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ON THE EXISTENCE, UNIQUENESS OF SOLUTION OF A NONLINEAR VIBRATIONS EQUATION

1. Introduction

We consider the following initial and boundary value problem

(1.1) $u_{tt} + \gamma\Delta^2 u - B(\|\nabla u\|^2)\Delta u + g(u, u_t) = f(x,t), \; x \in \Omega, \; t > 0,$

(1.2) $u = 0 \quad \text{on} \quad \partial\Omega,$

(1.3) $\dfrac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega,$

(1.4) $u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega,$

where Ω is a bounded domain in R^n with a sufficiently smooth boundary $\partial\Omega$, ν is the outward unit normal vector on the boundary $\partial\Omega$, γ is a positive constant, B, g, f, u_0, u_1 are the given functions. The precise hypotheses on these functions will be specified later. In Eq. (1.1), the function $B(\|\nabla u\|^2)$ depends on the integral

$$\|\nabla u\|^2 = \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i}(x,t) \right|^2 dx \tag{1.5}$$

and satisfy the conditions

(1.6) B is a continuous function defined on $R_+ = [0, +\infty)$;

$$\exists \lambda_0 > 0, D_0 > 0 : \int_0^{\lambda} B(s)ds \geq -D_0 \quad \text{for all} \quad \lambda \gtrsim \lambda_0. \tag{1.7}$$

In [1] the two-dimensional problem ($n = 2$), (1.1), (1.2), (1.4) and

$$\sum_{i=1}^{2} \frac{\partial^2 u}{\partial x_i^2} \nu_i = 0 \quad \text{on} \quad \partial\Omega, \tag{1.3'}$$

was considered, where

$$(1.8)\quad \nu_i = \cos(\nu, 0x_i), \quad \Omega = (0,\pi)\times(0,\pi), \quad \gamma = \frac{\pi^2 h^2}{6}, \quad B(s) = s,$$

$$g(u,u_t) = \epsilon u_t, \epsilon > 0 \quad \text{is a positive constant.}$$

In this case, problem (1.1), (1.2), (1.3′), (1.4) and (1.8) describes the nonlinear vibrations of a square plate with statistic load.

In [5] the following class of quasilinear hyperbolic equation was considered:

$$(1.9)\qquad u_{tt} + (-1)^m B\Big(\int_\Omega |\nabla^m u|^2 dx\Big)\Delta^m u = f(x,t),$$

where B satisfy the following conditions, which are stronger than (1.6), (1.7):

$$(1.10)\qquad B \in C^1(R_+), \quad B(s) \geq b_0 > 0 \forall s \geq 0.$$

In [3] the authors have studied the existence and uniqueness of the following equation

$$(1.11)\qquad u_{tt} + \Delta^2 u - B(\|\nabla u\|^2)\Delta u + |u_t|^{\alpha-1}u_t = f(x,t),$$

where $0 < \alpha < 1$ is a given constant.

In this paper, we use Galerkin and weak compactness method associated with a monotone operator to study the existence and uniqueness of the global solution of the problem (1.1)–(1.4) with respect to the conditions (1.6), (1.7). This result is a relative generalization of [1], [3], [4].

2. Notations

We omit the definitions of the usual function spaces which we will as follows

$$L^p = L^p(\Omega), \quad H^m = H^m(\Omega), \quad H_0^m = H_0^m(\Omega).$$

Let $\langle\cdot,\cdot\rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X.

We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$, a Banach space of the measurable functions $f : (0,T) \to X$ such that

$$\|f\|_{L^p(0,T;X)} = \Big(\int_0^T \|f(t)\|_X^p dt\Big)^{1/p} < \infty \quad \text{for} \quad 1 \leq p < \infty,$$

or

$$\|f\|_{L^\infty(0,T;X)} = \text{esssup}_{0<t<T} \|f(t)\|_X.$$

We make the following assumptions

(H_1) $u_0 \in H_0^2, u_1 \in L^2$,

(H_2) $f \in L^2(Q_T), Q_T = \Omega \times (0,T)$,

(H_3) the function $B : R_+ \to R$ satisfy the following conditions

(i) B is continuous,

(ii) there exist two positive constants λ_0 and D_0 such that

$$\int_0^\lambda B(s)ds \geq -D_0 \quad \text{for all} \quad \lambda \geq \lambda_0,$$

