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RESOLUTION OF A COMPLETE ABSTRACT SECOND ORDER DIFFERENTIAL EQUATION OF ELLIPTIC TYPE

Abstract. In this work we give some new results on the complete abstract second order differential equation of elliptic type in non homogeneous case. The existence, the uniqueness and the maximal regularity of the strict solution are proved under some natural assumptions which imply the ellipticity of the differential equation.

1. Introduction

In this paper, we study the following second order abstract differential equation

$$u''(t) + 2Bu'(t) + Au(t) = f(t), \qquad t \in (0,1) \tag{1}$$

under the non homogeneous boundary conditions

$$\begin{cases} u(0) = \varphi, \\ u(1) = \psi, \end{cases} \tag{2}$$

where φ, ψ and $f(t)$ belong to a complex Banach space E, and A, B are two closed linear operators with domains $D(A)$, $D(B)$. We are interested in the existence, uniqueness and maximal regularity of the strict solution u when f is regular (for instance f is Hölder continuous function). We recall that $u \in C([0,1]\,; E)$ is a strict solution of $(1)-(2)$ if and only if

$$u \in C^2([0,1]\,; E) \cap C^1([0,1]\,; D(B)) \cap C([0,1]\,; D(A))$$

and u verifies $(1)-(2)$.

Throughout this paper we assume that

(H_1) B generates a strongly continuous group in E,

(H_2) $\begin{cases} \exists K > 0 / \forall \lambda \geqslant 0 \\ \|(A - B^2 - \lambda I)^{-1}\|_{L(E)} \leqslant K/(1+\lambda), \end{cases}$

$$(H_3)\quad \begin{cases} (A-B^2-\lambda I)^{-1}(B-\mu I)^{-1}-(B-\mu I)^{-1}(A-B^2-\lambda I)^{-1}=0 \\ \forall \mu \in \mathbb{R},\ \forall \lambda \geqslant 0, \end{cases}$$

$$(H_4)\quad \begin{cases} \exists K>0/\forall \lambda \geqslant 0 \\ \text{i) } \|B(A-B^2-\lambda I)^{-1}\|_{L(E)} \leqslant K/(1+\sqrt{\lambda}), \\ \text{ii) } \|B^2(A-B^2-\lambda I)^{-1}\|_{L(E)} \leqslant K, \\ \text{iii) } \|A(A-B^2-\lambda I)^{-1}\|_{L(E)} \leqslant K. \end{cases}$$

Note that (H_1) implies $\overline{D(B)} = E$. But the domain $D(A-B^2)$ may be not dense. On the other hand, it is not difficult to see that assumptions (H_1), (H_2), (H_3) permit us to apply the Da Prato- Grisvard[1] sum theory and deduce that, necessarily,

$$(A-\lambda I)^{-1} \in L(E) \text{ and } \left\|(A-\lambda I)^{-1}\right\|_{L(E)} \leqslant K/(1+\lambda), \text{ for all } \lambda \geqslant 0.$$

(See the proof of Theorem 1, the end of statement i)).

Equation (1) can be illustrated, for instance, by the following simple differential problem

$$(3)\quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x)+2b\dfrac{\partial^2 u}{\partial x \partial t}(t,x)+a\dfrac{\partial^2 u}{\partial x^2}(t,x)=f(t,x), & (t,x)\in\Sigma, \\ u(0,x)=\varphi(x), \quad u(1,x)=\psi(x), & x\in\mathbb{R}, \end{cases}$$

where $\Sigma =]0,1[\times \mathbb{R}$ and

$$(4)\qquad a-b^2>0.$$

Then we can choose in $E = L^p(\mathbb{R})$, $1<p<\infty$, the operators A, B defined by

$$\begin{cases} D(B)=W^{1,p}(\mathbb{R}), & (Bu)(x)=bu'(x),\ \forall u\in D(B) \\ D(A)=W^{2,p}(\mathbb{R}), & (Au)(x)=au''(x),\ \forall\, u\in D(A). \end{cases}$$

So, our objective is to have an unified study for the problems of type (1)-(2) in general case.

Several authors have studied equation (1) when it is regarded as an abstract Cauchy problem, that is under the following initial data $u(0)=\varphi$, $u'(0)=\psi$. See, for instance, Favini[4], Neubrander[10], Liang and Xiao[7]. The techniques used in these papers are based on the parabolicity of (1), that is on the Cauchy's data. In our study, hypotheses (H_2) and (H_4) express the ellipticity of (1) and generalize the one used in Krein [5] in the case $B=0$. In example (3) these two assumptions are equivalent to (4). So it is difficult to reduce equations (1)–(2) to some first order system. When $B=0$, Labbas [6] has studied problem (1)–(2) and has given necessary and sufficient conditions on φ, ψ, $f(0)$ and $f(1)$ for having a unique strict solution when f is Hölder continuous function. In this work we generalize these results since all the hypotheses considered here, coincide with those used in [6] in the case $B=0$. Note that assumptions $(H_1), (H_2), (H_3)$ allow us to apply the sum theory of linear operators as in Da Prato-Grisvard [1] and we have $D_A(\theta;+\infty) = D_{(A-B^2)}(\theta;+\infty) \cap D_{B^2}(\theta;+\infty)$ (see Grisvard [3]).

In section 2, we build the natural representation of the eventual solution of the problem (1)–(2) by using the operational calculus and the Dunford's integral.

In section 3, we prove an essential lemma, which allow us to justify and to study the optimal smoothness of the previous representation; we then give necessary and sufficient conditions on φ, ψ, $f(0)$ and $f(1)$ for having a strict solution when f is Hölder continuous function.

Finally, in section 4, we give some concrete examples in different spaces, to which our abstract results can be applied.

2. Construction of the solution

If A and B are two scalars such that $B^2 - A = -\lambda$ is strictly positive, then the solution of (1) is given by

$$u(t) = \frac{\sinh\sqrt{-\lambda}(1-t)}{\sinh\sqrt{-\lambda}}e^{-tB}\varphi + \frac{\sinh\sqrt{-\lambda}t}{\sinh\sqrt{-\lambda}}e^{(1-t)B}\psi - \int_0^1 G_{\sqrt{-\lambda}}(t,s)f(s)\,ds,$$

where

$$G_{\sqrt{-\lambda}}(t,s) = \begin{cases} G^+_{\sqrt{-\lambda}}(t,s) = \dfrac{\sinh\sqrt{-\lambda}(1-t)\sinh\sqrt{-\lambda}s}{\sqrt{-\lambda}\sinh\sqrt{-\lambda}}, & 0 \leqslant s \leqslant t \\ G^-_{\sqrt{-\lambda}}(t,s) = \dfrac{\sinh\sqrt{-\lambda}(1-s)\sinh\sqrt{-\lambda}t}{\sqrt{-\lambda}\sinh\sqrt{-\lambda}}, & t \leqslant s \leqslant 1. \end{cases}$$

