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LINEAR BOUNDARY VALUE PROBLEM OF A DEGENERATE ELLIPTIC SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

Abstract. By application of function theoretic methods in partial differential equations (PDE), a nonlinear system of equations, elliptic in the sense of Lavrentiev with a linear boundary condition is investigated. Existence, uniqueness and stability for the boundary value problem (BVP) with degeneration of ellipticity have been proved.

In this paper, we consider the nonlinear system of first order type equations

$$w_{\bar{z}} = F(\theta(z) w_z) \quad \text{in } D, \tag{1}$$

where $w = w(z) = u(x,y) + iv(x,y)$, $z = x + iy$, $w_{\bar{z}} = \frac{\partial w}{\partial \bar{z}}$, $w_z = \frac{\partial w}{\partial z}$ and $\theta(z)$, $F(\eta)$ are complex valued functions. We assume that D is a simply connected Liapounov region with the boundary contour L. Clearly, equation (1) contains the complex form of the Cauchy–Riemann system $w_{\bar{z}} = 0$.

We suppose that system (1) satisfies the following condition.

Condition C_1:

(I) F fulfils the following inequality

$$|F(\eta_1) - F(\eta_2)| \le q(|\eta_1 - \eta_2|)|\eta_1 - \eta_2|, \tag{2}$$

where $q \le 1$;

(II) q as a function of $\beta = |\eta_1 - \eta_2|$ is continuous in $[0, \infty)$; $q(\beta) < 1$ in $(0, \infty)$; the function $\beta q^2(\beta)$ is increasing and concave;

(III) $\theta(z)$ is assumed to be measurable function belonging to the class $L_\infty(D)$.

Concerning the nonlinear system (1), we investigate the following boundary value problem:

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Problem A

$$\operatorname{Re}|\overline{a(t)}w(t)| = \gamma(t) \quad \text{on } L, \tag{3}$$

where the coefficients a, γ are given functions on the boundary L.

Similarly as in the Riemann-Hilbert problem for uniformly elliptic system of equations, the index corresponding to the problem A is defined as follows

$$n = \operatorname{ind} a = \frac{1}{2\pi i} \int_L d(\log a(t)). \tag{4}$$

REMARK 1. Let us recall that, in the classical Riemann-Hilbert boundary values problems of the type (1)–(3), relative to the uniform ellipticity of the nonlinear system of equations of Lavrentiev type, the solution w is sought in the Sobolev space $W_p^1(D)$, for some $p > 2$. For the equation (1) in the case of non-uniform ellipticity (2), we shall not apply the L_p-theory directly to the proof of the existence. Therefore formulation of the boundary values problem A, involves the weak boundary condition (see also [2]).

We shall make the usual assumptions for the coefficients a and γ, i.e.

Condition C_2: a, γ are Hölder continuous on the boundary $L, a \neq 0$, we assume also that $|\theta(z)| < 1$.

The number

$$q_0 = \lim_{\beta \to \infty} \sup(q(\beta)) < 1, \tag{5}$$

is called the ellipticity coefficient corresponding to the boundary values problem A and q_0 shows, how fast the gradient (see[2]) may approach the infinity and consequently this coefficient will influence the exponent $p > 2$ (see also Proposition 1) of integrability of the gradient. Moreover the exponent of the Hölder continuity of the solutions depends on q_0 (see Corollary 1).

REMARK 2. If $F = 0$ in (1), and the index n corresponding to the boundary values problem (1)–(3) is negative, then the non-homogeneous problem A is solvable if and only if the following condition holds

$$\int_L a(t)\psi(t)\gamma(t)dt = 0, \tag{6}$$

where ψ is an arbitrary solution of the homogeneous boundary value problem adjoint to the problem A (see for instance [5] or [6]).

PROPOSITION 1. *If conditions C_1, C_2 and* (6) *are satisfied, and $n < 0$;*

1) *then there exists a solution (in $W_p^1(D)$, for some $p > 2$) of the problem A;*

2) *this solution is unique.*

Proof. Making use of the representation formula for the solution w:

$$w = T(\omega) + \varrho(z), \tag{7}$$

(see for instance [1]), where $T(\omega) = (T(\omega))_D$, $\omega \in L_p(D)$, $p \geq 2$ and $\varrho(z)$ is the solution of the boundary value problem (1)–(3) in the case of $F = 0$, we observe that the operator T depending on the index n, fulfils the homogeneous boundary condition corresponding to (3) on L, when $z \to t$, ($z \in D$, $t \in L$). Moreover $\frac{\partial T(\omega)}{\partial \bar{z}} = \omega(z)$.