(H_4) the function $g : R^2 \to R$ satisfy the following conditions:

(i) g is continuous,

(ii) g is nondecreasing with respect to the second variable, i.e.,

$$(g(u,v) - g(u,\widetilde{v}))(v - \widetilde{v}) \geq 0 \quad \forall u, v, \widetilde{v} \in R,$$

(iii) there exist two positive constants λ_1 and D_1 such that

$$\int_0^\lambda g(s,0)ds \geq -D_1 \quad \text{for all} \quad \lambda \in R, |\lambda| \geq \lambda_1,$$

(4i) the Nemytsky operator $g : H_0^2 \times L^2 \to L^2$ takes bounded sets into bounded sets,

(5i) the Nemytsky operator $\widehat{g} : H_0^2 \to L^1$ where

$$\widehat{g}(\lambda) = \int_0^\lambda g(s,0)ds,$$

takes bounded sets of H_0^2 into bounded sets of L^1,

(H_5) for each bounded subset M of $H_0^2 \times L^2$ there exists a constant $k_M > 0$ such that

$$\|g(u,w) - g(v,w)\| \leq k_M \|\Delta u - \Delta v\| \quad \forall (u,w),(v,w) \in M,$$

(H_6) for each $r > 0$ there exists a constant $D_r > 0$ such that

$$|B(s_1) - B(s_2)| \leq D_r |s_1 - s_2| \forall s_1, s_2 \in [0,r].$$

REMARK 1. We consider the following function:

(i) $$g(u,u_t) = |u|^\alpha u - u + |u_t|^{\beta-1} u_t \quad \text{or}$$

(ii) $$g(u,u_t) = |u|^\alpha |u_t|^{\beta-1} u_t + |u|^\alpha u - u,$$

where α, β are the constants, with $0 < \beta < 1$, $0 \leq \alpha \leq \frac{2}{n-4}$ if $n \geq 5$ and $0 \leq \alpha < \infty$ if $n = 1,2,3,4$.

Then, g satisfies assumptions (H_4), (H_5).

We also use the notations $u' = u_t = \frac{\partial u}{\partial t}$, $u'' = u_{tt} = \frac{\partial^2 u}{\partial t^2}$.

3. The existence and uniqueness theorem

Without loss of generality, we can suppose that $\gamma = 1$.

THEOREM 1. *Let $T > 0$ be fixed. Let* (H_1)–(H_4) *hold. Then the problem* (1.1)–(1.4) *has at least one weak solution u such that*

$$u \in L^\infty(0,T;H_0^2) \quad \text{and} \quad u_t \in L^\infty(0,T;L^2). \tag{3.1}$$

Furthermore, if g, B satisfy (H_5), (H_6), *the solution is unique.*

Proof. The proof consists of several steps.

STEP 1. The Galerkin approximation (introduced by Lions [2]). Let $\{w_j\}$ be a denumerable base of H_0^2.

Put

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t) w_j,$$

where $c_{mj}(t)$ satisfy the system of nonlinear differential equations

$$\begin{aligned} &\langle u_m''(t), w_j\rangle + \langle \Delta u_m(t), \Delta w_j\rangle + B(\|\nabla u_m(t)\|^2)\langle \nabla u_m(t), \nabla w_j\rangle \\ &\quad + \langle g(u_m(t), u_m'(t)), w_j\rangle = \langle f(t), w_j\rangle, \quad 1 \le j \le m, \end{aligned} \tag{3.2}$$

$$u_m(0) = u_{0m}, \quad u_m'(0) = u_{1m}, \tag{3.3}$$

where

$$u_{0m} \to u_0 \quad \text{strongly in} \quad H_0^2, \tag{3.4}$$

$$u_{1m} \to u_1 \quad \text{strongly in} \quad L^2. \tag{3.5}$$

For fixed $T > 0$, from the assumptions of the theorem, system (3.2), (3.3) has solution $u_m(t)$ on an interval $0 \le t \le T_m$. The following estimates allow one to take $T_m = T$ for all m.