Now, it is well known that (H_2) implies the existence of $\delta_0 \in \,]0,\pi/2[$ and $\varepsilon_0 > 0$ such that the resolvent set of $A - B^2$ contains the following sector of the complex plane

$$S(\delta_0,\varepsilon_0) = \{z \in \mathbb{C}/|\arg(z)| \leqslant \delta_0\} \cup B(0,\varepsilon_0),$$

where $B(0,\varepsilon_0)$ is the open ball of radius ε_0. If γ denotes the sectorial boundary curve of $S(\delta_0,\varepsilon_0)$ oriented positively, then the natural representation of the solution of (1)-(2), in the abstract case, is given by Dunford's integral

$$\begin{aligned} (5) \qquad u(t) = {} & \frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}\varphi d\lambda \\ & + \frac{e^{(1-t)B}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(1-t)(L-\lambda I)^{-1}\psi\, d\lambda \\ & - \frac{1}{2\pi i}\int_\gamma\int_0^1 G_{\sqrt{-\lambda}}(t,s)e^{-(t-s)B}(L-\lambda I)^{-1}f(s)\,ds\,d\lambda, \end{aligned}$$

where

$$\begin{cases} L = A - B^2, \ D(L) = D(A) \cap D(B^2), \\ g_{\sqrt{-\lambda}}(t) = \dfrac{\sinh\sqrt{-\lambda}(1-t)}{\sinh\sqrt{-\lambda}} \quad t \in (0,1), \ \lambda \in \gamma. \end{cases}$$

Here $\sqrt{-\lambda}$ is the analytic determination defined by $Re\sqrt{-\lambda} > 0$. From the easy relation

$$\left|g_{\sqrt{-\lambda}}(t)\right| = \left|e^{-\sqrt{-\lambda}t}\left(\frac{1 - e^{-2\sqrt{-\lambda}(1-t)}}{1 - e^{-2\sqrt{-\lambda}}}\right)\right|,$$

we deduce that there exists two constants c_0 and K_0 such that

$$\text{(6)} \qquad \begin{cases} \left|g_{\sqrt{-\lambda}}(t)\right| \leqslant K_0 e^{-c_0|\lambda|^{1/2}t} \quad \forall\lambda \in \gamma, \quad \forall t \in]0,1], \\ \left|g_{\sqrt{-\lambda}}(1-t)\right| \leqslant K_0 e^{-c_0|\lambda|^{1/2}(1-t)} \quad \forall\lambda \in \gamma, \quad \forall t \in [0,1[, \end{cases}$$

where $c_0 = \cos(\pi/2 - \delta_0/2)$ and $K_0 = \dfrac{2}{1 - e^{-2c_0\sqrt{\varepsilon_0}}}$.

On the other hand, for any $f \in C([0,1];E)$, we can see that

$$\text{(7)} \qquad \left\|\int_0^1 G_{\sqrt{-\lambda}}(t,s)f(s)\,ds\right\|_E \leqslant \frac{1}{|\lambda|\cos(\delta_0/2)} \|f\|_{C(E)}, \quad \forall\lambda \in \gamma.$$

According to (H_1), (H_2) and estimates (6), (7) all the integrals in (5) converge absolutely for every $t \in]0,1[$.

3. Smoothness of the solution

Consider, for φ and ψ in E, the following two integrals

$$U(t,-B,L)\varphi = \frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}\varphi\,d\lambda \quad t \in]0,1],$$

$$U(1-t,B,L)\psi = \frac{e^{(1-t)B}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(1-t)(L-\lambda I)^{-1}\psi\,d\lambda \quad t \in [0,1[.$$

Then we have the essential lemma

Lemma 1. *Under the assumptions (H_1) and (H_2)*

i) *there exists a positive constant $K = K(\varepsilon_0,\delta_0)$ such that*

$$\begin{cases} \|U(t,-B,L)\varphi\|_E \leqslant K\|\varphi\|_E, \ \forall\,\varphi \in E, \ \forall\, t \in]0,1], \\ \|U(1-t,B,L)\psi\|_E \leqslant K\|\psi\|_E, \ \forall\,\psi \in E, \ \forall\, t \in [0,1[, \end{cases}$$

ii) $U(.,-B,L)\varphi \in C([0,1];E) \iff \varphi \in \overline{D(L)}$,

iii) $U(1-.,B,L)\psi \in C([0,1];E) \iff \psi \in \overline{D(L)}$.

P r o o f.) For $t > 0$ we can write

$$\begin{aligned} U(t,-B,L)\varphi &= \frac{e^{-tB}}{2\pi i}\int_{\gamma_+^t} g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}\varphi\, d\lambda \\ &\quad + \frac{e^{-tB}}{2\pi i}\int_{\gamma_-^t} g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}\varphi\, d\lambda \\ &= I_+ + I_-, \end{aligned}$$

where

$$\gamma_+^t = \left\{z\in\gamma;\ |z|\geqslant 1/t^2\right\};\quad \gamma_-^t = \left\{z\in\gamma;\ |z|\leqslant 1/t^2\right\}. \tag{8}$$

Then we have

$$\begin{aligned} \|I_+\|_E &\leqslant K_0\left(\int_{1/t^2}^{+\infty} \frac{e^{-c_0|\lambda|^{1/2}t}}{|\lambda|}\, d|\lambda|\right)\|\varphi\|_E \\ &\leqslant 2K_0\int_1^{+\infty}\frac{e^{-c_0\sigma}}{\sigma}d\sigma\|\varphi\|_E \leqslant K\|\varphi\|_E. \end{aligned}$$

The second integral can be written as

$$\begin{aligned} I_- &= \frac{e^{-tB}}{2\pi i}\int_{\gamma_-^t}\left(g_{\sqrt{-\lambda}}(t) - g_{\sqrt{-\lambda}}(0)\right)(L-\lambda I)^{-1}\varphi\, d\lambda \\ &\quad + \frac{e^{-tB}}{2\pi i}\int_{\gamma_-^t}(L-\lambda I)^{-1}\varphi\, d\lambda \\ &= -\frac{e^{-tB}}{2\pi i}\int_{\gamma_-^t}\left(\int_0^t \frac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}(1-s)}{\sinh\sqrt{-\lambda}}ds\right)(L-\lambda I)^{-1}\varphi\, d\lambda \\ &\quad - \frac{e^{-tB}}{2\pi i}\int_{C_{1/t^2}}(L-\lambda I)^{-1}\varphi\, d\lambda \\ &= I_-' + I_-'', \end{aligned}$$

where

$$C_{1/t^2} = \left\{z \,/\, |\arg z|\leqslant \delta_0 \text{ and } |z| = 1/t^2\right\}.$$

