta_n = a_n \varphi(x_{n+k})\psi(x_{n+p}) + b_n$$

By the convergence of the series $\sum u_j$, it follows that $\lim x_n = c$.

THEOREM 2. *If the functions φ, ψ are uniformly continuous and bounded, the series $\sum_{n=1}^{\infty} na_n$, $\sum_{n=1}^{\infty} nb_n$ are absolutely convergent, then for all $c, d \in R$ there exists a solution $\{x_n\}$ of (E) such that $x_n = cn + d + o(1)$.*

Proof. Let $c, d \in R$. Choose $M > 1$ such that $|\varphi(t)\psi(s)| < M$ for every $t, s \in R$. For $n \in N$, as in the proof of Theorem 1, let

$$\alpha_n = |a_n| + |b_n|, \qquad r_n = \sum_{j=n}^{\infty} \alpha_j, \quad \varrho_n = \sum_{j=n}^{\infty} r_j.$$

Let SQ denote the space of all sequences $x : N \to R$. Let

$$T = \{x \in \mathrm{BS} \ : \ |x_n| \le M\varrho, \quad n \in N\},$$
$$S = \{x \in \mathrm{SQ} : |x_n - (cn + d)| \le M\varrho_n, \quad n \in N\}.$$

Let $F : T \to S$ be defined by $F(x)(n) = x_n + cn + d$. Obviously, the formula $d(x, z) = \sup\{|x_n - z_n| : n \in N\}$ defines a metric on S such that F is an isometry of the set T onto S. Since T is compact and convex subset of the space BS, it follows, from Schauder's theorem and from the fact that the set S is homeomorphic to T, that every continuous map $A : S \to S$ has a fixed point.

For $x \in S$, $n \in N$ let $\beta_n = a_n\varphi(x_{n+k})\psi(x_{n+p}) + b_n$, $u_n = \sum_{j=n}^{\infty} \beta_j$,

$$A(x)(n) = cn + d + \sum_{j=n}^{\infty} u_j.$$

Then

$$|A(x)(n) - (cn + d)| = \Big|\sum_{j=n}^{\infty} u_j\Big| \le \sum_{j=n}^{\infty} |u_j| \le M\varrho_n$$

for any $n \in N$. Hence $A(x) \in S$. Thus $A(S) \subseteq S$.

Similarly as in the proof of Theorem 1 one can show that the mapping A is continuous. Therefore there exists a sequence $x \in S$ such that $A(x) = x$. So

$$x_n = cn + d + \sum_{j=n}^{\infty} u_j$$

for any $n \in N$. Since $\Delta^2(cn + d) = 0$, therefore we obtain

$$\Delta^2 x_n = a_n\varphi(x_{n+k})\psi(x_{n+p}) + b_n$$

for any $n \in N$. Moreover, by the convergence of the series $\sum_{j=1}^{\infty} u_j$ it follows that $x_n = cn + d + o(1)$.

THEOREM 3. *Assume there exists $\lambda \in R$ such that the restrictions $\varphi|[\lambda, \infty)$, $\psi|[\lambda, \infty)$ are uniformly continuous and bounded. If the series $\sum_{n=1}^{\infty} na_n$,*

$\sum_{n=1}^{\infty} nb_n$ are absolutely convergent then, for any $c > 0$ and all $d \in R$, there exists a solution $\{x_n\}$ of (E) such that

$$x_n = cn + d + o(1).$$

Proof. Fix $c > 0$, $d \in R$. Choose $M > 0$ such that $|\varphi(t)\psi(s)| < M$ for all $t, s \in [\lambda, \infty)$. For $n \in N$ we define the numbers α_n, r_n, ϱ_n as in the proof of Theorem 1. Choose $q \in N$ such that $cq + d \geq \lambda + M\varrho_1$. Let

$$T = \{x \in \mathrm{BS} : x_1 = \ldots = x_q = 0 \text{ and } |x_n| \leq M\varrho_n \text{ for } n \geq q\},$$

$$S = \{x \in \mathrm{SQ} : x_n = cn + d \text{ for } n \leq q \text{ and } |x_n - cn - d| \leq M\varrho_n \text{ for } n \geq q\}.$$

Similarly as in the proof of Theorem 2, the set S has the fixed point property (i.e., every continuous map $A : S \to S$ has a fixed point).

If $x \in S$ and $n \geq q$ then

$$x_n \geq cn + d - M\varrho_n \geq cq + d - M\varrho_n \geq \lambda + M\varrho_1 - M\varrho_n \geq \lambda.$$

Hence x_{n+k}, $x_{n+p} \in [\lambda, \infty)$ for all $x \in S$ and $n \geq q$. For $x \in S$ and $n \in N$ let $u_n = \sum_{j=n}^{\infty}(a_j\varphi(x_{j+k})\psi(x_{j+p}) + b_j)$ and

$$A(x)(n) = \begin{cases} cn + d & \text{for } n < q \\ cn + d + \sum_{j=n}^{\infty} u_j & \text{for } n \geq q. \end{cases}$$

The rest of the proof is similar to those of Theorems 1 and 2.

EXAMPLE. Let $\varphi(t) = \exp(-t)$, $\psi(t) = 1$ for $t \in R$, $a_n = n^{-3}$, $b_n = 0$ for $n \in N$. Then by Theorem 3 it follows that if $c > 0$ and $d \in R$ then there exists a solution $\{x_n\}$ of (E) such that $x_n = cn + d + o(1)$.