STEP 2. a priori estimates. Multiplying each equation in (3.2) by $c_{mj}'(t)$, summing up with respect to j and then integrating with respect to the time variable from 0 to t, we have

$$S_m(t) + 2\int_0^t \langle g(u_m(s), u_m'(s)), u_m'(s)\rangle ds = S_m(0) + 2\int_0^t \langle f(s), u_m'(s)\rangle ds, \tag{3.6}$$

where

$$S_m(t) = \|u_m'(t)\|^2 + \|\Delta u_m(t)\|^2 + \int_0^{\|\nabla u_m(t)\|^2} B(s)ds. \tag{3.7}$$

Using the monotonicity assumption (H_4, (ii)) with respect to the second variable, we have

$$(3.8)\quad 2\int_0^t \langle g(u_m(s),u'_m(s)),u'_m(s)\rangle ds \geq 2\int_0^t \langle g(u_m(s),0),u'_m(s)\rangle ds$$
$$= 2\int_\Omega \widehat{g}(u_m(x,t))dx - 2\int_\Omega \widehat{g}(u_{0m}(x))dx.$$

Note that from (H_4, (iii)) we obtain

$$(3.9)\qquad \widehat{g}(\lambda) = \int_0^\lambda g(s,0)ds \geq -\widetilde{C}_0 \equiv -\int_{-\lambda_1}^{\lambda_1} |g(s,0)|ds - D_1$$

for all $\lambda \in R$. Then we deduce, from (3.8), (3.9), that

$$(3.10)\quad 2\int_0^t \langle g(u_m(s),u'_m(s)),u'_m(s)\rangle ds \geq -2\widetilde{C}_0\mathrm{meas}\Omega - 2\int_\Omega \widehat{g}(u_{0m}(x))dx.$$

Similarly, from (H_3, (ii)) we also obtain

$$(3.11)\qquad \int_0^\lambda B(s)ds \geq -\widetilde{C}_1 \equiv -\int_0^{\lambda_0} |B(s)|ds - D_0 \quad \text{for all} \quad \lambda \geq 0.$$

It follows from (3.6), (3.10) and (3.11) that

$$(3.12)\quad \|u'_m(t)\|^2 + \|\Delta u_m(t)\|^2 \leq \widetilde{C}_1 + 2\widetilde{C}_0\mathrm{meas}\Omega + 2\int_\Omega \widehat{g}(u_{0m}(x))dx$$
$$+ S_m(0) + \int_0^t \|f(s)\|^2 ds + \int_0^t \|u'_m(s)\|^2 ds.$$

On the other hand, from (3.4), (3.5), using the assumptions (H_3, i) and (H_4,(5i)), we obtain

$$(3.13)\qquad S_m(0) + 2\int_\Omega \widehat{g}(u_{0m}(x))dx \leq C_2 \quad \text{for all} \quad m.$$

Hence, from (3.12), (3.13) we obtain

$$(3.14)\qquad X_m(t) \leq M_T + \int_0^t X_m(s)ds$$

where $X_m(t) = \|u'_m(t)\|^2 + \|\Delta u_m(t)\|^2$, M_T is a constant depending only on T.

By Gronwall's lemma, we obtain from (3.14) that

$$X_m(t) \le M_T e^t \le M_T e^T \quad \forall t \in [0, T_m]. \tag{3.15}$$

Therefore we can take $T_m = T$ for all m and hence

$$\{u_m\} \quad \text{in bounded in} \quad L^\infty(0, T; H_0^2), \tag{3.16}$$

$$\{u_m'\} \quad \text{in bounded in} \quad L^\infty(0, T; L^2). \tag{3.17}$$

Using (3.16), (3.17) and (H_4, (4i)) we get

$$g(u_m, u_m') \quad \text{is bounded in} \quad L^\infty(0, T; L^2). \tag{3.18}$$

On the other hand, from the inequality

$$\|\nabla v\|^2 \le C_0 \|\Delta v\|^2 \quad \forall v \in H_0^2 \tag{3.19}$$

we have

$$|B(\|\nabla u_m\|^2)| \le \max_{0 \le s \le C_0 M_T e^T} |B(s)| \tag{3.20}$$

hence

$$B(\|\nabla u_m\|^2) \nabla u_m \quad \text{is bounded in} \quad L^\infty(0, T; (L^2)^n). \tag{3.21}$$