Then

$$\|I_-'\|_E \leqslant \frac{K_0}{2\pi}\int_{\varepsilon_0}^{1/t^2}\frac{|\lambda|^{1/2}t}{|\lambda|}d|\lambda|\cdot\|\varphi\|_E \leqslant K\|\varphi\|_E,$$

$$\|I''_-\|_E \leqslant \frac{1}{2\pi}\int_{-\delta_0}^{+\delta_0} \|(L-\frac{e^{i\theta}}{t^2}I)^{-1}\varphi\|_E \frac{d\theta}{t^2} \leqslant K\|\varphi\|_E.$$

Similarly we obtain

$$\|U(1-t,B,L)\psi\|_E \leqslant K\,\|\psi\|_E\,, \quad \forall\,\psi\in E\;,\forall\,t\in[0,1[.$$

ii) Fix $\varepsilon>0$ and let $\varphi\in\overline{D(L)}$. Then there exists $y\in D(L)$ such that

(9) $$\|\varphi-y\|\leqslant\varepsilon.$$

Using the identity

$$(L-\lambda I)^{-1}y=\frac{(L-\lambda I)^{-1}Ly}{\lambda}-\frac{y}{\lambda},$$

we have

$$U(t,-B,L)y=\frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)\frac{(L-\lambda I)^{-1}}{\lambda}Ly\,d\lambda,$$

and this gives

(10) $$\|U(t,-B,L)y\|_E\leqslant\frac{1}{2\pi}\int_{\varepsilon_0}^{\infty}\frac{K}{|\lambda|^2}d|\lambda|\;\|Ly\|_E\leqslant\;K\,\|Ly\|_E\qquad\forall t\in[0,1].$$

Now, from the equality

$$U(t,-B,L)\varphi-\varphi=U(t,-B,L)\varphi-U(t,-B,L)y+U(t,-B,L)y-y+y-\varphi,$$

and the estimates (9), (10) we deduce that

$$U(t,-B,L)\varphi-\varphi\overset{t\to0^+}{\to}0.$$

The continuity in $t>0$ is easily verified.

Conversely, that $U(.,-B,L)\varphi\in C([0,1];E)$. Then $\lim_{t\to0^+}U(t,-B,L)\varphi$ exists and is necessarily equal to φ; however

$$g_{\sqrt{-\lambda}}(t)e^{-tB}(L-\lambda I)^{-1}\varphi\in D(L),$$

which implies that $U(t,-B,L)\varphi\in\overline{D(L)}$.

For statement iii) it is enough to substitute $1-t$ for t.

Let us consider, for $\theta\in\,]0,1[$, the well known real interpolation space between $D(L)$ and E characterized by

$$D_L(\theta;+\infty)=\{\varphi\in E/\sup_{r>0}r^\theta\left\|L(L-rI)^{-1}\varphi\right\|_E<\infty\}.$$

(See Grisvard [2]).

LEMMA 2. *Under assumptions (H_1), (H_2) and (H_3), and for $\theta\in\,]0,1/2[$ we have*

$$U(.,-B,L)\varphi\in C^{2\theta}([0,1]\,;E)\iff\varphi\in D_L(\theta;+\infty).$$

Proof. Let $\varphi \in D_L(\theta;+\infty)$ and $\tau, t \in]0,1[$ such that $\tau < t$, then

$$\begin{aligned}
U(t,-B,L)\varphi - U(\tau,-B,L)\varphi \\
&= \frac{e^{-tB}}{2\pi i}\int_\gamma (g_{\sqrt{-\lambda}}(t) - g_{\sqrt{-\lambda}}(\tau))\frac{L(L-\lambda I)^{-1}}{\lambda}\varphi\, d\lambda \\
&\quad + \frac{(e^{-tB} - e^{-\tau B})}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(\tau)\frac{L(L-\lambda I)^{-1}}{\lambda}\varphi\, d\lambda \\
&= I_1 + I_2.
\end{aligned}$$

I_1 may be written as

$$\begin{aligned}
I_1 &= \frac{e^{-tB}}{2\pi i}\int_{\gamma_+^{t-\tau}} (g_{\sqrt{-\lambda}}(t) - g_{\sqrt{-\lambda}}(\tau))\frac{L(L-\lambda I)^{-1}}{\lambda}\varphi\, d\lambda \\
&\quad + \frac{e^{-tB}}{2\pi i}\int_{\gamma_-^{t-\tau}} (g_{\sqrt{-\lambda}}(t) - g_{\sqrt{-\lambda}}(\tau))\frac{L(L-\lambda I)^{-1}}{\lambda}\varphi\, d\lambda \\
&= I_1' + I_1'',
\end{aligned}$$

(where $\gamma_+^{t-\tau}$ and $\gamma_-^{t-\tau}$ are defined in (8)), and

$$\|I_1'\|_E \leqslant 2K_0 \int_{\gamma_+^{t-\tau}} \frac{d\,|\lambda|}{|\lambda|^{\theta+1}}\, \|\varphi\|_{D_L(\theta;+\infty)},$$

$$\|I_1''\|_E \leqslant K_0 \int_{\gamma_-^{t-\tau}} \frac{|t-\tau|}{|\lambda|^{\theta+1/2}} d\,|\lambda|\, \|\varphi\|_{D_L(\theta;+\infty)}.$$

From the latter inequalities it follows that

$$\|I_1\|_E \leqslant K(t-\tau)^{2\theta}\, \|\varphi\|_{D_L(\theta;+\infty)}.$$

We may write I_2 as

$$I_2 = (t-\tau)^{2\theta}\, e^{-\tau B}\left(\frac{e^{-(t-\tau)B}\Phi_1 - \Phi_1}{(t-\tau)^{2\theta}}\right),$$

with

$$\Phi_1 = \frac{1}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(\tau)(L-\lambda I)^{-1}\varphi\, d\lambda.$$

Then, the classical operational calculus gives

$$L(L-rI)^{-1}\Phi_1 = \frac{1}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(\tau)\frac{L(L-\lambda I)^{-1}\varphi}{r-\lambda}\, d\lambda \quad \forall r > 0,$$

which implies that

$$\left\| L(L-rI)^{-1}\Phi_1 \right\|_E \leqslant \frac{K}{r^\theta} \left\| \varphi \right\|_{D_L(\theta;+\infty)} .$$

Therefore

$$\Phi_1 \in D_L(\theta;+\infty) \subset D_{B^2}(\theta;+\infty) = D_B(2\theta;+\infty),$$

the last equality holds by using the well known reiteration theorem in interpolation theory (see, Lions-Peetre[8]). So, we obtain

$$\left\| I_2 \right\|_E \leqslant K\,(t-\tau)^{2\theta} \left\| \Phi_1 \right\|_{D_B(2\theta;+\infty)} .$$

The main result in this work is the following

THEOREM 1. *Assume* $(H_1), (H_2), (H_3)$ *and* (H_4). *Let* $\varphi \in D(L), \psi \in D(L)$ *and* $f \in C^{2\theta}([0,1]\,;E)$ *with* $\theta \in \,]0,1/2[$. *Then* u *given in* (5) *verifies*

i) $u(t) \in D(A)$ $\forall t \in [0,1]$,

ii) $Au(.)$ *belongs to* $C([0,1]\,;E)$ *if and only if* $A\varphi, A\psi, f(0)$ *and* $f(1)$ *belong to* $\overline{D(L)}$,

iii) $Au(.)$ *belongs to* $C^{2\theta}([0,1]\,;E)$ *if and only if* $A\varphi, A\psi, f(0)$ *and* $f(1)$ *belong to* $D_L(\theta;+\infty)$.