Denoting $S\omega = S(\omega) = \frac{\partial T(\omega)}{\partial z}$, since $n < 0$, we conclude that the L_2-norm of S is equal to 1, also S is a bounded operator from $L_p(D)$, $p > 1$, into itself and continuity of $\|S\|_p$ with respect to $p > 1$ can be proved by applying the well-known Riesz–Thorin convexity theorem.

In view of (2), (7), we obtain the following equation for ω:

$$\omega = F(\theta S(\omega) + \theta \varrho'). \tag{8}$$

We solve the singular integral equation (8), using the successive approximation method (at first we assume $\omega \in L_2(D)$).

REMARK 3. The norm $\|\|$ is defined by

$$\|\omega\|_{L_2(D)} = \left(\frac{1}{|D|} \int_D |\omega(z)|^2 d\sigma_z \right)^{\frac{1}{2}}. \tag{9}$$

To prove the existence of ω, let us assume that

$$\omega_{k+1} = F(\theta S(\omega_k) + \theta \varrho') + F(\theta S \omega_k + \theta \varrho'), \tag{10}$$

ω_0, $k = 0, 1, \ldots$. Then from concavity property of $\eta q^2(\eta)$ and the well-known Jenssen inquality, since $n < 0$, $\|S\|_2 = 1$, we obtain (see also [2])

$$\|\omega_{k+1} - \omega_{j+1}\| < q(\|\omega_k - \omega_j\|)\|\omega_k - \omega_j\|, k, j = 0, 1, \ldots \tag{11}$$

In particular, (9) indicates that $\|\omega_{k+1} - \omega_k\|$ decreases and converges and converges to zero, since $q(\beta)$ is assumed to be continuous and less than 1 (see condition C_1) for $\beta \in (0, \infty)$.

In accordance with the above results, the Cauchy condition for the sequence ω_k in the topology of $L_2(D)$ has been proved. Since $F(\eta)$ is continuous, the function $\omega = \lim_{k \to \infty} \omega_k$ fulfils the equation (10).

Now assume that p satisfies the following inequality

$$q_0 \|S\|_p < 1. \tag{12}$$

Then we can conclude that ω belongs to $L_p(D)$ and the solution w of the problem A belongs to the Sobolev space $W_p^1(D)$, $p > 2$.

In order to prove the uniqueness, suppose that ω_1 and ω_2 are the solutions of equation (8), then, according to the condition C_1, we observe that

$$\|\omega_1 - \omega_2\| < q(\|\omega_1 - \omega_2\|)\|\omega_1 - \omega_2\|. \tag{13}$$

Now, in view of the properties of the function q, we have $\omega_1 = \omega_2$ almost everywhere.

COROLLARY 1. *If* $a, \gamma \in C_\alpha(L)$, $0 < \alpha < 1$, *then the solution* w *of the problem* A, *belongs to the class* $C_v(D + L)$, *where* $v = \min(\alpha, 1 - \frac{1}{p})$.

If we omit the assumption of simple connectedness of the domain, then under some suitable formulation, problem A can be extended to multiply-connected domains.

References

[1] H. Begehr and G. C. Hsiao, *The Hilbert boundary value problem for nonlinear elliptic systems*, Proc. Roy. Soc. of Edinburgh, 94A, (1983), 97–112.

[2] T. Iwaniec and A. Mamourian, *On the first order nonlinear differential systems with degeneration of ellipticicty*, Proc. of Sec. Finish-Polish Summer School in Complex Analysis. Jyvasdyla, Edited by J. Lawrynowicz and O. Martio, (1984), 40–52.

[3] E. Lanckau and W. Tutschke, *Complex Analysis, Methods, Trends and Applications*, North Oxford Acad. (1985).

[4] A. Mamourian, *First order nonlinear system of Lavrentiev type equations and Hilbert BVP*, in: Proc. of Asia Vibr. Conf. Edited by W. Bangchun, and T. Iwatsubo, Shenzehen, Northeast Univ. of Tech. (1989), 735–738.

[5] A. Mamourian, *On a mixed boundary values problem for Lavrentiev type equations*, Ann. Polon. Math., 45, (1985), 149–156.

[6] A. Mamourian, *BVP of a non-uniformly elliptic system of partial differential equations*, Demonstratio Math. 26, (1993), 735–741.

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