STEP 3. The limiting process. From (3.16), (3.17) and (3.18), we deduce that there exists a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$, such that

$$u_m \to u \quad \text{in} \quad L^\infty(0, T; H_0^2) \quad \text{weak } *, \tag{3.22}$$

$$u_m' \to u' \quad \text{in} \quad L^\infty(0, T; L^2) \quad \text{weak } *, \tag{3.23}$$

$$g(u_m, u_m') \to \chi \quad \text{in} \quad L^\infty(0, T; L^2) \quad \text{weak } *. \tag{3.24}$$

By the compactness lemma of Lions ([2], p. 57), we can deduce from (3.22), (3.23) that there exists a subsequence, still denoted by $\{u_m\}$, such that

$$u_m \to u \quad \text{in} \quad L^2(0, T; H_0^1) \quad \text{strongly and a.e.} \quad (x, t) \quad \text{in} \quad Q_T. \tag{3.25}$$

By the Riesz–Fischer theorem, from (3.25) we can take a subsequence, still denoted by $\{u_m\}$, such that

$$\|\nabla u_m\| \to \|\nabla u\| \quad \text{a.e.} \quad t \quad \text{in} \quad (0, T). \tag{3.26}$$

Because B is continuous

$$B(\|\nabla u_m\|^2) \to B(\|\nabla u\|^2) \quad \text{a.e.} \quad t \quad \text{in} \quad (0, T) \tag{3.27}$$

then

(3.28) $$B(\|\nabla u_m\|^2)\nabla u_m \to B(\|\nabla u\|^2)\nabla u \quad \text{a.e.} \quad (x,t) \quad \text{in} \quad Q_T.$$

Combining (3.21) and (3.28) with Lemma 1.3 in ([2], p. 12), we have

(3.29) $$B(\|\nabla u_m\|^2)\nabla u_m \to B(\|\nabla u\|^2)\nabla u \quad \text{in} \quad L^\infty(0,T;(L^2)^n) \quad \text{weak} *.$$

Passing to the limit in (3.2) by (3.22)–(3.24) and (3.29) we have

(3.30) $$\frac{d}{dt}\langle u'(t),v\rangle + \langle \Delta u(t), \Delta v\rangle + B(\|\nabla u(t)\|^2)\langle \nabla u(t), \nabla v\rangle + \langle \chi(t), v\rangle = \langle f(t), v\rangle \quad \text{a.e.} \quad t \quad \text{in} \quad (0,T), \quad \forall v \quad \text{in} \quad H_0^2.$$

Since u, $u_m \in C^0(0,T;L^2)$, we have $u_m(0) \to u(0)$ strongly in L^2. Thus

(3.31) $$u(0) = u_0.$$

On the other hand, $\langle u_m'(t), w_j\rangle$ and $\langle u'(t), w_j\rangle$ belong to $C^0(0,T)$. Therefore, $\langle u_m'(0) - u'(0), w_j\rangle \to 0$, as $m \to \infty$. Hence

(3.32) $$u'(0) = u_1.$$

Then, in order to prove the existence of weak solution of the problem (1.1)–(1.4), we only have to prove that: $\chi = g(u,u')$.

We shall now require the following lemma.

LEMMA 1. *Let u be the solution of the following problem*

(3.33) $$u'' + \Delta^2 u + \chi_1 = 0, \quad x \in \Omega, \quad t \in (0,T],$$

(3.34) $$u(x,0) = u_0(x), u'(x,0) = u_1(x),$$

(3.35) $$u \in L^\infty(0,T;H_0^2), \quad u' \in L^\infty(0,T;L^2).$$

Then we have

(3.36) $$\frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \int_0^t \langle \chi_1(s), u'(s)\rangle ds \geq \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\Delta u_0\|^2 \quad \text{a.e.} \quad t \in (0,T).$$

Furthermore, if $u_0 = u_1 = 0$ there is equality in (3.36).

The proof of Lemma 1 can be found in [3].

We now return to the proof of existence of a solution of the problem (1.1)–(1.4).