Furthermore, if

(11) $\quad B\varphi \in D_L(\theta+1/2;+\infty)$ *and* $B\psi \in D_L(\theta+1/2;+\infty)$,

then

iv) $u'(t) \in D(B)$ $\forall t \in [0,1]$,

v) $Bu'(.) \in C([0,1];E)$ *if and only if* $B^2\varphi$ *and* $B^2\psi$ *belong to* $\overline{D(L)}$,

vi) $Bu'(.) \in C^{2\theta}([0,1];E)$ *if and only if* $B^2\varphi, B^2\psi$ *belong to* $D_L(\theta;+\infty)$.

Proof. Statement i). The study of the first and second integral in representation (5) is identical since t and $(1-t)$ have the same role. So we can assume that $\psi = 0$. Moreover, if in the third integral in (5) we write $f(s) = (f(s)-f(t)) + f(t)$, then, after two integrations by parts, we get

$$\begin{aligned} u(t) = {} & \frac{e^{-tB}}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}\varphi \, d\lambda \\ & - \frac{e^{-tB}}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}(B^2+\lambda I)^{-1} f(t)\, d\lambda \end{aligned}$$

$$
\begin{aligned}
&- \frac{e^{(1-t)B}}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(1-t)(L-\lambda I)^{-1}(B^2+\lambda I)^{-1} f(t)\, d\lambda \\
&- \frac{1}{2\pi i} \int_\gamma \int_0^1 G_{\sqrt{-\lambda}}(t,s) e^{-(t-s)B}(L-\lambda I)^{-1}(f(s)-f(t))\, ds\, d\lambda \\
&+ \frac{1}{2\pi i} \int_\gamma (B^2+\lambda I)^{-1}(L-\lambda I)^{-1} f(t)\, d\lambda \\
= &\sum_{i=1}^{5} a_i .
\end{aligned}
$$

All these integrals converge absolutely. From Lemma 1 and (H_4), we deduce that the first integral a_1 belongs to $D(A)$. Writing $A = L + B^2 = (L - \lambda I) + \big(B^2 + \lambda I\big)$ and using (H_4) we obtain

$$A(L-\lambda I)^{-1}(B^2+\lambda I)^{-1} f(t) = (L-\lambda I)^{-1} f(t) + (B^2+\lambda I)^{-1} f(t),$$

which implies, as for the first integral, that a_2 and a_3 belong to $D(A)$. For a_4, we use *iii*) of (H_5), getting

$$
\begin{aligned}
&\left\| \frac{1}{2\pi i} \int_\gamma \int_0^1 G_{\sqrt{-\lambda}}(t,s) e^{-(t-s)B} A (L-\lambda I)^{-1}(f(s)-f(t))\, ds\, d\lambda \right\|_E \\
&\qquad \leqslant K \int_{\varepsilon_0}^{\infty} \Big(\sup_{0 \leqslant t \leqslant 1} \int_0^1 \big|G_{\sqrt{-\lambda}}(t,s)\big|\, |t-s|^{2\theta}\, ds \Big) d|\lambda|\, \|f\|_{C^{2\theta}(E)} \\
&\qquad \leqslant K \Big(\int_{\varepsilon_0}^{\infty} \frac{1}{|\lambda|^{1+\theta}} d|\lambda| \Big) \|f\|_{C^{2\theta}(E)} ,
\end{aligned}
$$

(the last estimate follows in virtue of Hölder's inequality). For a_5 we apply the Da Prato-Grisvard's[1] sums theory to B^2 and L. In fact, from (H_1) we deduce that B^2 generates a bounded holomorphic semigroup in E (see Stone[12]); then $(B^2 + L)$ is closable and

$$(\overline{B^2 + L})^{-1} = S = \frac{1}{2\pi i} \int_\gamma (B^2+\lambda I)^{-1}(L-\lambda I)^{-1}\, d\lambda,$$

which implies that

$$Sx = A^{-1}x, \ \forall x \in E,$$

so

$$a_5 = A^{-1} f(t).$$

Statements ii) and iii). We have

$$\begin{aligned}
Au(t) = & \frac{e^{-tB}}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1} A\varphi \, d\lambda \\
& - \frac{e^{-tB}}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(t)(B^2+\lambda I)^{-1} f(t) \, d\lambda \\
& - \frac{e^{-tB}}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1} f(t) \, d\lambda \\
& - \frac{e^{(1-t)B}}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(1-t)(B^2+\lambda I)^{-1} f(t) \, d\lambda \\
& - \frac{e^{(1-t)B}}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(1-t)(L-\lambda I)^{-1} f(t) \, d\lambda \\
& - \frac{1}{2\pi i} \int_\gamma \int_0^1 G_{\sqrt{-\lambda}}(t,s) e^{-(t-s)B} A(L-\lambda I)^{-1}(f(s)-f(t)) \, ds \, d\lambda + f(t) \\
= & \sum_{i=1}^{6} v_i(t) + f(t).
\end{aligned}$$

As in Lemmas 1–2, $v_1(.) \in C([0,1]\,;E)$ if and only if $A\varphi \in \overline{D(L)}$ and $v_1(.) \in C^{2\theta}([0,1]\,;E)$ if and only if $A\varphi \in D_L(\theta;+\infty)$. Let us write for $0 \leqslant \tau < t \leqslant 1$

$$\begin{aligned}
v_2(t) - v_2(\tau) = & - \frac{e^{-tB}}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(t)(B^2+\lambda I)^{-1}(f(t)-f(\tau)) \, d\lambda \\
& - \frac{e^{-\tau B}}{2\pi i} \int_\gamma (g_{\sqrt{-\lambda}}(t) - g_{\sqrt{-\lambda}}(\tau))(B^2+\lambda I)^{-1}(f(\tau)-f(0)) \, d\lambda \\
& - \frac{(e^{-tB} - e^{-\tau B})}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(t)(B^2+\lambda I)^{-1}(f(\tau)-f(0)) \, d\lambda \\
& + \frac{e^{-tB}}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(t)(B^2+\lambda I)^{-1} f(0) \, d\lambda \\
& - \frac{e^{-\tau B}}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(\tau)(B^2+\lambda I)^{-1} f(0) \, d\lambda \\
= & \sum_{i=1}^{5} b_i.
\end{aligned}$$