Assume now that $c < 0$, $d \in R$ and $x_n = cn + d + o(1)$ is a solution of (E). Then

$$\Delta^2(x_n) = n^{-3}\exp(-x_{n+k}) = n^{-3}\exp(-(cn + ck + d + o(1))) \to \infty.$$

On the other hand

$$\Delta^2(x_n) = \Delta^2(cn + d + o(1)) = \Delta^2(o(1)) = o(1).$$

From this it follows that $\{x_n\}$ is not a solution of (E).

The proof of the following theorem is similar to that of Theorem 3 and hence it is omitted.

THEOREM 4. *Assume that $\varphi|(-\infty, \lambda]$, $\psi|(-\infty, \lambda]$ are uniformly continuous and bounded for some $\lambda \in R$. If the series $\sum_{n=1}^{\infty} na_n$, $\sum_{n=1}^{\infty} nb_n$ are absolutely convergent then for any $c < 0$ and all $d \in R$ there exists solution $\{x_n\}$ of (E) such that*

$$x_n = cn + d + o(1).$$

LEMMA 2. *If $\{x_n\}$ is a sequence such that the sequence $\{\Delta x_n\}$ is bounded then $\{x_n/n\}$ is also bounded.*

Proof. Assume $|\Delta x_n| \leq M$ for any $n \in N$. Let $z_n = x_n - x_1$. Then $z_1 = 0$ and $\Delta z_n = \Delta x_n$. Since $z_1 \in [-M, M]$ and $|z_2 - z_1| \leq M$, it follows that $z_2 \in [-2M, 2M]$. Similarly, from $|z_3 - z_2| \leq M$ it follows that $z_3 \in [-3M, 3M]$. Analogously $z_n \in [-nM, nM]$ for all $n \in N$. Hence $|z_n| \leq nM$ for any $n \in N$. Thus the sequence $\{z_n/n\}$ is bounded. Therefore the sequence

$$x_n/n = z_n/n + x_1/n$$

is bounded, too.

THEOREM 5. *Assume that the functions φ, ψ are bounded, the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and let a sequence $\{x_n\}$ be a solution of (E). Then*

(a) *if the sequence of partial sums of the series $\sum_{n=1}^{\infty} b_n$ is bounded then the sequence $\{x_n/n\}$ is bounded,*

(b) *if the series $\sum_{n=1}^{\infty} b_n$ is convergent, then $\{x_n/n\}$ is convergent,*

(c) *if $\sum_{n=1}^{\infty} b_n = \infty$, then $\lim(x_n/n) = \infty$,*

(d) *if $\sum_{n=1}^{\infty} b_n = -\infty$, then $\lim(x_n/n) = -\infty$.*

Proof. For $n \in N$ let $u_n = a_n\varphi(x_{n+k})\psi(x_{n+p})$. Since the functions φ, ψ are bounded and the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then the series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent. Since $\Delta^2 x_n = u_n + b_n$, we obtain

$$\begin{aligned}\Delta x_n - \Delta x_1 &= \Delta^2 x_1 + \Delta^2 x_2 + \ldots + \Delta^2 x_{n-1} \\ &= (u_1 + b_1) + \ldots + (u_{n-1} + b_{n-1}) \\ &= (u_1 + \ldots + u_{n-1}) + (b_1 + \ldots + b_{n-1}).\end{aligned}$$

Now, if the sequence of partial sums of the series $\sum_{n=1}^{\infty} b_n$ is bounded then the sequence $\{\Delta x_n\}$ is bounded and, by Lemma 3 it follows that the sequence $\{x_n/n\}$ is also bounded. Moreover, if the series $\sum_{n=1}^{\infty} b_n$ is convergent then the sequence $\{\Delta x_n\}$ is convergent, too. If $\lim \Delta x_n = c$ then

$$\lim(\Delta x_n/\Delta n) = \lim \Delta x_n = c$$

and $\lim(x_n/n) = c$ by the Stolz theorem ([5] p. 55 or [1] Theorem 1.7.9). Analogously, if $\sum_{n=1}^{\infty} b_n = \infty$ then $\lim(\Delta x_n) = \infty$, hence $\lim(x_n/n) = \infty$ by the Stolz theorem. Analogously, if $\sum_{n=1}^{\infty} b_n = -\infty$ then $\lim(x_n/n) = -\infty$.

THEOREM 6. *Assume that the functions φ, ψ are bounded, $q \in N$, $a_n = o(n^q)$, and let a sequence $\{x_n\}$ be a solution of (E). Then*

(a) *if the sequence $\{b_n/n^q\}$ is convergent then $\{x_n/n^{q+2}\}$ is also convergent,*

(b) *if $b_n = o(n^q)$ then $x_n = o(n^{q+2})$.*

Proof. Since $\Delta^2 x_n = a_n \varphi(x_{n+k})\psi(x_{n+p}) + b_n$, the sequence $\{\Delta^2 x_n / n^q\}$ is convergent. If $\lim(\Delta^2 x_n / n^q) = c$, then

$$\lim \frac{\Delta^2 x_n}{\Delta n^{q+1}} = \lim \frac{\Delta^2 x_n}{n^q(g+1+o(1))} = \frac{c}{g+1}\ .$$

By the Stolz theorem we obtain $\lim(\Delta x_n / n^{q+1}) = \frac{c}{q+1}$. Hence

$$\lim \frac{\Delta^2 x_n}{\Delta n^{q+2}} = \lim \frac{\Delta x_n}{n^{q+1}(g+2+o(1))} = \frac{c}{(g+1)(q+2)}.$$

By the Stolz theorem it follows that the sequence $\{x_n / n^{q+2}\}$ is convergent. Obviously, if $b_n = o(n^q)$, then $c = 0$ and $x_n = o(n^{q+2})$.

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