It follows from (3.2), (3.3) that

$$(3.37)\qquad \int_0^t \langle g(u_m(s), u_m'(s)), u_m'(s)\rangle ds$$

$$= \frac{1}{2}\|u_{1m}\|^2 + \frac{1}{2}\|\Delta u_{0m}\|^2 + \frac{1}{2}\int_0^{\|\nabla u_{0m}\|^2} B(s)ds - \frac{1}{2}\|u_m'(t)\|^2 - \frac{1}{2}\|\Delta u_m(t)\|^2$$

$$-\frac{1}{2}\int_0^{\|\nabla u_m(t)\|^2} B(s)ds + \int_0^t \langle f(d), u_m'(s)\rangle ds.$$

Passing to the limit as $m \to \infty$, by using (3.4), (3.5), (3.22)–(3.24), (3.26) and Lemma 1 with

$$\chi_1 = -B(\|\nabla u\|^2)\Delta u + \chi - f,$$

we obtain

$$(3.38)\qquad \lim_{m\to\infty} \sup \int_0^t \langle g(u_m(s), u_m'(s)), u_m'(s)\rangle ds \le \int_0^t \langle \chi(s), u'(s)\rangle ds,$$

a.e. t in $(0,T)$.

By using the same arguments as in [4] we can show that

$$\chi = g(u,u') \quad \text{a.e. in} \quad Q_T.$$

STEP 3. Uniqueness of the solution. Let u and v be two solutions of the problem (1.1)–(1.4). Then $w = u - v$ satisfies the following problem

$$w'' + \Delta^2 w - B(\|\nabla u\|^2)\Delta w - [B(\|\nabla u\|^2) - B(\|\nabla v\|^2)]\Delta v + g(u,u') - g(v,v') = 0,$$

$$w(0) = w'(0) = 0,$$

$$u, v, w \in L^\infty(0,T; H_0^2), u', v', w' \in L^\infty(0,T; L^2).$$

Using Lemma 1 with $u_0 = u_1 = 0$ we have equality

$$(3.39)\qquad \frac{1}{2}\|w'(t)\|^2 + \frac{1}{2}\|\Delta w(t)\|^2 = -\int_0^t \langle g(u(s), u'(s))$$

$$- g(v(s), v'(s)), w'(s)\rangle ds + \int_0^t B(\|\nabla u(s)\|^2 \langle \Delta w(s), w'(s)\rangle ds$$

$$+ \int_0^t [B(\|\nabla u(s)\|^2) - B(\|\nabla v(s)\|^2)]\langle \Delta v(s), w'(s)\rangle ds.$$

Let

$$
\begin{aligned}
&X(t) = \|w'(t)\|^2 + \|\Delta w(t)\|^2,\\
&R = \max\{\|u'\|_{L^\infty(0,T;L^2)} + \|\Delta u\|_{L^\infty(0,T;H_0^2)}; \|v'\|_{L^\infty(0,T;L^2)} \\
&\qquad\qquad + \|\Delta v\|_{L^\infty(0,T;H_0^2)}\},\\
&M = \{(\emptyset, q) \in H_0^2 \times L^2 : \|\Delta \emptyset\| + \|q\| \le R\},\\
&b_M = \max_{0\le s\le C_0R^2} |B(s)|, r = C_0 R^2,
\end{aligned}
$$

where C_0 is constant as in (3.19).

Noticing that the function g is nondecreasing with respect to the second variable, we have from (3.39) that

$$
\begin{aligned}
(3.41)\quad X(t) \le\ & 2\int_0^t \|g(u(s), v'(s)) - g(v(s), v'(s))\| \|w'(s)\| ds\\
&+ 2\int_0^t |B(\|\nabla u(s)\|^2)| \|\Delta w(s)\| \|w'(s)\| ds\\
&+ 2\int_0^t |B(\|\nabla u(s)\|^2) - B(\|\nabla v(s)\|^2)| \|\Delta v(s)\| \|w'(s)\| ds.
\end{aligned}
$$

Using the assumptions (H_5) and (H_6) it follows from (3.41) that

$$
X(t) \le (k_M + b_M + 2C_0R^2D_r)\int_0^t X(s) ds, \tag{3.42}
$$

i.e., $X = 0$ by Gronwall's lemma.

Theorem 1 is proved completely.

In the case $1 \le n \le 3$, using the imbedding theorem of Sobolev: $H^2 \hookrightarrow C^0(\overline{\Omega})$, it follows that g satisfies the assumption (H_4, (5i)).

Then, we have the following theorem.

THEOREM 2. *Let fix* $T > 0$. *Let* (H_1–H_3), (H_4, (i)–(4i)) *hold.*

Then, the problem (1.1)–(1.4) *has at least one weak solution* u *satisfying* (3.1).

Furthermore, if g, B *satisfy* (H_5), (H_6), *the solution is unique.*

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