Then, as in the proof of Lemma 1, we have

$$\|b_1\| \leqslant K(t-\tau)^{2\theta}\,\|f\|_{C^{2\theta}(E)}\,.$$

Writing b_2 as

$$b_2 = -\frac{e^{-\tau B}}{2\pi i}\int_\tau^t\int_\gamma \frac{\sqrt{-\lambda}\cosh\sqrt{-\lambda}(1-s)}{\sinh\sqrt{-\lambda}}(B^2+\lambda I)^{-1}(f(\tau)-f(0))\,d\lambda\,ds,$$

we get

$$\begin{aligned}
\|b_2\|_E &\leqslant K\int_\tau^t\int_{\varepsilon_0}^{+\infty}\frac{e^{-|\lambda|^{1/2}c_0 s}\tau^{2\theta}}{|\lambda|^{1/2}}d\,|\lambda|\,ds\,\|f\|_{C^{2\theta}(E)}\\
&\leqslant K\int_\tau^t\int_{\sqrt{\varepsilon_0}s}^{+\infty}\frac{e^{-\sigma c_0}\tau^{2\theta}}{s}d\sigma ds\,\|f\|_{C^{2\theta}(E)}\\
&\leqslant K\int_\tau^t\Big(\int_0^{+\infty}e^{-\sigma c_0}d\sigma\Big)s^{2\theta-1}\,ds\,\|f\|_{C^{2\theta}(E)}\\
&\leqslant K(t^{2\theta}-\tau^{2\theta})\,\|f\|_{C^{2\theta}(E)}\\
&\leqslant K(t-\tau)^{2\theta}\,\|f\|_{C^{2\theta}(E)}\,.
\end{aligned}$$

For b_3, we have

$$b_3 = \frac{1}{2\pi i}\int_\tau^t\int_\gamma g_{\sqrt{-\lambda}}(t)e^{-sB}B(B^2+\lambda I)^{-1}(f(\tau)-f(0))\,d\lambda\,ds,$$

and

$$\begin{aligned}
\|b_3\|_E &\leqslant K\int_\tau^t\int_{\varepsilon_0}^{+\infty}\frac{e^{-|\lambda|^{1/2}c_0 t}\tau^{2\theta}}{|\lambda|^{1/2}}d\,|\lambda|\,ds\,\|f\|_{C^{2\theta}(E)}\\
&\leqslant K\int_\tau^t\int_{\varepsilon_0}^{+\infty}\frac{e^{-|\lambda|^{1/2}c_0 s}\tau^{2\theta}}{|\lambda|^{1/2}}d\,|\lambda|\,ds\,\|f\|_{C^{2\theta}(E)}\\
&\leqslant K(t-\tau)^{2\theta}\,\|f\|_{C^{2\theta}(E)}\,.
\end{aligned}$$

The sum b_4+b_5 can be estimated by the same method used in Lemma 2. Then we obtain that

$$\|b_4+b_5\|_E \leqslant K(t-\tau)^{2\theta}\,\|f(0)\|_{D_{B^2}(\theta;+\infty)}\,,$$

if and only if $f(0)\in D_{B^2}(\theta;+\infty)$. Similarly $v_3(.)$ belongs to $C^{2\theta}([0,1]\,;E)$ if and only if $f(0)$ belongs to $D_L(\theta;+\infty)$ and $v_4(.)+v_5(.)$ belongs to $C^{2\theta}([0,1]\,;E)$ if and only if $f(1)$ belongs to $D_L(\theta;+\infty)$.

Finally

$$
\begin{aligned}
&v_6(t) - v_6(\tau) \\
= &-\frac{1}{2\pi i}\int_\gamma\int_\tau^t G^+_{\sqrt{-\lambda}}(t,s)e^{-(t-s)B}A(L-\lambda I)^{-1}(f(s)-f(t))\,ds\,d\lambda \\
&-\frac{1}{2\pi i}\int_\gamma\int_0^\tau (G^+_{\sqrt{-\lambda}}(t,s) - G^+_{\sqrt{-\lambda}}(\tau,s)) \\
&\times e^{-(t-s)B}A(L-\lambda I)^{-1}(f(s)-f(t))\,ds\,d\lambda \\
&-\frac{1}{2\pi i}\int_\gamma\int_0^\tau G^+_{\sqrt{-\lambda}}(t,s)(e^{-(t-s)B} - e^{-(\tau-s)B}) \\
&\times A(L-\lambda I)^{-1}(f(s)-f(t))\,ds\,d\lambda \\
&-\frac{1}{2\pi i}\int_\gamma\int_\tau^1 G^-_{\sqrt{-\lambda}}(t,s)(e^{(s-t)B} - e^{(s-\tau)B})A(L-\lambda I)^{-1}(f(s)-f(t))\,ds\,d\lambda \\
&+\frac{1}{2\pi i}\int_\gamma\int_\tau^t G^-_{\sqrt{-\lambda}}(t,s)e^{(s-t)B}A(L-\lambda I)^{-1}(f(s)-f(t))\,ds\,d\lambda \\
&-\frac{1}{2\pi i}\int_\gamma\int_\tau^1 (G^-_{\sqrt{-\lambda}}(t,s) - G^-_{\sqrt{-\lambda}}(\tau,s)) \\
&\times e^{(s-t)B}A(L-\lambda I)^{-1}(f(s)-f(t))\,ds\,d\lambda \\
&+\frac{e^{-\tau B}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(\tau)A(L-\lambda I)^{-1}(B^2+\lambda I)^{-1}(f(t)-f(\tau))\,d\lambda \\
&+\frac{e^{(1-\tau)B}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(1-\tau)A(L-\lambda I)^{-1}(B^2+\lambda I)^{-1}(f(t)-f(\tau))\,d\lambda \\
&-(f(t)-f(\tau)) \\
= &\sum_{i=1}^{8} c_i - (f(t)-f(\tau)).
\end{aligned}
$$

Using the fact that in the first integral c_1 (since $0 \leqslant \tau < s < t \leqslant 1$)

$$
|G^+_{\sqrt{-\lambda}}(t,s)| \leqslant K\frac{e^{-|\lambda|^{1/2}c_0(t-s)}}{|\lambda|^{1/2}},
$$

have

$$\|c_1\|_E \leqslant K \int_{\tau}^{t} \int_{\varepsilon_0}^{+\infty} \frac{e^{-|\lambda|^{1/2} c_0 (t-s)}}{|\lambda|^{1/2}} (t-s)^{2\theta} d|\lambda|\, ds\, \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K \int_{\tau}^{t} (t-s)^{2\theta-1} \Big(\int_{\sqrt{\varepsilon_0}(t-s)}^{+\infty} e^{-\sigma c_0} d\sigma \Big) ds\, \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K (t-\tau)^{2\theta} \|f\|_{C^{2\theta}(E)} \cdot$$

iting c_2 as

$$c_2 = -\frac{1}{2\pi i} \int_0^{\tau} \int_{\gamma} \int_{\tau}^{t} \frac{\cosh \sqrt{-\lambda}(1-\xi) \sinh \sqrt{-\lambda} s}{\sinh \sqrt{-\lambda}}$$

$$\times e^{-(\tau - s)B} A (L - \lambda I)^{-1} (f(s) - f(t))\, d\xi\, d\lambda\, ds,$$

obtain

$$\|c_2\|_E \leqslant K \int_0^{\tau} \int_{\varepsilon_0}^{+\infty} \int_{\tau}^{t} e^{-|\lambda|^{1/2} c_0 (\xi - s)} (t-s)^{2\theta}\, d\xi d|\lambda|\, ds\, \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K \int_0^{\tau} \int_{\tau - s}^{t-s} \int_{\varepsilon_0}^{+\infty} e^{-|\lambda|^{1/2} c_0 \eta} (t-s)^{2\theta}\, d|\lambda|\, d\eta ds\, \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K \int_0^{\tau} \int_{\tau - s}^{t-s} \frac{(t-s)^{2\theta}}{\eta^2} \Big(\int_{\sqrt{\varepsilon_0}\eta}^{+\infty} \sigma e^{-\sigma c_0} d\sigma \Big) d\eta\, ds\, \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K \int_0^{\tau} (t-s)^{2\theta} \Big(\int_{\tau - s}^{t-s} \frac{d\eta}{\eta^2} \Big) ds\, \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K \int_0^{\tau} (t-s)^{2\theta} \Big(\frac{1}{\tau - s} - \frac{1}{t-s} \Big) ds\, \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K (t-\tau) \int_0^{\tau} \frac{|t-s|^{2\theta-1}}{\tau - s} ds\, \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K (t-\tau)^{2\theta} \Big(\int_0^{+\infty} \frac{z^{2\theta-1}}{1+z} dz \Big) \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K (t-\tau)^{2\theta} \|f\|_{C^{2\theta}(E)} \cdot$$

$c_3 + c_4$, we write

$$c_3 + c_4 = e^{-\tau B} (e^{-(t-\tau)B} \Phi_2 - \Phi_2),$$

where

$$\Phi_2 = \frac{-1}{2\pi i}\int_\gamma\int_0^\tau G^+_{\sqrt{-\lambda}}(t,s)e^{sB}A(L-\lambda I)^{-1}(f(s)-f(t))\,ds\,d\lambda$$
$$-\frac{1}{2\pi i}\int_\gamma\int_\tau^1 G^-_{\sqrt{-\lambda}}(t,s)e^{sB}A(L-\lambda I)^{-1}(f(s)-f(t))\,ds\,d\lambda.$$

Using the fact that for $r>0$ and $\lambda\in\gamma$

$$L(L-rI)^{-1}A(L-\lambda I)^{-1} = \frac{r}{r-\lambda}A(L-rI)^{-1} - \frac{\lambda}{r-\lambda}A(L-\lambda I)^{-1}$$

we get

$$\begin{aligned}
&L(L-rI)^{-1}\Phi_2\\
&\quad= \frac{1}{2\pi i}\int_\gamma\int_0^\tau G^+_{\sqrt{-\lambda}}(t,s)e^{sB}\frac{\lambda}{r-\lambda}A(L-\lambda I)^{-1}(f(s)-f(t))\,ds\,d\lambda\\
&\qquad+\frac{1}{2\pi i}\int_\gamma\int_\tau^1 G^-_{\sqrt{-\lambda}}(t,s)e^{sB}\frac{\lambda}{r-\lambda}A(L-\lambda I)^{-1}(f(s)-f(t))\,ds\,d\lambda,
\end{aligned}$$

and

$$\begin{aligned}
&\left\|L(L-rI)^{-1}\Phi_2\right\|_E\\
&\quad\leqslant K\int_{\varepsilon_0}^{+\infty}\Big(\sup_{0\leqslant t\leqslant 1}\int_0^1 |G_{\sqrt{-\lambda}}(t,s)||s-t|^{2\theta}ds\Big)\frac{|\lambda|}{|r-\lambda|}\,d|\lambda|\,\|f\|_{C^{2\theta}(E)}\\
&\quad\leqslant K\left(\int_{\varepsilon_0}^{+\infty}\frac{d|\lambda|}{|\lambda|^\theta|r-\lambda|}\right)\|f\|_{C^{2\theta}(E)}\\
&\quad\leqslant \frac{K}{r^\theta}\|f\|_{C^{2\theta}(E)}.
\end{aligned}$$

Therefore $\Phi_2\in D_L(\theta;+\infty)\subset D_{B^2}(\theta;+\infty)=D_B(2\theta;+\infty)$. Thus we have

$$\begin{aligned}
\|c_3+c_4\|_E &\leqslant K(t-\tau)^{2\theta}\sup_{t-\tau>0}\left\|\frac{e^{-(t-\tau)B}\Phi_2-\Phi_2}{(t-\tau)^{2\theta}}\right\|_E\\
&\leqslant K(t-\tau)^{2\theta}\,\|\Phi_2\|_{D_B(2\theta;+\infty)}\,.
\end{aligned}$$

By reasoning in the same manner for the last integral we obtain

$$\|v_6(t)-v_6(\tau)\|_E = O\left((t-\tau)^{2\theta}\right).$$

Statement iv). We have

$$
\begin{aligned}
u'(t) = {} & \frac{e^{-tB}}{2\pi i}\int_\gamma g'_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}\varphi\,d\lambda \\
& - \frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}B\varphi\,d\lambda \\
& - \frac{e^{-tB}}{2\pi i}\int_\gamma g'_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}(B^2+\lambda I)^{-1}f(t)\,d\lambda \\
& + \frac{e^{-tB}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(t)B(B^2+\lambda I)^{-1}(L-\lambda I)^{-1}f(t)\,d\lambda \\
& - \frac{e^{(1-t)B}}{2\pi i}\int_\gamma g'_{\sqrt{-\lambda}}(1-t)(L-\lambda I)^{-1}(B^2+\lambda I)^{-1}f(t)\,d\lambda \\
& + \frac{e^{(1-t)B}}{2\pi i}\int_\gamma g_{\sqrt{-\lambda}}(1-t)B(B^2+\lambda I)^{-1}(L-\lambda I)^{-1}f(t)\,d\lambda \\
& + \frac{1}{2\pi i}\int_\gamma\int_0^1 G_{\sqrt{-\lambda}}(t,s)e^{-(t-s)B}B(L-\lambda I)^{-1}(f(s)-f(t))\,ds\,d\lambda \\
& - \frac{1}{2\pi i}\int_\gamma\int_0^1 \partial_t G_{\sqrt{-\lambda}}(t,s)e^{-(t-s)B}(L-\lambda I)^{-1}(f(s)-f(t))\,ds\,d\lambda \\
= {} & \sum_{i=1}^{8} w_i(t).
\end{aligned}
$$

The derivative of the last integral in (5) is obtained by differentiation the integrand then writing $f(s) = (f(s) - f(t)) + f(t)$ and carrying out two integrations by parts. We verify that all these integrals converge absolutely. The first integral $w_1(t) \in D(B)$. In fact, writing

$$
\frac{e^{-tB}}{2\pi i}\int_\gamma g'_{\sqrt{-\lambda}}(t)(L-\lambda I)^{-1}B\varphi\,d\lambda = \frac{e^{-tB}}{2\pi i}\int_\gamma g'_{\sqrt{-\lambda}}(t)\frac{L(L-\lambda I)^{-1}B\varphi}{\lambda}\,d\lambda,
$$

we have

$$
\begin{aligned}
&\left\|\frac{e^{-tB}}{2\pi i}\int_\gamma g'_{\sqrt{-\lambda}}(t)\frac{L(L-\lambda I)^{-1}B\varphi}{\lambda}\,d\lambda\right\|_E \\
&\qquad \leqslant K\Big(\int_{\varepsilon_0}^{+\infty}\frac{e^{-|\lambda|^{1/2}c_0 t}}{|\lambda|^{1+\theta}}d|\lambda|\Big)\,\|B\varphi\|_{D_L(\theta+1/2;+\infty)}
\end{aligned}
$$

$$\leqslant K \left\| B\varphi \right\|_{D_L(\theta+1/2;+\infty)} .$$

Due to Lemma 1 and (H_4), we have $w_2(t) \in D(B)$. Moreover, we can write

$$\begin{aligned}
& w_3(t) + w_4(t) \\
&= -\frac{e^{-tB}}{2\pi i} \int_\gamma \left(\sqrt{-\lambda} g_{\sqrt{-\lambda}}(t) - g'_{\sqrt{-\lambda}}(t) \right) (L - \lambda I)^{-1} (B^2 + \lambda I)^{-1} f(t)\, d\lambda \\
&\quad + \frac{e^{-tB}}{2\pi i} \int_\gamma \sqrt{-\lambda} g_{\sqrt{-\lambda}}(t) (L - \lambda I)^{-1} (B^2 + \lambda I)^{-1} f(t)\, d\lambda \\
&\quad + \frac{e^{-tB}}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(t) B (B^2 + \lambda I)^{-1} (L - \lambda I)^{-1} f(t)\, d\lambda \\
&= \frac{e^{-tB}}{\pi i} \int_\gamma \frac{\sqrt{-\lambda} e^{-\sqrt{-\lambda}(2-t)}}{1 - e^{-2\sqrt{-\lambda}}} (L - \lambda I)^{-1} (B^2 + \lambda I)^{-1} f(t)\, d\lambda \\
&\quad + \frac{e^{-tB}}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(t) \left(B - \sqrt{-\lambda} I \right)^{-1} (L - \lambda I)^{-1} f(t)\, d\lambda,
\end{aligned}$$

which implies that $w_3(t) + w_4(t) \in D(B)$. Similarly, we show that the sum $w_5(t) + w_6(t) \in D(B)$. As in the proof of statement i), we also obtain that $w_7(t)$ and $w_8(t)$ belong to $D(B)$.

Statements v) and vi). We have

$$\begin{aligned}
Bu'(t) &= \frac{e^{-tB}}{2\pi i} \int_\gamma g'_{\sqrt{-\lambda}}(t) (L - \lambda I)^{-1} B\varphi\, d\lambda \\
&\quad - \frac{e^{-tB}}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(t) (L - \lambda I)^{-1} B^2 \varphi\, d\lambda \\
&\quad - \frac{e^{-tB}}{2\pi i} \int_\gamma g'_{\sqrt{-\lambda}}(t) B (L - \lambda I)^{-1} (B^2 + \lambda I)^{-1} f(t)\, d\lambda \\
&\quad + \frac{e^{-tB}}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(t) B^2 (B^2 + \lambda I)^{-1} (L - \lambda I)^{-1} f(t)\, d\lambda \\
&\quad - \frac{e^{(1-t)B}}{2\pi i} \int_\gamma g'_{\sqrt{-\lambda}}(1-t) B (L - \lambda I)^{-1} (B^2 + \lambda I)^{-1} f(t)\, d\lambda \\
&\quad + \frac{e^{(1-t)B}}{2\pi i} \int_\gamma g_{\sqrt{-\lambda}}(1-t) B^2 (B^2 + \lambda I)^{-1} (L - \lambda I)^{-1} f(t)\, d\lambda
\end{aligned}$$

$$+ \frac{1}{2\pi i} \int_\gamma \int_0^1 G_{\sqrt{-\lambda}}(t,s) e^{-(t-s)B} B^2 (L - \lambda I)^{-1} (f(s) - f(t))\, ds\, d\lambda$$

$$- \frac{1}{2\pi i} \int_\gamma \int_0^1 \partial_t G_{\sqrt{-\lambda}}(t,s) e^{-(t-s)B} B^2 (L - \lambda I)^{-1} (f(s) - f(t))\, ds\, d\lambda$$

$$= \sum_{i=1}^{8} x_i(t),$$

and for $0 \leqslant \tau < t \leqslant 1$, we get

$$\begin{aligned} x_1(t) - x_1(\tau) = & - \frac{e^{-tB}}{2\pi i} \int_\tau^t \int_\gamma g_{\sqrt{-\lambda}}(s) L (L - \lambda I)^{-1} B\varphi\, d\lambda\, ds \\ & + \frac{(e^{-tB} - e^{-\tau B})}{2\pi i} \int_\gamma g'_{\sqrt{-\lambda}}(\tau) \frac{L(L - \lambda I)^{-1} B\varphi}{\lambda}\, d\lambda \\ = & \, I_1 + I_2. \end{aligned}$$

Then

$$\begin{aligned} \|I_1\|_E & \leqslant K_0 \int_\tau^t \int_{\varepsilon_0}^{+\infty} \frac{e^{-|\lambda|^{1/2} c_0 s}}{|\lambda|^{\theta+1/2}} d|\lambda|\, ds \, \|B\varphi\|_{D_L(\theta+1/2;+\infty)} \\ & \leqslant K_0 \int_\tau^t \int_{\sqrt{\varepsilon_0} s}^{+\infty} \frac{2\sigma e^{-\sigma c_0}}{\left(\frac{\sigma}{s}\right)^{2\theta+1} s^2} d\sigma ds \, \|B\varphi\|_{D_L(\theta+1/2;+\infty)} \\ & \leqslant K \int_\tau^t s^{2\theta-1} \left(\int_0^{+\infty} \frac{e^{-\sigma c_0}}{\sigma^{2\theta}} d\sigma \right) ds \, \|B\varphi\|_{D_L(\theta+1/2;+\infty)} \\ & \leqslant K(t-\tau)^{2\theta} \|B\varphi\|_{D_L(\theta+1/2;+\infty)} . \end{aligned}$$

Writing I_2 as

$$I_2 = e^{-\tau B} (e^{-(t-\tau)B} \Phi_3 - \Phi_3),$$

with

$$\Phi_3 = \frac{1}{2\pi i} \int_\gamma g'_{\sqrt{-\lambda}}(\tau) \frac{L(L - \lambda I)^{-1} B\varphi}{\lambda}\, d\lambda,$$

we obtain

$$\left\| L(L - rI)^{-1} \Phi_3 \right\|_E \leqslant \frac{K}{r^\theta} \|B\varphi\|_{D_L(\theta+1/2;+\infty)} \quad \forall r > 0.$$

Therefore $\Phi_3 \in D_L(\theta; +\infty) \subset D_B(2\theta; +\infty)$. Thus we have

$$\|I_2\|_E \leqslant K(t-\tau)^{2\theta} \sup_{t-\tau>0} \left\| \frac{e^{-(t-\tau)B} \Phi_3 - \Phi_3}{(t-\tau)^{2\theta}} \right\|_E$$

$$\leqslant K(t-\tau)^{2\theta}\left\|\Phi_3\right\|_{D_B(2\theta;+\infty)}.$$

The techniques for the rest of the proof is quite similar to those used in the preceeding statements.

REMARK 1. *In the same manner we have a similar result to Theorem* 1 *when the second member f has a spatial smoothness, that is*

$$f \in \{f \in L^\infty(E)/f(t) \in D_L(\sigma,\infty)\ \forall t \in [0,1],\ \sup_{t\in(0,1)} \|f(t)\|_{D_L(\sigma,\infty)} < \infty\}$$

where $L^\infty(E)$ is the set of the strongly measurable and bounded functions f defined on $[0,1]$. *We omit to prove this assertion in this work.*

REMARK 2. *We can consider other hypotheses instead of* (11) *involving a comparison of $D(B)$ and $D\left((-L)^\alpha\right)$ for some $\alpha \in]0,1[$.*

4. Examples

Let $a, b \in \mathbb{R}$, $\theta \in]0,1/2[$ be given.

4.1. In Hilbertian spaces

In $E = L^2(\mathbb{R})$, we define A and B by

$$\begin{cases} D(A) = H^4(\mathbb{R}) \\ Au = au^{(4)} \end{cases} ; \qquad \begin{cases} D(B) = H^2(\mathbb{R}) \\ Bu = ibu'' \end{cases}$$

where $a > 0$. Then B generates a C_0-group (see Pazy [11], p. 224). Moreover, it is well known that

$$\begin{aligned} D_L(\theta;+\infty) &= (D(L);E)_{1-\theta,\infty} \\ &= (H^4(\mathbb{R});L^2(\mathbb{R}))_{1-\theta,\infty} \\ &= B^{4\theta}_{2,\infty}(\mathbb{R}). \end{aligned}$$

The last space is a Besov space defined, for instance, in Grisvard[3]. It is not difficult to see that $(H_1), (H_2), (H_3)$, and (H_4) are verified. Applying Theorem 1 we have

THEOREM 2. *Let $f \in C^{2\theta}([0,1];L^2(\mathbb{R}))$ be such that for $j = 0,1$ the mappings $x \mapsto f(j,x)$ belong to $B^{4\theta}_{2,\infty}(\mathbb{R})$ and assume that φ, ψ belong to $H^4(\mathbb{R})$ and $\varphi^{(4)}$,$\psi^{(4)}$ belong to $B^{4\theta}_{2,\infty}(\mathbb{R})$. Then the problem*

$$\begin{cases} \partial_t^2 u + 2ib\partial_x^2\partial_t u + a\partial_x^4 u = f, & \text{in }]0,1[\times\mathbb{R}, \\ u(0,x) = \varphi(x), \\ u(1,x) = \psi(x), \end{cases}$$

has a unique solution u verifying

i) $u \in C^2([0,1];L^2(\mathbb{R})) \cap C^1([0,1],H^2(\mathbb{R})) \cap C([0,1],H^4(\mathbb{R}))$,
ii) $\partial_x^4 u, \partial_x^2\partial_t u$ *belong to* $C^{2\theta}([0,1];L^2(\mathbb{R}))$.

Analogous results can be given in all the followings cases

$$\begin{cases} D(A) = H^4(\mathbb{R}^2) \\ Au = a\Delta^2 u \end{cases}; \qquad \begin{cases} D(B) = H^2(\mathbb{R}^2) \\ Bu = ib\Delta u \end{cases}.$$

4.2. In L^p spaces

$E = L^p(\mathbb{R})$, $1 < p < \infty$ and

$$\begin{cases} D(A) = W^{2,p}(\mathbb{R}) \\ Au = au'' \end{cases}; \qquad \begin{cases} D(B) = W^{1,p}(\mathbb{R}) \\ Bu = bu' \end{cases}$$

4.3. In continuous functions spaces

In $E = C_0(\mathbb{R}) = \{u \in C(\mathbb{R}) / \lim_{|t|\to\infty} u(t) = 0\}$, consider the following operators defined by

$$\begin{cases} D(A) = \{u \in C^2(\mathbb{R}) / u, u'' \in C_0(\mathbb{R})\} \\ Au = au'', \quad u \in D(A) \end{cases}$$

and

$$\begin{cases} D(B) = \{u \in C_0(\mathbb{R}) / u' \in C_0(\mathbb{R})\} \\ Bu = bu', \quad u \in D(B). \end{cases}$$

Then B generates a bounded C_0-group on $C_0(\mathbb{R})$ (see Nagel [9], p. 9)). In this case we have $D_L(\theta; +\infty) = C^{2\theta}(\mathbb{R}) \cap C_0(\mathbb{R})$.

We can also consider cases of spaces defined on bounded sets.

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